On the Navier–Stokes equations in a curved thin domain

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1 Introduction

We consider the incompressible Navier–Stokes equations in a three-dimensional curved thin domain with Navier's slip boundary conditions

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon - \nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ \operatorname{div} u^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ u^\varepsilon \cdot n_\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ \nu[D(u^\varepsilon)n_\varepsilon]_{\tan} + \gamma_\varepsilon u^\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ u^\varepsilon|_{t=0} = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon. \end{cases} \tag{1.1}$$

Here Ω_ε is a curved thin domain in $\mathbb{R}^3$ with very small width of order $\varepsilon \in (0,1)$ around a given closed two-dimensional surface Γ and Γ_ε is the boundary of Ω_ε (for precise definitions see Section 2). Also, $\nu > 0$ is the viscosity coefficient, n_ε is the unit outward normal vector to Γ_ε, $D(u^\varepsilon) := \{\nabla u^\varepsilon + (\nabla u^\varepsilon)^T\}/2$ is the strain rate tensor, $[D(u^\varepsilon)n_\varepsilon]_{\tan}$ is the tangential component of the stress vector $D(u^\varepsilon)n_\varepsilon$ on Γ_ε, and γ_ε is the friction coefficient.

Fluid flows in a thin domain appear in many problems of natural sciences, e.g. flow of water in a large lake and the geophysical dynamics such as the ocean and atmosphere dynamics. In the study of the Navier–Stokes equations in a three-dimensional thin domain mathematical researchers are mainly interested in the global-in-time existence of a strong solution for large data, since a three-dimensional thin domain with very small width can be seen as almost two-dimensional. It is also important to analyze singular limit problems for degeneration of a thin domain and compare the dynamics of bulk flows in a thin domain and limit flows in its degenerate set. There is a large number of literature studying such problems in a flat thin domain [5, 6, 7, 10, 12] of the form

$$\Omega_\varepsilon = \{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \mid (x_1,x_2) \in \omega,\ \varepsilon g_0(x_1,x_2) < x_3 < \varepsilon g_2(x_1,x_2)\},$$

where ω is a domain in $\mathbb{R}^2$ and g_0, g_1 are functions on ω. A thin spherical domain $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid a < |x| < a(1+\varepsilon)\}$, $a > 0$ was also investigated in [13]. However, the mathematical study of the Navier–Stokes equations in a thin domain has not been done in the case where the degenerate set of a thin domain has more complicated

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geometry (see [9] for the study of a reaction-diffusion equation in a thin domain around a lower dimensional manifold). Recently, the present author established the global-in-time existence of a strong solution to (1.1) for large data of order $\varepsilon^{-1/2}$ when the degenerate set is a general closed smooth surface [8]. In this paper we give a result of [8] in a restricted case and show an outline of its proof. By $\mathbb{P}_\varepsilon$ and A_ε we denote the Helmholtz–Leray projection on $L^2(\Omega_\varepsilon)^3$ and the Stokes operator on $L^2(\Omega_\varepsilon)^3$ associated with slip boundary conditions (see Section 3.2). Also, we write M_τ for the tangential component (with respect to the surface Γ) of the average operator in the thin direction (see Section 3.3 for a precise definition).

Theorem 1.1. *Suppose that there exists a constant $c > 0$ such that*

$$c^{-1}\varepsilon \leq \gamma_\varepsilon \leq c\varepsilon \quad \text{for all} \quad \varepsilon \in (0,1). \tag{1.2}$$

Then there exist constants $\varepsilon_0 \in (0,1)$ and $c_0 > 0$ such that for each $\varepsilon \in (0,\varepsilon_0)$ if given data $u_0^\varepsilon \in D(A_\varepsilon^{1/2})$ and $f^\varepsilon \in L^\infty(0,\infty;L^2(\Omega_\varepsilon)^3)$ satisfy

$$\begin{aligned}\|A_\varepsilon^{1/2}u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|M_\tau u_0^\varepsilon\|_{L^2(\Gamma)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2 \\ + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Gamma))}^2 \leq c_0\varepsilon^{-1}\end{aligned} \tag{1.3}$$

then there exists a global-in-time strong solution

$$u^\varepsilon \in C([0,\infty);D(A_\varepsilon^{1/2})) \cap L^2_{loc}([0,\infty);D(A_\varepsilon))$$

to the Navier–Stokes equations (1.1).

Note that here we only consider the partial slip boundary conditions by making the assumption (1.2). It is required to make the bilinear form corresponding to the Stokes problem with slip boundary conditions continuous and corecive uniformly in ε on $D(A_\varepsilon^{1/2}) = L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ (see Lemma 3.4). In [8] the perfect slip boundary conditions (i.e. $\gamma_\varepsilon = 0$) are also studied with another assumption on the degenerate surface Γ.

Main tools of analysis are the average operator and its extension to Ω_ε tangential on Γ_ε (see Section 3.3). We use them and the slip boundary conditions to get a good estimate of the trilinear term $((u \cdot \nabla)u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)}$ for $u \in D(A_\varepsilon)$, which is crucial for our proof of Theorem 1.1 (see Lemma 4.1). A key idea for the estimate is to decompose $u \in D(A_\varepsilon)$ into the average part, which is almost two-dimensional, and the residual part, which satisfies the impermeable boundary condition on Γ_ε. Such decomposition enables us to use an $L^2(\Omega_\varepsilon)$-estimate for the product of functions on Γ and Ω_ε and an $L^\infty(\Omega_\varepsilon)$-estimate deduced by combination of the Poincarè and Agmon inequalities.

Finally, we note that throughout our arguments it is important to determine the dependency of constants on ε explicitly in all inequalities. Here we do not discuss on this point since it requires a lot of calculations of surface quantities on Γ and Γ_ε (see [8] for detailed calculations).

