Remark on the strong solvability of the Navier-Stokes equations in the weak L^n space

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Abstract

The initial value problem of the incompressible Navier-Stokes equations with non-zero forces in $L^{n,\infty}(\mathbb{R}^n)$ is investigated. Even though the Stokes semigroup is not strongly continuous on $L^{n,\infty}(\mathbb{R}^n)$, with the qualitative condition for the external forces, it is clarified that the mild solution of the Naiver-Stokes equations satisfies the differential equations in the topology of $L^{n,\infty}(\mathbb{R}^n)$. Inspired by the conditions for the forces, we characterize the maximal complete subspace in $L^{n,\infty}(\mathbb{R}^n)$ where the Stokes semigroup is strongly continuous at $t=0$. By virtue of this subspace, we also show local well-posedness of the strong solvability of the Cauchy problem without any smallness condition on the initial data in the subspace.

This aritcle is based on the joint work with Professor Yohei Tsutsui.

1 Introduction

Let $n \geq 3$. We consider the Cauchy problem to the incompressible Naiver-Stokes equations in the whole space $\mathbb{R}^n$:

$$\text{(N-S)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = f & \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(\cdot, 0) = a & \text{in } \mathbb{R}^n. \end{cases}$$

Here $u = u(x,t) = \big(u_1(x,t), \dots, u_n(x,t)\big)$ and $\pi = \pi(x,t)$ are the unknown velocity and the pressure of the incompressible fluid at $(x,t) \in \mathbb{R}^n \times (0,T)$, respectively. While, $a = a(x) = \big(a_1(x), \dots, a_n(x)\big)$ and $f = f(x,t) = \big(f_1(x,t), \dots, f_n(x,t)\big)$ are the given initial data and external force, respectively.

In this article, we study the strong solvability of the Naiver-Stokes equations in the framework of the weak Lebesgue space $L^{n,\infty}(\mathbb{R}^n)$ with non-zero external forces. In particular, introducing the maximal subspace in $L^{n,\infty}(\mathbb{R}^n)$ where the Stokes operator is strongly continuous, we consider the local in time well-posedness of the strong solvability of initial value problem of (N-S) in the subspace.

The strong solvability of the (N-S) in the Lebesgue and the Sobolev spaces, in terms of the semigroup theory, was developed by Fujita and Kato [7], Kato [11] and Giga and Miyakawa [9], and so on. However, it is well-known that the weak Lebesgue space $L^{n,\infty}(\mathbb{R}^n)$ has lack of the density of compact-supported functions $C_0^\infty(\mathbb{R}^n)$ and that the Stokes operator $\{e^{t\Delta}\}_{t\geq 0}$ is not strongly continuous at $t=0$ in $L^{n,\infty}(\mathbb{R}^n)$. Therefore, there are difficulty for the validity of the differential equation:

$$\frac{d}{dt}u - \mathbb{P}\Delta u + \mathbb{P}[u\cdot\nabla u] = \mathbb{P}f, \qquad t>0$$

in the critical topology of $L^{n,\infty}(\mathbb{R}^n)$, especially, with non-trivial external forces and for the verification of the the local in time existence and also the uniqueness of mild solutions of (N-S) for initial data in $L^{n,\infty}(\mathbb{R}^n)$, where $\mathbb{P}$ denotes the Leray-Hopf, the Weyl-Helmholtz or the Fujita-Kato bounded projection. For the Cauchy problem, in case $f\equiv 0$, Miyakawa and Yamada [19] constructed the mild solution $u\in C\big((0,\infty)\,;\,L^{2,\infty}(\mathbb{R}^2)\big)$ with $u(t)\rightharpoonup a$ weakly $*$ in $L^{2,\infty}(\mathbb{R}^2)$. Barraza [1] proved the existence of a global mild solution $u\in BC\big((0,\infty)\,;\,L^{n,\infty}(\mathbb{R}^n)\big)$ with small initial data. As for local in time solution, Kozono and Yamazaki [14] constructed a regular solution $u(t)$ in the framework of $L^{n,\infty}(\Omega)+L^r(\Omega)$ where $r>n$ and Ω is a exterior domain. By the lack of the density of $C_0^\infty(\mathbb{R}^n)$, Kozono and Yamazaki [15] gave the uniqueness criterion for the mild solution $u\in C\big((0,T)\,;\,L^{n,\infty}(\Omega)\cap L^r(\Omega)\big)$ of (N-S) under the assumption

$$\limsup_{t\searrow 0} t^{\frac{1}{2}-\frac{n}{2r}}\|u(t)\|_r \leq \kappa \tag{1.1}$$

for some sufficiently small $\kappa>0$. In case $f\equiv f(x)$, Borchers and Miyakawa [3] refered to the existence of a strong solution of (N-S) with $u(t)\rightharpoonup a$ in weakly $*$ in $L^{n,\infty}(\Omega)$, as a solution of the purterbed equations (P) below from the stationary solution v associated with the force f:

$$\frac{d}{dt}w - \Delta w + \mathbb{P}[v\cdot\nabla w + w\cdot\nabla v] + \mathbb{P}[w\cdot\nabla w] = 0, \qquad t>0, \tag{P}$$

which has apparently no forces. In [3], they consider the stability in $L^{n,\infty}(\Omega)$ introducing the subspace $L_0^{n,\infty}(\Omega)$ of the completion of $C_0^\infty(\Omega)$ in $L^{n,\infty}(\Omega)$ where the Stokes semigroup is strongly continuous. Recently, with the subspace $L_0^{n,\infty}(\Omega)$, Koba [12] and Maremonti [17] considered the existence of the strong solution of (N-S) and (P), the stability and the uniqueness of mild solution of (N-S) without (1.1).

In case of non-trivial force $f=f(x,t)$, we need the essential treatment of the Duhamel terms which comes from f. Yamazaki [23] consider the global existence and the stability of the weak mild solution of (N-S) in $L^{n,\infty}(\Omega)$ for small a and $f=\nabla\cdot F$ with small $F(t)\in L^{\frac{n}{2},\infty}(\Omega)$. See also Definition 2.1 below. On the other hand, our previous work [20] construct a time periodic strong solution in $BC\big(\mathbb{R}\,;\,L^{n,\infty}(\mathbb{R}^n)\big)$ by a different approach from [12, 17], assuming a qualitative condition only on f which satisfies Hölder continuous on $\mathbb{R}$ with value in $L^{n,\infty}(\mathbb{R}^n)$ such as

$$\lim_{\varepsilon\searrow 0}\|e^{\varepsilon\Delta}\mathbb{P}f(t) - \mathbb{P}f(t)\|_{n,\infty} = 0, \qquad \text{for a.e. } t\in\mathbb{R}. \tag{A}$$

The aim of this article is to investigate the global and the local well-posedness of the strong solvability for the Cauchy problem of (N-S) with non-trivial external forces. Firstly, we construct a global weak mild and mild solution $u \in BC\big((0,\infty)\,;\,L^{n,\infty}(\mathbb{R}^n)\big)$ of (N-S) for small $a \in L^{n,\infty}(\mathbb{R}^n)$ and small $f \in BC\big([0,\infty)\,;\,L^{\frac{n}{3},\infty}(\mathbb{R}^n)\big)$, $n \geq 4$ and $f \in BC\big([0,\infty)\,;\,L^1(\mathbb{R}^3)\big)$ which are scale invariant classes for initial data and external forces, respectively. Here, the key is the Meyer's estimate based on the K-method on $L^{n,\infty}(\mathbb{R}^n)$ which enables us to deal with external forces with the critical regularity. Then we observe this mild solution of (N-S) becomes a strong solutions with the aid of (A).

Secondly, inspired by the condition (A) above, we are successful to characterize the subspace $X^{n,\infty}_\sigma$ in $L^{n,\infty}(\mathbb{R}^n)$ which is equivalent to the condition (A). Here we note that $X^{n,\infty}_\sigma$ is the maximal subspace where the Stokes semigroup is strongly continuous at $t=0$ and that $X^{n,\infty}_\sigma$ is a strictly wider class than that in [12, 17], see Remark 2.4 below.

Finally, by the virtue of $X^{n,\infty}_\sigma$, we establish the local well-posedness of the Cauchy problem of (N-S) in $X^{n,\infty}_\sigma$. We construct a local weak mild solution $u \in BC\big([0,T)\,;\,X^{n,\infty}_\sigma\big)$ of (N-S) for every $a \in X^{n,\infty}_\sigma$ and $f \in BC\big([0,T)\,;\,L^{\frac{n}{3},\infty}(\mathbb{R}^n)\big)$, $n \geq 4$ and $f \in BC\big([0,T)\,;\,L^1(\mathbb{R}^3)\big)$ with less spatial singularity. In this case, since f has just critical regularity, there is a difficulty that weak L^n-norm is only one which is applicable to the iteration scheme. Hence, as a different way from the usual Fujita-Kato (auxiliary norm) approach, we introduce another iteration scheme where $a \in X^{n,\infty}_\sigma$ is much effective. The existence of a local solution of (N-S) yields the uniqueness of weak mild solution in $BC\big([0,T)\,;\,L^{n,\infty}(\mathbb{R}^n)\big)$ as long as a and f have less singularity within the scale critical spaces, respectively.

2 Results

Before stating results, we introduce the following notations and some function spaces. Let $C^\infty_{0,\sigma}(\mathbb{R}^n)$ denotes the set of all C^∞-solenoidal vectors ϕ with compact support in $\mathbb{R}^n$, i.e., $\operatorname{div}\phi = 0$ in $\mathbb{R}^n$. $L^r_\sigma(\mathbb{R}^n)$ is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n)$ with respect to the L^r-norm $\|\cdot\|_r$, $1 < r < \infty$. $(\cdot,\cdot)$ is the duality pairing between $L^r(\mathbb{R}^n)$ and $L^{r'}(\mathbb{R}^n)$, where $1/r + 1/r' = 1$, $1 \leq r < \infty$. $L^r(\mathbb{R}^n)$ and $W^{m,r}(\mathbb{R}^n)$ denote the usual (vector-valued) L^r-Lebesgue space and L^r-Sobolev space over $\mathbb{R}^n$, respectively. Moreover, $\mathcal{S}(\mathbb{R}^n)$ denotes the set of all of the Schwartz functions. $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions. When X is a Banach space, $\|\cdot\|_X$ denotes the norm on X. Moreover, $C(I;X)$, $BC(I;X)$ and $L^r(I;X)$ denote the X-valued continuous and bounded continuous functions over the interval $I \subset \mathbb{R}$, and X-valued L^r functions, respectively.