2 Notations on a surface and a thin domain

In this section we briefly introduce notations on a surface and a curved thin domain. Let Γ be a two-dimensional closed (i.e. compact and without boundary), connected, oriented, and smooth surface in $\mathbb{R}^3$. By n and d we denote the unit outward normal vector of Γ and the signed distance function from Γ increasing in the direction of n. Also, we write κ_1 and κ_2 for the principal curvatures of Γ and define (twice) the mean curvature of Γ as $H := \kappa_1 + \kappa_2$. By the compactness and smoothness of Γ we may take a tubular neighborhood $N = \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x, \Gamma) < \delta\}$, $\delta > 0$ that admits the normal coordinate system around Γ, i.e. for each $x \in N$ there exists a unique point $\pi(x) \in \Gamma$ such that

$$x = \pi(x) + d(x)n(\pi(x)), \quad \nabla d(x) = n(\pi(x)). \tag{2.1}$$

For a C^1 function η on Γ we define the tangential gradient and derivatives by

$$\nabla_\Gamma \eta(y) := P(y)\nabla\tilde{\eta}(y), \quad \underline{D}_i\eta(y) := \sum_{j=1}^{3}\{\delta_{ij} - n_i(y)n_j(y)\}\partial_j\tilde{\eta}(y)$$

for $y \in \Gamma$ and $i = 1, 2, 3$, where $\tilde{\eta}$ is an extension of η to N satisfying $\tilde{\eta}|_\Gamma = \eta$ and $P := I_3 - n \otimes n$ is the orthogonal projection onto the tangent plane of Γ. Note that the values of $\nabla_\Gamma \eta$ and $\underline{D}_i\eta$ are independent of the choice of an extension of η (see e.g. [3, Lemma 2.4]). For $\eta, \xi \in C^1(\Gamma)$ the integration by parts formula

$$\int_\Gamma \{\eta\,\underline{D}_i\xi + \xi\,\underline{D}_i\eta\}\,d\mathcal{H}^2 = \int_\Gamma \eta\xi H n_i\,d\mathcal{H}^2, \quad i = 1, 2, 3$$

holds, where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure (see e.g. [3, Theorem 2.10]). Based on this identity we say that $\eta \in L^2(\Gamma)$ has the weak tangential derivative $\underline{D}_i\eta \in L^2(\Gamma)$ if there exists $\eta_i\ (= \underline{D}_i\eta) \in L^2(\Gamma)$ such that

$$\int_\Gamma \eta\,\underline{D}_i\xi\,d\mathcal{H}^2 = -\int_\Gamma \eta_i\xi\,d\mathcal{H}^2 + \int_\Gamma \eta\xi H n_i\,d\mathcal{H}^2$$

for all $\xi \in C^1(\Gamma)$. Then we define the Sobolev spaces on Γ by

$$\begin{aligned} H^1(\Gamma) &:= \{\eta \in L^2(\Gamma) \mid \underline{D}_i\eta \in L^2(\Gamma) \text{ for all } i = 1, 2, 3\}, \\ H^2(\Gamma) &:= \{\eta \in H^1(\Gamma) \mid \underline{D}_i\underline{D}_j\eta \in L^2(\Gamma) \text{ for all } i, j = 1, 2, 3\}. \end{aligned}$$

The norms of $H^1(\Gamma)$ and $H^2(\Gamma)$ are given by

$$\begin{aligned} \|\eta\|_{H^1(\Gamma)}^2 &:= \|\eta\|_{L^2(\Gamma)}^2 + \sum_{i=1}^{3}\|\underline{D}_i\eta\|_{L^2(\Gamma)}^2, \\ \|\eta\|_{H^2(\Gamma)}^2 &:= \|\eta\|_{H^1(\Gamma)}^2 + \sum_{i,j=1}^{3}\|\underline{D}_i\underline{D}_j\eta\|_{L^2(\Gamma)}^2. \end{aligned}$$

Next we give notations on a thin domain. Let g_0 and g_1 be smooth functions on Γ satisfying $|g_i| < \delta$ on Γ, $i = 0, 1$. Based on the normal coordinate system (2.1) we define a curved thin domain in $\mathbb{R}^3$ by

$$\Omega_\varepsilon := \{y + rn(y) \mid y \in \Gamma,\ \varepsilon g_0(y) < r < \varepsilon g_1(y)\}, \quad \varepsilon \in (0, 1).$$

By Γ_ε and n_ε we denote the boundary of Ω_ε and its unit outward normal vector. For a function φ on Ω_ε we have the change of variables formula (see e.g. [4, Section 14.6])

$$\int_{\Omega_\varepsilon} \varphi(x)\,dx = \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y+rn(y))J(y,r)\,dr\,d\mathcal{H}^2(y), \tag{2.2}$$

where $J(y,r) := \{1-r\kappa_1(y)\}\{1-r\kappa_2(y)\}$ for $y \in \Gamma$ and $r \in (-\delta,\delta)$. By the formula (2.2) we easily see that there exists $c>0$ independent of ε such that

$$c^{-1}\varepsilon^{1/2}\|\eta\|_{L^2(\Gamma)} \leq \|\bar{\eta}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|\eta\|_{L^2(\Gamma)} \tag{2.3}$$

for all $\eta \in L^2(\Gamma)$. Here and in what follows we write $\bar{\eta} := \eta \circ \pi$ for the constant extension of a function η on Γ in the normal direction of Γ.

3 Fundamental tools and inequalities

In this section we give fundamental facts for the proof of Theorem 1.1, especially for the estimate of the trilinear term. In what follows, we denote by c a general positive constant independent of ε.

3.1 Basic inequalities for functions on the curved thin domain

For a function φ on Ω_ε we define the derivative of φ in the normal direction of Γ by

$$\partial_n\varphi(x) := \frac{d}{dr}\Big(\varphi(y+rn(y))\Big)\Big|_{y=\pi(x),\,r=d(x)} = n(\pi(x))\cdot\nabla\varphi(x), \quad x \in \Omega_\varepsilon.$$

Based on the formula (2.2) we can show Poincaré's inequalities on Ω_ε.