Moreover, for $1 < p < \infty$ and $1 \leq q \leq \infty$ let $L^{p,q}(\mathbb{R}^n)$ be the space of all locally integrable functions with (quasi) norm $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \begin{cases} \left(\displaystyle\int_0^\infty \big(\lambda\, |\{x \in \mathbb{R}^n\,;\, |f(x)| > \lambda\}|^{\frac{1}{p}}\big)^q \frac{d\lambda}{\lambda} \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \displaystyle\sup_{\lambda > 0} \lambda\, |\{x \in \mathbb{R}^n\,;\, |f(x)| > \lambda\}|^{\frac{1}{p}}, & q = \infty, \end{cases}$$

where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$. For the case $q = \infty$, $L^{p,\infty}(\mathbb{R}^n)$ is a

Banach space with the following norm: with any $1 \le r < p$

$$\|f\|_{L^{p,\infty}} = \sup_{0<|E|<\infty} |E|^{-\frac{1}{r}+\frac{1}{p}} \Big(\int_E |f(x)|^r \, dx \Big)^{\frac{1}{r}}.$$

Here, we note that $\|\cdot\|_{L^{n,\infty}}$ is equivalent to $\|\cdot\|_{n,\infty}$. Since $\mathbb{P}$ is a bounded operator on $L^{p,\infty}(\mathbb{R}^n)$ for $1 < p < \infty$, we introduce the set of solenoidal vectors in $L^{p,\infty}(\mathbb{R}^n)$ as $L^{p,\infty}_\sigma(\mathbb{R}^n) = \mathbb{P}L^{n,\infty}(\mathbb{R}^n)$.

Definition 2.1 (Weak mild solution). *Let $a \in L^{n,\infty}_\sigma(\mathbb{R}^n)$ and $f \in BC\big([0,\infty)\,;\, L^{p,\infty}(\mathbb{R}^n)\big)$ for some $n/3 \le p \le n$. We call a function $u \in BC\big((0,\infty)\,;\, L^{n,\infty}_\sigma(\mathbb{R}^n)\big)$ weak (generalized) mild solution of (N-S), if*

$$\text{(IE}^*\text{)} \quad u(t) = e^{t\Delta}a + \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\, ds - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}(u \otimes u)(s)\, ds, \qquad 0 < t < T.$$

Remark 2.1. In case of $n = 3$, we modify the condition as $f \in BC([0,\infty)\,;\, L^1(\mathbb{R}^3))$ and

$$\text{(IE}^{**}\text{)} \quad u(t) = e^{t\Delta}a + \int_0^t \mathbb{P}e^{(t-s)\Delta}f(s)\, ds - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}(u \otimes u)(s)\, ds, \qquad 0 < t < T.$$

Moreover, a weak mild solution u satisfies

$$\big(u(t), \varphi\big) = (e^{t\Delta}a, \varphi) + \int_0^t \big(e^{(t-s)\Delta}f(s), \varphi\big)\, ds + \int_0^t \big(u(s) \cdot \nabla e^{(t-s)\Delta}\varphi, u(s)\big)\, ds,$$

$0 < t < T$, $\varphi \in C^\infty_{0,\sigma}(\mathbb{R}^n)$. See also, Kozono and Yamazaki [14], Yamazaki [23].

Definition 2.2 (Mild solution). *Let $a \in L^{n,\infty}_\sigma(\mathbb{R}^n)$ and $f \in BC\big([0,T)\,;\, L^{p,\infty}(\mathbb{R}^n)\big)$ for some $n/3 \le p \le n$. Then a function $u \in BC\big((0,T)\,;\, L^{n,\infty}_\sigma(\mathbb{R}^n)\big)$ which satisfies $\nabla u \in C\big((0,T)\,;\, L^{q,\infty}(\mathbb{R}^n)\big)$ with $\limsup_{t\to 0} t^{1-\frac{n}{2q}}\|\nabla u(t)\|_q < \infty$ for some $q \ge n/2$ is called a mild solution of (N-S), if*

$$\text{(IE)} \qquad u(t) = e^{t\Delta}a + \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\, ds - \int_0^t e^{(t-s)\Delta}\mathbb{P}[u \cdot \nabla u](s)\, ds, \qquad 0 < t < T.$$

Remark 2.2. In case of $n = 3$, we introduce a similar modification for f as in Remark 2.1. We note that $u(t)$ tends to a as $t \searrow 0$ in the sense of distributions, i.e., $\big(u(t), \varphi\big) \to (a, \varphi)$ as $t \searrow 0$ for all $\varphi \in C^\infty_0(\mathbb{R}^n)$. Moreover, if, additionally, $u \in BC\big((0,T)\,;\, L^r_\sigma(\mathbb{R}^n)\big)$ with some $r > n$ or $\nabla u \in C\big((0,T)\,;\, L^q(\mathbb{R}^n)\big)$ with $\limsup_{t\to 0} t^{1-\frac{n}{2q}}\|\nabla u(t)\|_q = 0$ for some $q > n/2$, then it holds that $u(t) \rightharpoonup a$ weakly $*$ in $L^{n,\infty}(\mathbb{R}^n)$ as $t \searrow 0$. However, we are unable to obtain $u(t) \to a$ in $L^{n,\infty}$ as $t \searrow 0$ in general, since $\{e^{t\Delta}\}$ is not strongly continuous at $t = 0$ in $L^{n,\infty}(\mathbb{R}^n)$.