Lemma 3.1. *There exists a constant $c>0$ independent of ε such that*

$$\begin{aligned} \|\varphi\|_{L^2(\Omega_\varepsilon)} &\leq c\left(\varepsilon^{1/2}\|\varphi\|_{L^2(\Gamma_\varepsilon)} + \varepsilon\|\partial_n\varphi\|_{L^2(\Omega_\varepsilon)}\right), \\ \|\varphi\|_{L^2(\Gamma_\varepsilon)} &\leq c\left(\varepsilon^{-1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/2}\|\partial_n\varphi\|_{L^2(\Omega_\varepsilon)}\right) \end{aligned} \tag{3.1}$$

for all $\varphi \in H^1(\Omega_\varepsilon)$. Moreover, if $u \in H^1(\Omega_\varepsilon)^3$ satisfies $u\cdot n_\varepsilon = 0$ on Γ_ε, then

$$\|u\cdot\bar{n}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{H^1(\Omega_\varepsilon)}. \tag{3.2}$$

By the anisotropic Agmons' inequality on $(0,1)^3$ (see [12, Proposition 2.2]) and a localization argument with a partition of unity of Γ we have Agmon's inequality on Ω_ε with explicit dependence on ε.

Lemma 3.2. *There exists a constant $c>0$ independent of ε such that*

$$\begin{aligned} \|\varphi\|_{L^\infty(\Omega_\varepsilon)} \leq{}& c\varepsilon^{-1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/4}\|\varphi\|_{H^2(\Omega_\varepsilon)}^{1/2} \\ &\times \left(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\partial_n\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon^2\|\partial_n^2\varphi\|_{L^2(\Omega_\varepsilon)}\right)^{1/4} \end{aligned} \tag{3.3}$$

for all $\varphi \in H^2(\Omega_\varepsilon)$.

In Section 3.2 we see that the bilinear form corresponding to the Stokes problem with slip boundary conditions is given by the $L^2(\Omega_\varepsilon)$-inner product of the strain rate tensors of vector fields instead of that of their gradient matrices. The following Korn type inequality shows that the bilinear form is uniformly corecive in ε on an appropriate function space.

Lemma 3.3. *For all $u \in H^1(\Omega_\varepsilon)^3$ satisfying $u \cdot n_\varepsilon = 0$ on Γ_ε we have*

$$\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \leq 4\|D(u)\|^2_{L^2(\Omega_\varepsilon)} + c\|u\|^2_{L^2(\Omega_\varepsilon)} \tag{3.4}$$

with a constant $c > 0$ independent of ε.

3.2 Stokes operator associated with slip boundary conditions

For $u \in H^2(\Omega_\varepsilon)^3$ and $v \in H^1(\Omega_\varepsilon)^3$ we have

$$\int_{\Omega_\varepsilon} \{\Delta u + \nabla(\operatorname{div} u)\} \cdot v\, dx = -2\int_{\Omega_\varepsilon} D(u) : D(v)\, dx + 2\int_{\Gamma_\varepsilon} [D(u)n_\varepsilon] \cdot v\, d\mathcal{H}^2$$

by integration by parts. In particular, if u satisfies $\operatorname{div} u = 0$ in Ω_ε and

$$u \cdot n_\varepsilon = 0, \quad \nu[D(u)n_\varepsilon]_{\tan} + \gamma_\varepsilon u = 0 \quad \text{on} \quad \Gamma_\varepsilon, \tag{3.5}$$

and v satisfies $v \cdot n_\varepsilon = 0$ on Γ_ε then from the above identity it follows that

$$\nu\int_{\Omega_\varepsilon} \Delta u \cdot v\, dx = -2\nu\int_{\Omega_\varepsilon} D(u) : D(v)\, dx - 2\gamma_\varepsilon\int_{\Gamma_\varepsilon} u \cdot v\, d\mathcal{H}^2.$$

Hence the bilinear form corresponding to the Stokes problem with slip boundary conditions (3.5) is given by

$$a_\varepsilon(u, v) := 2\nu\int_{\Omega_\varepsilon} D(u) : D(v)\, dx + 2\gamma_\varepsilon\int_{\Gamma_\varepsilon} u \cdot v\, d\mathcal{H}^2$$

for $u, v \in V_\varepsilon := L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$, where $L^2_\sigma(\Omega_\varepsilon)$ is the solenoidal space on Ω_ε, i.e. $L^2_\sigma(\Omega_\varepsilon) = \{u \in L^2(\Omega_\varepsilon)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega_\varepsilon,\ u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}$. Moreover, by (1.2), (3.1), and (3.4) we observe that a_ε is uniformly continuous and coercive on V_ε in ε.

Lemma 3.4. *Under the assumption* (1.2) *there exist $\varepsilon_1 \in (0,1)$ and $c > 0$ such that*

$$c^{-1}\|u\|^2_{L^2(\Omega_\varepsilon)} \leq a_\varepsilon(u, u) \leq c\|u\|^2_{L^2(\Omega_\varepsilon)} \tag{3.6}$$

for all $\varepsilon \in (0, \varepsilon_1)$ and $u \in V_\varepsilon$.