Definition 2.3 (Strong solution). *Let $a \in L^{n,\infty}_\sigma(\mathbb{R}^n)$ and $f \in BC\big([0,T)\,;\, L^{n,\infty}(\mathbb{R}^n)\big)$ Then a function u is called a strong solution of (N-S), if* *(i)* $u \in BC\big((0,T)\,;\, L^{n,\infty}_\sigma(\mathbb{R}^n)\big) \cap C^1\big((0,T)\,;\, L^{n,\infty}_\sigma(\mathbb{R}^n)\big)$,

(ii) $u(t) \in \{u \in L^{n,\infty}_\sigma\,;\, \Delta u \in L^{n,\infty}(\mathbb{R}^n)\}$ *for* $0 < t < T$ *and* $\Delta u \in C\big((0,T)\,;\, L^{n,\infty}(\mathbb{R}^n)\big)$,

(iii) u *satisfies (N-S) in the following sense.*

$$\text{(DE)} \qquad \begin{cases} \dfrac{du}{dt} - \Delta u + \mathbb{P}[u\cdot\nabla u] = \mathbb{P}f & \text{in } L^{n,\infty}_\sigma(\mathbb{R}^n), \quad 0<t<T, \\ \lim\limits_{t\to\infty}\big(u(t), \varphi\big) = (a,\varphi) & \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n). \end{cases}$$

Remark 2.3. The strong solution u in the class (i) and (ii) in Definition 2.3 necessarily satisfies the initial condition in the sense of distributions. Indeed, the strong solution u satisfies (IE). Then, noting that for each t, $u(t) \in L^r(\mathbb{R}^n)$ for some $r > n$ by the Sobolev embedding and by the real interpolation, we see that for $\phi \in C^\infty_{0,\sigma}(\mathbb{R}^n)$ and for $p > n/(n-1)$

$$\begin{aligned} \left|\left(\int_0^t e^{(t-s)\Delta}\mathbb{P}[u\cdot\nabla u](s)\,ds, \phi\right)\right| &= \left|-\int_0^t \big(u(s)\cdot\nabla e^{(t-s)\Delta}\phi, u(s)\big)ds\right| \\ &\le t^{-\frac{n}{2p}+\frac{n}{2}-\frac{1}{2}} \sup_{0<s<T}\|u(s)\|^2_{n,\infty}\|\phi\|_p \\ &\to 0 \quad \text{as } t\searrow 0, \end{aligned}$$

since $-\frac{n}{2p}+\frac{n}{2}-\frac{1}{2} > 0$.

The following theorem characterizes the functions which satisfies the condition (A). For this purpose, we introduce the domain of the Stokes operator $-\Delta$ in $L^{n,\infty}_\sigma(\mathbb{R}^n)$ as $D(-\Delta) = \{u \in L^{n,\infty}_\sigma(\mathbb{R}^n)\,;\, \Delta u \in L^{n,\infty}(\mathbb{R}^n)\}$

Theorem 2.1 (Lunardi). *Let* $f \in L^{n,\infty}_\sigma(\mathbb{R}^n)$. *Then it holds that*

$$\lim_{\varepsilon\searrow 0}\|e^{\varepsilon\Delta}f - f\|_{n,\infty} = 0 \quad \text{if and only if} \quad f \in \overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}.$$

Consequently, $\{e^{t\Delta}\}_{t\ge 0}$ *is a bounded* C_0*-analytic semigroup on* $\overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}$. *In other words,* $\overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}$ *is the maximal subspace in* $L^{n,\infty}_\sigma(\mathbb{R}^n)$ *where the Stokes semigroup is* C_0*-semigroup.*

Remark 2.4. (i) Let $A = -\Delta$ be the Stokes operator on $\overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}$. Then we easily see that $D(A) \subsetneqq D(-\Delta)$, since $D(A) = \{u \in D(-\Delta)\,;\, \Delta u \in \overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}\}$. Hence, we may need more specific structure of the operator and the semigroup in order to confirm that $D(A)$ is dense in $\overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}$ or $\{e^{-tA}\}$ is the C_0-semigroup on $\overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}$.

(ii) We note that $\overline{C^\infty_{0,\sigma}(\mathbb{R}^n)}^{\|\cdot\|_{n,\infty}} \subsetneqq \overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}$. Indeed, take $f(x) \sim 1/|x|$ for $|x| \gg 1$. Then we see that $f \in D(-\Delta)$, but $f \notin \overline{C^\infty_{0,\sigma}(\mathbb{R}^n)}^{\|\cdot\|_{n,\infty}}$.

(iii) The condition (A) is not necessary for the strong solvability of the Stokes equations and the Naiver-Stokes equations in $L^{n,\infty}_\sigma(\mathbb{R}^n)$. Indeed, take $f \in \big(L^{n,\infty}(\mathbb{R}^n) \setminus$

$\overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}) \cap L^{\frac{n}{3},\infty}(\mathbb{R}^n)$ for $n \geq 4$ and consider $u = (-\Delta)^{-1}\mathbb{P}f$ for the Stokes equations and $u = (-\Delta)^{-1}\mathbb{P}f - (-\Delta)^{-1}\mathbb{P}[u \cdot \nabla u]$ for the Navier-Stokes equations. Then we see that $u \in D(-\Delta)$ and satisfies the equations for the strong sense.

(iv) With our method to prove the Theorem 2.1, in a general Banach space X, for every bounded analytic semigroup $\{e^{tL}\}$ on X with the property that $e^{tL}a$ is weakly or weakly$*$ continuous at $t = 0$ for all $a \in X$, we also characterize the maximal subspace as $\overline{D(L)}^{X}$ where $\{e^{tL}\}$ is strongly continuous, as a different approach from Lunardi [16].

Next, we consider the existence of a local in time solution of (N-S) by the virtue of the subspace in Theorem 2.1. So we shall introduce a notation such as $X_\sigma^{n,\infty} := \overline{D(-\Delta)}^{\|\cdot\|_{n,\infty}}$. On the other hand, for local existence of a weak mild or a mild solution, $f \in BC([0,T)\,;\, L^{\frac{n}{3},\infty}(\mathbb{R}^n))$ is not enough. We restrict $f(t) \in \widetilde{L}^{\frac{n}{3},\infty}(\mathbb{R}^n) = \overline{L^{\frac{n}{3},\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)}^{\|\cdot\|_{\frac{n}{3},\infty}}$ for $t \geq 0$ as a treatment of a spatial singularity of the force. For this space see, see, for instance, Farwig and Nakatsuka and Taniuchi [5, 6].

Theorem 2.2. *Let $n \geq 3$ and $a \in X_\sigma^{n,\infty}$.*

(i) Suppose $f \in BC([0,\infty)\,;\, \widetilde{L}^{\frac{n}{3},\infty}(\mathbb{R}^n))$ for $n \geq 4$ and $f \in BC([0,\infty); L^1(\mathbb{R}^3))$. Then there exist $T > 0$ and a weak mild solution $u \in BC([0,T)\,;\, X_\sigma^{n,\infty})$ of (N-S) with

$$u(t) \to a \quad \text{in } L_\sigma^{n,\infty} \quad \text{as } t \searrow 0.$$

(ii) Suppose $f \in BC([0,\infty)\,;\, L^{p,\infty}(\mathbb{R}^n))$ with some $\frac{n}{3} < p \leq n$. Then there exist $T > 0$ and a weak mild solution $u \in BC([0,T)\,;\, X_\sigma^{n,\infty})$ of (N-S) with $u(t) \to a$ in $L^{n,\infty}(\mathbb{R}^n)$ as $t \searrow 0$.

(iii) Furthermore, if additionally $\mathbb{P}f$ is Hölder continuous on $[0,T)$ in $L_\sigma^{n,\infty}(\mathbb{R}^n)$ and satisfies (A), i.e., $\mathbb{P}f(t) \in X_\sigma^{n,\infty}$ for almost every $0 < t < T$, then the weak mild solution u obtained by (i) or (ii) above becomes the strong solution of (N-S) with $u(t) \to a$ in $L_\sigma^{n,\infty}(\mathbb{R}^n)$ as $t \searrow 0$.

Remark 2.5. (i) If $n \geq 4$ and $\nabla a \in L^{\frac{n}{2},\infty}(\mathbb{R}^n)$, the weak mild solution u obtained by (i) of Theorem 2.2 is actually a mild solution with $\nabla u \in BC([0,T)\,;\, L^{\frac{n}{2},\infty}(\mathbb{R}^n))$. Similarly, if $p = n$ or $\nabla a \in L^{q,\infty}(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, $\frac{n}{3} < p < n$, then the weak mild solution u obtained by (ii) of Theorem 2.2 is actually a mild solution of (N-S).

(ii) The solution class $BC([0,T)\,;\, X_\sigma^{n,\infty})$ is well known for the uniqueness of weak mild or mild solutions of (N-S) since the Stokes semigroup is strongly continuous for $t \geq 0$.

(iii) For the local in time solvability, $a \in \widetilde{L}_\sigma^{n,\infty}(\mathbb{R}^n) = \overline{L_\sigma^{n,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)}^{\|\cdot\|_{n,\infty}}$ and $f \in BC([0,T); L^{p,\infty}(\mathbb{R}^n))$ for $\frac{n}{3} < p \leq n$ are also valid. Since $X_\sigma^{n,\infty} \subset \widetilde{L}_\sigma^{n,\infty}(\mathbb{R}^n)$, $\widetilde{L}_\sigma^{n,\infty}(\mathbb{R}^n)$ is a wider class of initial data a for local weak mild or mild solutions $u \in BC((0,T)\,;\, L_\sigma^{n,\infty}(\mathbb{R}^n))$ of (N-S) with $u(t) \rightharpoonup a$ weakly $*$ in $L^{n,\infty}(\mathbb{R}^n)$.

(iv) Borchers and Miyakawa [3], and Koba [12] consider the stability of the stationary solution of (N-S) in $L_{0,\sigma}^{n,\infty}(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{n,\infty}}$. In the framework of $L_{0,\sigma}^{n,\infty}(\Omega)$, we expect that the asymptotic stability of the solution u in the critical norm, i.e., $\lim_{t\to\infty} \|u(t)\|_{n,\infty} = 0$. However, $X_\sigma^{n,\infty}$ seems not to allow the time decay of the solution with the $L^{n,\infty}$-norm.

The uniqueness theorem is expected within the solution class such as $BC([0,T)\,;\,X^{n,\infty}_\sigma)$, see, for instance, [22]. Therefore, the natural question is that when $u \in BC([0,T)\,;\,X^{n,\infty}_\sigma)$. The following theorem implies that if the singularity of data are well-controled, then the orbit of the solution is unique and stays in $X^{n,\infty}_\sigma$.