Hereafter we assume $\varepsilon \in (0, \varepsilon_1)$. By Lemma 3.4 and the Lax–Milgram theory we see that the bilinear form a_ε induces a bounded linear operator A_ε from V_ε into its dual space. As an unbounded operator on $L^2(\Omega_\varepsilon)^3$ the Stokes operator A_ε has its domain

$$D(A_\varepsilon) = \{u \in L^2_\sigma(\Omega_\varepsilon) \cap H^2(\Omega_\varepsilon)^3 \mid \nu[D(u)n_\varepsilon]_{\tan} + \gamma_\varepsilon u = 0 \text{ on } \Gamma_\varepsilon\}$$

and representation $A_\varepsilon u = -\nu\mathbb{P}_\varepsilon\Delta u$ for $u \in D(A_\varepsilon)$, which follows from a regularity result for the Stokes problem with slip boundary conditions (see [1]). Note that

$$c^{-1}\|u\|_{H^1(\Omega_\varepsilon)} \leq \|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)} \tag{3.7}$$

for all $u \in D(A_\varepsilon^{1/2}) = V_\varepsilon$ by (3.6) and $a_\varepsilon(u,u) = \|A_\varepsilon^{1/2}u\|^2_{L^2(\Omega_\varepsilon)}$. We also have

$$\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)} \leq c\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \tag{3.8}$$

for all $u \in D(A_\varepsilon)$ by $\|A_\varepsilon^{1/2}u\|^2_{L^2(\Omega_\varepsilon)} = (u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)}$ and (3.7). By the slip boundary conditions (3.5) and analysis of surface quantities on Γ_ε we get an integration by parts formula for the rotation of $u \in D(A_\varepsilon)$ with an auxiliary vector field bounded by u independently of ε.

Lemma 3.5. *For $u \in D(A_\varepsilon)$ and $\Phi \in L^2(\Omega_\varepsilon)^3$ with* $\operatorname{curl}\Phi \in L^2(\Omega_\varepsilon)^3$ *we have*

$$\begin{aligned}\int_{\Omega_\varepsilon} &\operatorname{curl}\operatorname{curl} u \cdot \Phi\, dx \\ &= -\int_{\Omega_\varepsilon} \operatorname{curl} G(u) \cdot \Phi\, dx + \int_{\Omega_\varepsilon} \{\operatorname{curl} u + G(u)\} \cdot \operatorname{curl}\Phi\, dx,\end{aligned} \tag{3.9}$$

where $G(u)$ is a vector field on Ω_ε satisfying

$$|G(u)| \leq c|u|, \quad |\nabla G(u)| \leq c(|\nabla u| + |u|) \quad \text{on} \quad \Omega_\varepsilon. \tag{3.10}$$

Based on the integration by parts identity (3.9) we can derive an estimate for the difference between the Stokes and Laplace operators.

Lemma 3.6. *There exists a constant $c > 0$ independent of ε such that*

$$\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)} \tag{3.11}$$

for all $u \in D(A_\varepsilon)$.

Note that in (3.11) the $L^2(\Omega_\varepsilon)$-norm of the difference between $A_\varepsilon u$ and $-\nu\Delta u$ is estimated by the $H^1(\Omega_\varepsilon)$-norm of u, not by its $H^2(\Omega_\varepsilon)$-norm.

By a regularity result of the Stokes problem we easily observe that the norm $\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}$ is equivalent to the canonical $H^2(\Omega_\varepsilon)$-norm on $D(A_\varepsilon)$. However, it is difficult to show the uniform equivalence of these norms in ε.

Lemma 3.7. *There exist constants $\varepsilon_0 \in (0, \varepsilon_1)$ and $c > 0$ such that*

$$c^{-1}\|u\|_{H^2(\Omega_\varepsilon)} \leq \|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^2(\Omega_\varepsilon)} \tag{3.12}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $u \in D(A_\varepsilon)$.

The right-hand inequality of (3.12) is an immediate consequence of (3.11). To prove the left-hand inequality we first show that

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq c\left(\|\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \tag{3.13}$$

for all $u \in D(A_\varepsilon)$ and then use (3.7), (3.8), and (3.11). The proof of (3.13) is technical and requires the notion of the Riemannian connection on the surface Γ_ε.

In what follows, we assume $\varepsilon \in (0, \varepsilon_0)$ with ε_0 given in Lemma 3.7.

3.3 Average operators in the thin direction

In the study of the Navier–Stokes equations in thin domains it is useful to transform a three-dimensional vector field into a two-dimensional one. To this end we introduce the average operator M in the thin direction. For a function φ on Ω_ε we set

$$M\varphi(y) := \frac{1}{\varepsilon g(y)} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y + rn(y))\, dr, \quad y \in \Gamma.$$

Also, for a vector field u on Ω_ε we write $M_\tau u := P(Mu)$ for the tangential component (with respect to the surface Γ) of the average of u. The average operator is a bounded linear operator from $H^m(\Omega_\varepsilon)$ into $H^m(\Gamma)$ for each $m = 0, 1, 2$. Indeed, we have

$$\|M\varphi\|_{H^m(\Gamma)} \leq c\varepsilon^{-1/2}\|\varphi\|_{H^m(\Omega_\varepsilon)}, \quad \|M_\tau u\|_{H^m(\Gamma)} \leq c\varepsilon^{-1/2}\|u\|_{H^m(\Omega_\varepsilon)} \tag{3.14}$$

for all $\varphi \in H^m(\Omega_\varepsilon)$ and $u \in H^m(\Omega_\varepsilon)^3$. Moreover, by the change of variables formula (2.2) we can get an estimate for the difference between φ and $M\varphi$.

Lemma 3.8. *There exists a constant $c > 0$ independent of ε such that*

$$\|\varphi - \overline{M\varphi}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\varphi\|_{H^1(\Omega_\varepsilon)} \tag{3.15}$$

for all $\varphi \in H^1(\Omega_\varepsilon)$.

Since the average of a function on Ω_ε is defined on the two-dimensional surface Γ, the two-dimensional Sobolev inequalities are applicable. In particular, we can use the product estimate for functions on Γ and Ω_ε.

Lemma 3.9. *For $\eta \in H^1(\Gamma)$ and $\varphi \in H^1(\Omega_\varepsilon)$ we have*

$$\|\bar{\eta}\varphi\|_{L^2(\Omega_\varepsilon)} \leq c\|\eta\|_{L^2(\Gamma)}^{1/2}\|\eta\|_{H^1(\Gamma)}^{1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}^{1/2}. \tag{3.16}$$

Here $c > 0$ is a constant independent of ε, η, and φ.