Theorem 2.3. *Let $n \geq 3$ and let $a \in X^{n,\infty}_\sigma$ and $f \in BC([0,T)\,;\,\widetilde{L}^{\frac{n}{3},\infty}(\mathbb{R}^n))$ for $n \geq 4$ and $f \in BC([0,T)\,;\,L^1(\mathbb{R}^3))$. Suppose u, v are two weak mild solutions of (N-S) with $u|_{t=0} = v|_{t=0} = a$. If*

$$u, v \in BC([0,T)\,;\,L^{n,\infty}_\sigma(\mathbb{R}^n)),$$

then it holds

$$u, v \in BC([0,T)\,;\,X^{n,\infty}_\sigma) \quad \text{and} \quad u \equiv v.$$

In the above theorem, we restrict the singularity of data. Next, we shall give the uniqueness criterion for wider class of data, especially, for general initial data. For this purpose, we focus on the continuity $\lim_{t\to 0} u(t) = a$ in $L^{n,\infty}(\mathbb{R}^n)$ and additional regularity $u \in BC([0,T)\,;\,\widetilde{L}^{n,\infty}(\mathbb{R}^n))$. Furthermore, in such a case we see that the structure of the Navier-Stokes equation requires the restriction for the initial data.

Theorem 2.4. *Let $a \in L^{n,\infty}_\sigma(\mathbb{R}^n)$ and $f \in BC([0,T)\,;\,L^{n,\infty}(\mathbb{R}^n))$. Further let u, v be two (weak) mild solution of (N-S) with $u|_{t=0} = v|_{t=0} = a$ in the class*

$$u, v \in BC([0,T)\,;\,\widetilde{L}^{n,\infty}_\sigma(\mathbb{R}^n)). \tag{2.1}$$

Then $a \in X^{n,\infty}_\sigma$ and $u \equiv v$.

Remark 2.6. Without smallness condition, Theorem 2.4 ensures that a local in time strong solution u in the class $BC([0,T)\,;\,L^{n,\infty}_\sigma(\mathbb{R}^n))$ is unique, while the existence of such a local strong solution of (N-S) is guaranteed, provided $a \in X^{n,\infty}_\sigma$ and $f \in BC([0,T); X^{n,\infty}_\sigma)$ for some $T > 0$.

3 Key lemmata

3.1 Critical estimates

In order to construct mild solutions of (N-S) we deal with

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\,ds = \int_0^\infty e^{s\Delta}\mathbb{P}f(t-s)\chi_{[0,t]}(s)\,ds, \qquad t > 0.$$

Here, χ_A is the usual characteristic function on the set A, i.e., $\chi_A(x) = 1$ if $x \in A$, otherwise $\chi_A(x) = 0$. For this aim, we introduced the following lemmas in the previous work [20], besed on the real interpolation approach by Meyer [18] and Yamazaki [23].

Lemma 3.1 ([20, Lemma 4.1]). *Let $n \geq 3$ and $1 \leq p < \frac{n}{2}$, and define $p < q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{2}{n}$. Then it holds*

$$\left\| \int_0^\infty \mathbb{P}e^{s\Delta}g(s)\,ds \right\|_{q,\infty} \leq A_p \begin{cases} \sup_{s>0} \|g(s)\|_{p,\infty}, & \text{if } p > 1, \\ \sup_{s>0} \|g(s)\|_1, & \text{if } p = 1. \end{cases}$$

Remark 3.1. (i) If $g \in L^{p,\infty}(\mathbb{R}^n)$ for some $1 < p < \infty$, it is easy to see that $\mathbb{P}e^{s\Delta}g(s) = e^{s\Delta}\mathbb{P}g(s)$ for a.e. $x \in \mathbb{R}^n$.

(ii) The bound $\|\mathbb{P}e^{s\Delta}g(s)\|_{q,\infty} \leq cs^{-1}\|g(s)\|_{p,\infty}$ is not enough for the convergence of the integral at both $s = 0$ and $s = \infty$.

We also apply Meyer's estimate [18] for the non-linear term. See also, Yamazaki [23].

Lemma 3.2 ([18], [23], [20, Lemma 4.2]). *Let $n \geq 2$ and $1 \leq p < n$. Denote $\frac{n}{n-1} \leq q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. then*

$$\left\| \int_0^\infty \nabla \mathbb{P}e^{s\Delta}g(s)\, ds \right\|_{q,\infty} \leq B_p \begin{cases} \sup_{s>0} \|g(s)\|_{p,\infty} & \text{if } p > 1, \\ \sup_{s>0} \|g(s)\|_1 & \text{if } p = 1. \end{cases}$$

As an application, Lemma 3.1 and Lemma 3.2 yields the continuity of the Duhamel terms associated with the forces and the nonlinear term.

Lemma 3.3. *Let $n \geq 3$ and $1 < p < \frac{n}{2}$. For $f \in BC\big((0,\infty)\,;\, L^{p,\infty}(\mathbb{R}^n)\big)$ it holds that*

$$\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\, ds \in BC\big((0,\infty)\,;\, L^{q,\infty}(\mathbb{R}^n)\big) \quad \text{with } \frac{1}{q} = \frac{1}{p} - \frac{2}{n}.$$

Remark 3.2. In case $p = 1$, Lemma 3.3 is also valid with a slight modification as $\int_0^t \mathbb{P}e^{(t-s)\Delta}f(s)\, ds$ for $f \in BC\big((0,\infty)\,;\, L^1(\mathbb{R}^n)\big)$. Moreover, for $\int_0^t \nabla e^{(t-s)\Delta}\mathbb{P}f(s)\, ds$ we similarly obtain the continuity.