To analyze the difference between a vector field on Ω_ε and its average part it is convenient to consider an extension of the average to Ω_ε satisfying the impermeable boundary condition on Γ_ε. By the definition of Ω_ε we can take a smooth vector field Ψ_ε on Ω_ε such that

$$|\Psi_\varepsilon| \leq c\varepsilon, \quad |\nabla\Psi_\varepsilon| \leq c \quad \text{on} \quad \Omega_\varepsilon, \quad \Psi_\varepsilon = \frac{1}{n_\varepsilon \cdot \bar{n}}\overline{P}n_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon. \tag{3.17}$$

For a vector field u on Ω_ε we define the extension of the tangential average

$$u^a(x) := \overline{M_\tau u}(x) + \left\{\overline{M_\tau u}(x) \cdot \Psi_\varepsilon(x)\right\}\bar{n}(x), \quad x \in \Omega_\varepsilon. \tag{3.18}$$

Then from the last equality of (3.17) it immediately follows that $u^a \cdot n_\varepsilon = 0$ on Γ_ε, even if u itself does not satisfy the same impermeable boundary condition. Moreover, from (3.14), (3.16), and (3.17) we can deduce a product estimate for a function on Ω_ε and u^a, which can be considered as an almost two-dimensional vector field.

Lemma 3.10. *For $\varphi \in H^1(\Omega_\varepsilon)$, $u \in H^1(\Omega_\varepsilon)^3$, and u^a given by* (3.18) *we have*

$$\| |u^a| \varphi \|_{L^2(\Omega_\varepsilon)} \le c\varepsilon^{-1/2} \|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2} \|\varphi\|_{H^1(\Omega_\varepsilon)}^{1/2} \|u\|_{L^2(\Omega_\varepsilon)}^{1/2} \|u\|_{H^1(\Omega_\varepsilon)}^{1/2} \tag{3.19}$$

with a constant independent of ε, φ, and u. If in addition $u \in H^2(\Omega_\varepsilon)^3$, then

$$\| |\nabla u^a| \varphi \|_{L^2(\Omega_\varepsilon)} \le c\varepsilon^{-1/2} \|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2} \|\varphi\|_{H^1(\Omega_\varepsilon)}^{1/2} \|u\|_{H^1(\Omega_\varepsilon)}^{1/2} \|u\|_{H^2(\Omega_\varepsilon)}^{1/2}. \tag{3.20}$$

When u satisfies $u \cdot n_\varepsilon = 0$ on Γ_ε, the residual term $u^r := u - u^a$ also satisfies the same impermeable boundary condition on Γ_ε by the definition of u^a. This property enables us to get Poincaré's inequality for u^r and its first order derivatives.

Lemma 3.11. *Let $u \in H^1(\Omega_\varepsilon)^3$ satisfy $u \cdot n_\varepsilon = 0$ on Γ_ε. Then we have*

$$\|u^r\|_{L^2(\Omega_\varepsilon)} \le c\varepsilon \|\partial_n u^r\|_{L^2(\Omega_\varepsilon)} \tag{3.21}$$

for $u^r := u - u^a$, where u^a is given by (3.18) *and $c > 0$ is a constant independent of ε and u. Moreover, if $u \in D(A_\varepsilon)$, then we have*

$$\|\nabla u^r\|_{L^2(\Omega_\varepsilon)} \le c \left(\varepsilon \|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)} \right). \tag{3.22}$$

Combining Agmon's inequality (3.3) and Poincaré's inequalities (3.21)–(3.22) we can deduce an $L^\infty(\Omega_\varepsilon)$-estimate for the residual term u^r, which is useful for dealing with the trilinear term $((u \cdot \nabla)u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)}$.

Lemma 3.12. *For $u \in D(A_\varepsilon)$ let u^a be given by* (3.18) *and $u^r := u - u^a$. Then*

$$\|u^r\|_{L^\infty(\Omega_\varepsilon)} \le c \left(\varepsilon^{1/2} \|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2} \|u\|_{H^2(\Omega_\varepsilon)}^{1/2} \right) \tag{3.23}$$

with a constant $c > 0$ independent of ε and u.

4 Estimate for the trilinear form

Based on the results in Section 3 we derive an estimate for the trilinear term, which is crucial for our proof of the global-in-time existence of a strong solution.

Lemma 4.1. *For given $\alpha > 0$ there exist $c_\alpha^1, c_\alpha^2 > 0$ independent of ε such that*

$$\begin{aligned} \left| \left((u \cdot \nabla)u, A_\varepsilon u \right)_{L^2(\Omega_\varepsilon)} \right| &\le \left(\alpha + c_\alpha^1 \varepsilon^{1/2} \|u\|_{H^1(\Omega_\varepsilon)} \right) \|u\|_{H^2(\Omega_\varepsilon)}^2 \\ &\quad + c_\alpha^2 \left(\|u\|_{L^2(\Omega_\varepsilon)}^2 \|u\|_{H^1(\Omega_\varepsilon)}^4 + \varepsilon^{-1} \|u\|_{L^2(\Omega_\varepsilon)}^2 \|u\|_{H^1(\Omega_\varepsilon)}^2 \right) \end{aligned} \tag{4.1}$$

for all $u \in D(A_\varepsilon)$. (In fact, c_α^1 does not depend on α.)

Outline of the proof. For $u \in D(A_\varepsilon)$ let $\omega := \operatorname{curl} u$, u^a be given by (3.18), and $u^r := u - u^a$. Since $(u \cdot \nabla)u = \omega \times u + \nabla(|u|^2)/2$ and $(\nabla(|u|^2), A_\varepsilon u)_{L^2(\Omega_\varepsilon)} = 0$ by $A_\varepsilon u \in L_\sigma^2(\Omega_\varepsilon)$ and $\nabla(|u|^2) \in L_\sigma^2(\Omega_\varepsilon)^\perp$, we have

$$\left((u \cdot \nabla)u, A_\varepsilon u \right)_{L^2(\Omega_\varepsilon)} = (\omega \times u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)} = I_1 + I_2 + I_3,$$

where I_1, I_2, and I_3 are given by

$$I_1 := (\omega \times u^r, A_\varepsilon u)_{L^2(\Omega_\varepsilon)},$$
$$I_2 := (\omega \times u^a, A_\varepsilon u + \nu\Delta u)_{L^2(\Omega_\varepsilon)}, \quad I_3 := -(\omega \times u^a, \nu\Delta u)_{L^2(\Omega_\varepsilon)}.$$

Let us estimate I_1, I_2, and I_3 separately. By (3.12) and (3.23) we have

$$\begin{aligned}|I_1| &\le \|u^r\|_{L^\infty(\Omega_\varepsilon)}\|\omega\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \\ &\le c\left(\varepsilon^{1/2}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\right)\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} \\ &\le c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^2 + c\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^{3/2}.\end{aligned}$$

Applying Young's inequality $ab \le \alpha a^{4/3} + c_\alpha b^4$ to the second term we get

$$|I_1| \le \left(\alpha + c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}^2 + c_\alpha\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^4. \tag{4.2}$$

Next we estimate I_2. By (3.19) we see that

$$\begin{aligned}\|\omega \times u^a\|_{L^2(\Omega_\varepsilon)} &\le c\varepsilon^{-1/2}\|\omega\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\omega\|_{H^1(\Omega_\varepsilon)}^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^{1/2} \\ &\le c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}.\end{aligned}$$

Combining this with (3.11) we have

$$\begin{aligned}|I_2| &\le \|\omega \times u^a\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} \\ &\le c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^2\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}.\end{aligned}$$

Moreover, the inequalities (3.7) and (3.12) yield that

$$\begin{aligned}\|u\|_{H^1(\Omega_\varepsilon)}^2 &\le c\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}^2 = c(u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)} \\ &\le c\|u\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \le c\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}.\end{aligned}$$

Using this inequality and Young's inequality $ab \le \alpha a^2 + c_\alpha b^2$ we obtain

$$\begin{aligned}|I_2| &\le c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} \\ &\le \alpha\|u\|_{H^2(\Omega_\varepsilon)}^2 + c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2.\end{aligned} \tag{4.3}$$

It is more difficult to derive an estimate for I_3. Here let us just explain an idea for dealing with it. Using $\Delta u = -\mathrm{curl}\,\omega$ by $\mathrm{div}\,u = 0$ and (3.9) we get

$$I_3 = \nu(\mathrm{curl}\,\omega, \omega \times u^a)_{L^2(\Omega_\varepsilon)} = J_1 + J_2 + J_3,$$

where J_1, J_2, and J_3 are given by

$$J_1 := -\nu(\mathrm{curl}\,G(u), \omega \times u^a)_{L^2(\Omega_\varepsilon)},$$
$$J_2 := \nu\big(G(u), \mathrm{curl}(\omega \times u^a)\big)_{L^2(\Omega_\varepsilon)}, \quad J_3 = \nu\big(\omega, \mathrm{curl}(\omega \times u^a)\big)_{L^2(\Omega_\varepsilon)}.$$

We apply (3.10), (3.19), (3.20), and Young's inequality to J_1 and J_2 to obtain

$$|J_i| \le \alpha\|u\|_{H^2(\Omega_\varepsilon)}^2 + c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2, \quad i = 1, 2. \tag{4.4}$$

To deal with J_3 we observe that

$$\operatorname{curl}(\omega \times u^a) = (u^a \cdot \nabla)\omega - (\omega \cdot \nabla)u^a + (\operatorname{div} u^a)\omega - (\operatorname{div}\omega)u^a,$$

$$\operatorname{div}\omega = 0, \quad (\omega, (u^a \cdot \nabla)\omega)_{L^2(\Omega_\varepsilon)} = -\frac{1}{2}(\operatorname{div} u^a, |\omega|^2)_{L^2(\Omega_\varepsilon)},$$

where the last equality follows from integration by parts and $u^a \cdot n_\varepsilon = 0$ on Γ_ε. From these equalities we deduce that

$$J_3 = \frac{\nu}{2}(\operatorname{div} u^a, |\omega|^2)_{L^2(\Omega_\varepsilon)} - \nu(\omega, (\omega \cdot \nabla)u^a)_{L^2(\Omega_\varepsilon)}$$

and estimate the right-hand side by analyzing $\omega = \operatorname{curl} u$ and the divergence of u^a and using the inequalities (3.16), (3.19), and (3.20). Here we omit details and the resulting estimate is

$$|J_3| \leq c\left(\alpha + \varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|^2_{H^2(\Omega_\varepsilon)} + c_\alpha\varepsilon^{-1}\|u\|^2_{L^2(\Omega_\varepsilon)}\|u\|^2_{H^1(\Omega_\varepsilon)}. \tag{4.5}$$

Finally, we apply (4.2), (4.3), (4.4), and (4.5) to

$$\big((u \cdot \nabla)u, A_\varepsilon u\big)_{L^2(\Omega_\varepsilon)} = I_1 + I_2 + I_3 = I_1 + I_2 + (J_1 + J_2 + J_3)$$

to obtain (4.1) (after replacing the constant α). ☐

Using (3.7) and (3.12) we can express (4.1) in terms of the Stokes operator.

Corollary 4.2. *There exist $d_1, d_2 > 0$ independent of ε such that*

$$\begin{aligned}\left|\big((u \cdot \nabla)u, A_\varepsilon u\big)_{L^2(\Omega_\varepsilon)}\right| &\leq \left(\frac{1}{4} + d_1\varepsilon^{1/2}\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}\right)\|A_\varepsilon u\|^2_{L^2(\Omega_\varepsilon)} \\ &\quad + d_2\left(\|u\|^2_{L^2(\Omega_\varepsilon)}\|A_\varepsilon^{1/2}u\|^4_{L^2(\Omega_\varepsilon)} + \varepsilon^{-1}\|u\|^2_{L^2(\Omega_\varepsilon)}\|A_\varepsilon^{1/2}u\|^2_{L^2(\Omega_\varepsilon)}\right)\end{aligned} \tag{4.6}$$

for all $u \in D(A_\varepsilon)$.