The following lemmas play an important role for the local existence and for the uniqueness criterion of weak mild solutions of (N-S). For this aim, we recall the space $\widetilde{L}^{p,\infty}(\mathbb{R}^n) := \overline{L^{p,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)}^{\|\cdot\|_{p,\infty}}$ for $1 < p < \infty$ and introduce a space $Y_{p'}^{p,\infty} = \{f \in L^{p,\infty}(\mathbb{R}^n)\,;\, f \in L^{p'}(\mathbb{R}^n)\}$ for $1 < p < p' < \infty$.

Lemma 3.4. *Let $1 < p < p' < \infty$. For every $\varepsilon > 0$ and $f \in BC\big([0,\infty)\,;\, \widetilde{L}^{p,\infty}(\mathbb{R}^n)\big)$ there exists $f_\varepsilon \in BC\big([0,\infty)\,;\, Y_{p'}^{p,\infty}\big)$ such that*

$$\sup_{0 \leq s < \infty} \|f(s) - f_\varepsilon(s)\|_{p,\infty} < \varepsilon,$$

i.e., $BC\big([0,\infty)\,;\, Y_{p'}^{p,\infty}\big)$ is a dense subspace within $BC\big([0,\infty)\,;\, \widetilde{L}^{p,\infty}(\mathbb{R}^n)\big)$.

Remark 3.3. For a finite interval $[0,T]$, we easily obtain the same density property. Moreover, it is easy to see $BC\big([0,\infty)\,;\, C_0^\infty(\mathbb{R}^3)\big)$ is a dense subspace within $BC\big([0,\infty)\,;\, L^1(\mathbb{R}^3)\big)$.

Lemma 3.5. *Let $n \geq 3$. Suppose $f \in BC\big([0,\infty)\,;\, \widetilde{L}^{\frac{n}{3},\infty}(\mathbb{R}^n)\big)$ for $n \geq 4$ and $f \in BC\big([0,\infty)\,;\, L^1(\mathbb{R}^3)\big)$. Then it holds that*

(3.1)
$$\int_0^t \mathbb{P}e^{(t-s)\Delta}f(s)\, ds \in BC\big([0,\infty)\,;\, X_\sigma^{n,\infty}\big) \quad \text{with } \lim_{t\to 0}\left\| \int_0^t \mathbb{P}e^{(t-s)\Delta}f(s)\, ds \right\|_{n,\infty} = 0.$$

Similarly, for a tensor $g=\left(g_{jk}\right)_{j,k=1}^{n}$, $g\in BC\left([0,\infty);\widetilde{L}^{\frac{n}{2},\infty}(\mathbb{R}^n)\right)$ *it holds that*

$$\int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}g(s)\,ds\in BC\left([0,\infty);X_\sigma^{n,\infty}\right)\quad\textit{with}\quad \lim_{t\to0}\left\|\int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbb{P}g(s)\,ds\right\|_{n,\infty}=0.$$

Remark 3.4. Lemma 3.5 plays an crucial role to construct a weak mild solution $u\in BC\left([0,\infty);X_\sigma^{n,\infty}\right)$ by the iteration scheme, where the uniqueness is guaranteed. For $f\in BC\left([0,\infty);L^{p,\infty}(\mathbb{R}^n)\right)$ with some $\frac{n}{3}<p\le n$, we easily see that $\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\,ds\in BC\left([0,T);X_\sigma^{n,\infty}\right)$ with $\lim_{t\to0}\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\,ds\right\|_{n,\infty}=0$ for finite $T>0$ instead of (3.1), estimating F^1, F^2 and F^3 below just by L^p-L^q estimate.

Proof. Put $F(t)=\int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\,ds$. We firstly show that $F(t)\to0$ in $L^{n,\infty}(\mathbb{R}^n)$ as $t\to0$. Take $\eta>0$. By Lemma 3.4 choose $f_\eta\in BC\left([0,\infty);Y_p^{\frac{n}{3},\infty}\right)$ with some $\frac{n}{3}<p<\infty$ such that $\sup_{0\le s<\infty}\|f(s)-f_\eta(s)\|_{\frac{n}{3},\infty}<\frac{\eta}{2A_{\frac{n}{3}}}$, where $A_{\frac{n}{3}}$ is the constant in Lemma 3.1 when $p=\frac{n}{3}$. By Lemma 3.1 we have

$$\begin{aligned}\|F(t)\|_{n,\infty}&\le\left\|\int_0^\infty e^{(t-s)\Delta}\mathbb{P}[f(s)-f_\eta(s)]\chi_{[0,t]}(s)\,ds\right\|_{n,\infty}+\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}f_\eta(s)\,ds\right\|_{n,\infty}\\&\le A_{\frac{n}{3}}\sup_{0\le s<\infty}\|f(s)-f_\eta(s)\|_{\frac{n}{3},\infty}+C\int_0^t(t-s)^{\frac12-\frac{n}{2p}}\|f_\eta(s)\|_p\,ds\\&\le\frac{\eta}{2}+Ct^{\frac32-\frac{n}{2p}}\sup_{0\le s<\infty}\|f_\eta(s)\|_{\frac{n}{3},\infty}.\end{aligned}$$

Since $\frac32-\frac{n}{2p}>0$, there exists $\delta>0$ such that if $0<t<\delta$ then $\|F(t)\|_{n,\infty}<\eta$. This prove the continuity of $F(t)$ at $t=0$.