5 Outline of the proof of Theorem 1.1

Now let us give an outline of the proof of the global-in-time existence of a strong solution to (1.1) for large data. First we recall the well-known local-in-time existence result on a strong solution to the Navier–Stokes equations (see e.g. [2, 11]).

Theorem 5.1. *For $u_0^\varepsilon \in D(A_\varepsilon^{1/2})$ and $f^\varepsilon \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$ there exist $T_0 > 0$ depending on Ω_ε, ν, u_0^ε, and f^ε and a strong solution u^ε to (1.1) on $[0, T_0)$ with*

$$u^\varepsilon \in C([0,T]; D(A_\varepsilon^{1/2})) \cap L^2(0,T; D(A_\varepsilon)) \quad \textit{for all} \quad T \in (0, T_0).$$

If u^ε is maximally defined on the time interval $[0, T_{\max})$ and $T_{\max}$ is finite, then

$$\lim_{t \to T_{\max}^-} \|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} = \infty.$$

To prove $T_{\max} = \infty$ in the above theorem we will show that the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon^{1/2}u^\varepsilon(t)$ is bounded uniformly in $t \in [0, T_{\max})$. We argue by a standard energy method and use the uniform Gronwall inequality (see [11, Lemma D.3]).

Lemma 5.2 (Uniform Gronwall inequality). *Let z, ξ, ζ be nonnegative functions in $L^1_{loc}([0,T);\mathbb{R})$ with $T \in (0,\infty]$. Suppose that $z \in C(0,T;\mathbb{R})$ and*

$$\frac{dz}{dt}(t) \leq \xi(t)z(t) + \zeta(t) \quad \textit{for almost all} \quad t \in (0,T).$$

Then $z \in L^\infty_{loc}(0,T;\mathbb{R})$ and

$$z(t_2) \leq \left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} z(s)\,ds + \int_{t_1}^{t_2} \zeta(s)\,ds\right)\exp\left(\int_{t_1}^{t_2}\xi(s)\,ds\right)$$

for all $t_1, t_2 \in (0,T)$ with $t_1 < t_2$.

Outline of the proof of Theorem 1.1. Following the idea of the proofs of [5, Theorem 7.4] and [6, Theorem 3.1] we prove $T_{\max} = \infty$. For a vector field u on Ω_ε we write $u_\tau := \overline{P}u$ and $u_n := (u \cdot \bar{n})\bar{n}$ for the tangential and normal components (with respect to Γ) of u. Also, we denote by c a general positive constant independent of ε, c_0, and $T_{\max}$.

Let $u_0^\varepsilon \in D(A_\varepsilon^{1/2})$ and $f^\varepsilon \in L^\infty(0,\infty;L^2(\Omega_\varepsilon)^3)$ satisfy (1.3), where $c_0 \in (0,1)$ is determined later. Noting that $M_\tau u_0^\varepsilon = M u^\varepsilon_{0,\tau}$ and u_0^ε satisfies $u_0^\varepsilon \cdot n_\varepsilon = 0$ on Γ_ε we split $u_0^\varepsilon = (u^\varepsilon_{0,\tau} - \overline{Mu^\varepsilon_{0,\tau}}) + \overline{M_\tau u_0^\varepsilon} + u^\varepsilon_{0,n}$, apply (2.3), (3.2), and (3.15), and then use (1.3) and $c_0 < 1$ to get

$$\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c c_0^{1/2} \leq c. \tag{5.1}$$

Let u^ε be a strong solution to (1.1) defined on the maximal time interval $[0, T_{\max})$. It satisfies the abstract evolutionary equation

$$\partial_t u^\varepsilon + A_\varepsilon u^\varepsilon = -\mathbb{P}_\varepsilon(u^\varepsilon \cdot \nabla)u^\varepsilon + \mathbb{P}_\varepsilon f^\varepsilon \quad \text{on} \quad [0,T_{\max}). \tag{5.2}$$

Taking the $L^2(\Omega_\varepsilon)$-inner product of (5.2) and u^ε we get

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|A_\varepsilon^{1/2}u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = (\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon)_{L^2(\Omega_\varepsilon)} \quad \text{on} \quad [0,T_{\max}). \tag{5.3}$$

We decompose the right-hand side of the above equality into

$$(\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon)_{L^2(\Omega_\varepsilon)} = (\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon_n)_{L^2(\Omega_\varepsilon)} + \left(\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon_\tau - \overline{M_\tau u^\varepsilon}\right)_{L^2(\Omega_\varepsilon)} + \left(\mathbb{P}_\varepsilon f^\varepsilon, \overline{M_\tau u^\varepsilon}\right)_{L^2(\Omega_\varepsilon)}$$

and apply (3.2) and (3.15) to the first and second terms on the right-hand side, respectively, and calculate the last term with the aid of the change of variables formula (2.2). Then we use (3.7) and Young's inequality to get

$$|(\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon)_{L^2(\Omega_\varepsilon)}| \leq \frac{1}{2}\|A_\varepsilon^{1/2}u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + c\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^2(\Gamma)}\right).$$

Applying this inequality to (5.3) we find that

$$\frac{d}{dt}\|u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|A_\varepsilon^{1/2}u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq c\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^2(\Gamma)}\right) \tag{5.4}$$

on $[0, T_{\max})$, which further yields by (3.7) that

$$\frac{d}{dt}\|u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{a_1}\|u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq c\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^2(\Gamma)}\right) \tag{5.5}$$

on $[0, T_{\max})$, where a_1 is a positive constant independent of ε, c_0, and $T_{\max}$. For each $t \in [0, T_{\max})$ we integrate (5.4) over $[t, t_*)$ with $t_* := \min\{t+1, T_{\max}\}$. Also, we multiply both sides of (5.5) at $s \in [0, t)$ by $e^{(s-t)/a_1}$ and integrate them over $[0, t)$. Then we apply (1.3) and (5.1) to the resulting inequalities to obtain

$$\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 + \int_t^{t_*} \|A_\varepsilon^{1/2} u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2 \, ds \leq c c_0 \quad \text{for all} \quad t \in [0, T_{\max}). \tag{5.6}$$

Now let us prove the uniform boundedness in time of the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon^{1/2} u^\varepsilon$. Let d_1 be the positive constant given in Corollary 4.2. Our goal is to show that

$$\varepsilon^{1/2} \|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} < d_3 := \frac{1}{4d_1} \quad \text{for all} \quad t \in [0, T_{\max}) \tag{5.7}$$

if we take $c_0 \in (0,1)$ in (1.3) appropriately. To this end we assume to the contrary that there exists $T \in (0, T_{\max})$ such that

$$\varepsilon^{1/2} \|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} < d_3 \quad \text{for all} \quad t \in [0, T), \tag{5.8}$$

$$\varepsilon^{1/2} \|A_\varepsilon^{1/2} u^\varepsilon(T)\|_{L^2(\Omega_\varepsilon)} = d_3. \tag{5.9}$$

We consider (5.2) on $[0, T]$ and take its $L^2(\Omega_\varepsilon)$-inner product with $A_\varepsilon u^\varepsilon$ to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \\ &\qquad \leq \left| \left((u^\varepsilon \cdot \nabla) u^\varepsilon, A_\varepsilon u^\varepsilon \right)_{L^2(\Omega_\varepsilon)} \right| + |(\mathbb{P}_\varepsilon f^\varepsilon, A_\varepsilon u^\varepsilon)_{L^2(\Omega_\varepsilon)}| \end{aligned} \tag{5.10}$$

on $[0, T]$. To the first term on the right-hand side we apply (4.6) and (5.8)–(5.9). Then by $d_3 = 1/4d_1$ we have

$$\begin{aligned} \left| \left((u^\varepsilon \cdot \nabla) u^\varepsilon, A_\varepsilon u^\varepsilon \right)_{L^2(\Omega_\varepsilon)} \right| &\leq \frac{1}{2} \|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \\ &\quad + d_2 \left(\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^4 + \varepsilon^{-1} \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \right). \end{aligned}$$

Also, Young's inequality implies that

$$|(\mathbb{P}_\varepsilon f^\varepsilon, A_\varepsilon u^\varepsilon)_{L^2(\Omega_\varepsilon)}| \leq \frac{1}{4} \|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2.$$

Using these inequalities to (5.10) we obtain

$$\frac{d}{dt} \|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{2} \|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq \xi \|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \zeta \tag{5.11}$$

on $[0, T]$, where

$$\begin{aligned} \xi(t) &:= 2d_2 \|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2, \\ \zeta(t) &:= 2 \left(d_2 \varepsilon^{-1} \|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \right) \end{aligned}$$

for $t \in [0, T]$. By (1.3), (5.6), and (5.8)–(5.9) we see that

$$\xi \leq c c_0 \varepsilon^{-1}, \quad \zeta \leq c c_0 \varepsilon^{-1} \left(\|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + 1 \right) \quad \text{on} \quad [0, T].$$

From these estimates, (3.7), and (5.11) we deduce that

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2+\frac{1}{a_2}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\le cc_0\varepsilon^{-1}\left(\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2+1\right)$$

on $[0,T]$, where a_2 is a positive constant independent of ε, c_0, and T. When $t\le\min\{1,T\}$, we multiply both sides of the above inequality at $s\in[0,t)$ by $e^{(s-t)/a_2}$, integrate them over $[0,t)$, and use (1.3), (5.6), and $c_0<1$ to get

$$\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2\le cc_0(1+c_0)\varepsilon^{-1}\le cc_0\varepsilon^{-1}\quad\text{for all}\quad t\in[0,T_*],\tag{5.12}$$

where $T_*:=\min\{1,T\}$. In the case $T\ge1$ we see by (5.11) that

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\le\xi\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2+\zeta\quad\text{on}\quad[0,T]$$

and thus we can apply Lemma 5.2 to $z(t)=\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2$ to deduce that

$$\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2\le\left(\int_{t-1}^t\|A_\varepsilon^{1/2}u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds+\int_{t-1}^t\zeta(s)\,ds\right)\exp\left(\int_{t-1}^t\xi(s)\,ds\right)$$

for all $t\in[1,T]$. Applying (1.3), (5.6), and $c_0<1$ to the right-hand side we get

$$\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2\le cc_0\varepsilon^{-1}\quad\text{for all}\quad t\in[1,T].\tag{5.13}$$

Now we combine (5.12) and (5.13) to observe that

$$\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2\le d_4c_0\varepsilon^{-1}\quad\text{for all}\quad t\in[0,T]$$

with some constant $d_4>0$ independent of ε, c_0, and T. Hence if we set

$$c_0:=\frac{1}{4}\min\left\{1,\frac{d_3^2}{d_4}\right\}=\frac{1}{4}\min\left\{1,\frac{1}{16d_1^2d_4}\right\}$$

and take $t=T$ in the above inequality, then it follows that

$$\|A_\varepsilon^{1/2}u^\varepsilon(T)\|_{L^2(\Omega_\varepsilon)}^2\le\frac{d_3^2\varepsilon^{-1}}{4}\quad\text{i.e.}\quad\varepsilon^{1/2}\|A_\varepsilon^{1/2}u^\varepsilon(T)\|_{L^2(\Omega_\varepsilon)}\le\frac{d_3}{2}<d_3,$$

which contradicts with (5.9). Hence the inequality (5.7) is valid for all $t\in[0,T_{\max})$ and we conclude by Theorem 5.1 that $T_{\max}=\infty$, i.e. the strong solution u^ε to (1.1) exists on the whole time interval $[0,\infty)$. ☐

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