Next we show $F(t)\in X_\sigma^{n,\infty}$ for each $t>0$. It holds that for sufficiently small $\varepsilon>0$

$$\begin{aligned}e^{\varepsilon\Delta}F(t)-F(t)&=\int_0^t e^{(s+\varepsilon)\Delta}\mathbb{P}f(t-s)\,ds-\int_0^t e^{s\Delta}\mathbb{P}f(t-s)\,ds\\&=\int_\varepsilon^{t+\varepsilon}e^{s\Delta}f(t+\varepsilon-s)\,ds-\int_0^t e^{s\Delta}\mathbb{P}f(t-s)\,ds\\&=\int_\varepsilon^t e^{s\Delta}\mathbb{P}[f(t+\varepsilon-s)-f(t-s)]\,ds\\&\quad+\int_t^{t+\varepsilon}e^{s\Delta}\mathbb{P}f(t+\varepsilon-s)\,ds+\int_0^\varepsilon e^{s\Delta}f(t-s)\,ds\\&=:F^1+F^2+F^3.\end{aligned}$$

By Lemma 3.1 with $p=\frac{n}{3}$, we have

$$\|F^1\|_{n,\infty}\le A_{\frac{n}{3}}\sup_{\varepsilon<s<t}\|f(t+\varepsilon-s)-f(t-s)\|_{\frac{n}{3},\infty}\le A_{\frac{n}{3}}\sup_{0<s<t-\varepsilon}\|f(s+\varepsilon)-f(s)\|_{\frac{n}{3},\infty}.$$

Hence the uniform continuity of f on $[0,t]$ yields $\|F^1\|_{n,\infty}\to0$ as $\varepsilon\to0$. Next we see that

$$\|F^2\|_{n,\infty}\le C\int_t^{t+\varepsilon}\frac1s\sup_{0\le s<\infty}\|f(s)\|_{\frac{n}{3},\infty}\,ds\le C\log\left(\frac{t+\varepsilon}{t}\right)\sup_{0\le s<\infty}\|f(s)\|_{\frac{n}{3},\infty}\to0\quad\text{as }\varepsilon\to0.$$

Finally, we estimate F^3. Take arbitrary $\eta > 0$ and take f_η as above. Then it holds that

$$\begin{aligned}\|F^3\|_{n,\infty} &\le \left\| \int_0^\varepsilon e^{s\Delta}\mathbb{P}[f(t-s) - f_\eta(t-s)]\,ds \right\|_{n,\infty} + \left\| \int_0^\varepsilon e^{s\Delta}\mathbb{P}f_\eta(t-s)\,ds \right\|_{n,\infty} \\ &\le \frac{\eta}{2} + C\varepsilon^{\frac{3}{2}-\frac{n}{2p}} \sup_{0\le s<\infty} \|f_\eta(s)\|_p.\end{aligned}$$

Then for sufficiently small $\varepsilon > 0$ we obtain $\|F^3\|_{n,\infty} < \eta$. Therefore, $e^{\varepsilon\Delta}F(t) - F(t) \to 0$ in $L^{n,\infty}(\mathbb{R}^n)$ as $\varepsilon \to 0$. By Theorem 2.1, $F(t) \in X_\sigma^{n,\infty}$ for each $t > 0$. Moreover, we note that $F(0) = 0 \in X_\sigma^{n,\infty}$.

For the case $f \in BC([0,\infty); L^1(\mathbb{R}^3))$, take $f_\eta \in BC([0,\infty); C_{0,\sigma}^\infty(\mathbb{R}^3))$ as above. Then same procedure above holds true.

We remark that the same argument is applicable to $\int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}g(s)\,ds$. This completes the proof. □

3.2 Abstract evolution equations

In this subsection, we develop a theory of abstract evolution equations with the semigroup which is not strongly continuous at $t = 0$, introduced by the previous work [20].

For a while, let A be a general closed operator on a Banach space X and $\{e^{tA}\}$ a bounded and analytic on X with the estimates

$$\sup_{0<t<\infty} \|e^{tA}\|_{\mathcal{L}(X)} \le N, \quad \|Ae^{tA}\|_{\mathcal{L}(X)} \le \frac{M}{t}, \quad t > 0, \tag{3.2}$$

where $\mathcal{L}(X)$ is the space of all bounded linear operators on X equipped with the operator norm. Especially, we note that e^{tA} is strongly continuous in X for $t \neq 0$.

Definition 3.1. *Let $\theta \in (0,1]$. We call f is the Hölder continuous on $[0,\infty)$ with value in X with the order θ, if for every $T > 0$ there exists $K_T > 0$ such that*

$$\|f(t) - f(s)\|_X \le K_T|t-s|^\theta, \qquad 0 \le t \le T,\ 0 \le s \le T.$$

Assumption. *Let $f : [0,\infty) \to X$. We assume for every $t > 0$*

$$\lim_{\varepsilon \searrow 0} \|e^{\varepsilon A}f(t) - f(t)\|_X = 0. \tag{A}$$

Lemma 3.6 ([20, Lemma 3.1]). *Let $a \in X$ and let $f \in C([0,\infty); X)$ be the Hölder continous on $[0,\infty)$ with value in X with order $\theta > 0$ and satisfy Assumption. Then*

$$u(t) = e^{tA}a + \int_0^t e^{(t-s)A}f(s)\,ds$$

satisfies $u \in C^1((0,\infty); X)$, $Au \in C((0,\infty); X)$ and

$$\frac{d}{dt}u = Au + f \quad \text{in } X \quad t > 0.$$

Remark 3.5. We note that we need a restriction only on the external force f not on initial data a. Moreover, Lemma 3.6 gives no information on the verification of the initial condition. While, property of the adjoint operator A^* and the dual space X^* has the possibility to verify of the initial condition with a suitable sense.

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