Large exponent asymptotics for one dimensional fully nonlinear diffusions of power type

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Abstract

In this note, we discuss limit behavior for fully nonlinear diffusion of power type in one space dimension. It turns out that, as the exponent tends to infinity, the solution converges locally uniformly to a unique limit function that is independent of the time variable. We rescale the time variable to characterize the limit as a unique viscosity solution of a fully nonlinear singular parabolic equation with jump discontinuity. Such asymptotic behavior is closely related to applications in math models of image denoising and collapsing sandpiles.

1 Introduction

This note is a simplified presentation of our work on large exponent behavior for power curvature flow [12, 10]. Although some parts of our results apply to the Euclidean space of general dimensions, in this paper we restrict ourselves in one space dimension so as to give a clearer view of the topic. Moreover, we choose to consider a heat-type problem rather than a general parabolic operator so that we can better clarify our basic idea.

The equation we are concerned with is as follows:

$$\text{(PH)} \quad \begin{cases} u_t - |u_{xx}|^{\alpha-1}u_{xx} = 0 & \text{in } \mathbb{R}\times(0,\infty), \quad (1.1) \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}, \quad (1.2) \end{cases}$$

where $\alpha > 0$ is a given exponent and u_0 is a given Lipschitz initial value. (Below we denote by $C^{0,1}(\mathbb{R})$ the class of k times differentiable functions with k-th derivatives Lipschitz continuous.) We are interested in the limit behavior of the unique solution of u_α of (PH) as $\alpha \to \infty$.

In addition to the Lipschitz continuity, we need to further assume that u_0 has small second derivatives near space infinity, that is, $u_0 \in C^{0,1}_b(\mathbb{R})$, where

$$C^{0,1}_b(\mathbb{R}) = \Big\{ f \in C^{0,1}(\mathbb{R}) : \text{there exist } \lambda\in(0,1) \text{ and a bounded open } \mathcal{K}\subset\mathbb{R} \text{ such that } -\lambda \le f_{xx} \le \lambda \text{ in the viscosity sense in } \mathbb{R}\setminus\overline{\mathcal{K}} \Big\}. \quad (1.3)$$

Our main result is as follows.

Theorem 1.1 (Large exponent behavior)**.** *Assume that* $u_0 \in C_b^{0,1}(\mathbb{R})$*. Let* u_α *be the unique solution of (PH). Then there exists* $U_\infty \in C_b^{0,1}(\mathbb{R})$ *such that* $u_\alpha(x,t) \to U_\infty(x)$ *locally uniformly for all* $(x,t) \in \mathbb{R} \times (0,\infty)$ *as* $\alpha \to \infty$*.*

This result amounts to saying that the large exponent limit does not depend on t and an initial layer forms in this limit process. When $u_0 \in C^{1,1}(\mathbb{R}) \cap C_b^{0,1}(\mathbb{R})$, the large exponent limit U_∞ can be characterized via a layer analysis as follows. Let

$$L_0 = \max\left\{\|(u_0)_{xx}\|_{L^\infty(\mathbb{R})}, 1\right\}. \tag{1.4}$$

We can rescale the solution u_α of (PH) by choosing arbitrarily

$$0 < \tau < \frac{1}{L_0} \tag{1.5}$$

and setting

$$U_\alpha(x,t) = u_\alpha\left(x, \frac{t^\alpha}{\alpha}\right) \quad \text{for } (x,t) \in \mathbb{R} \times [\tau, \infty). \tag{1.6}$$

Then it is not difficult to verify that U_α is the unique solution of

$$U_t - |tU_{xx}|^{\alpha-1} U_{xx} = 0 \quad \text{in } \mathbb{R} \times (\tau, \infty) \tag{1.7}$$

with the initial condition

$$U_\alpha(x,\tau) = u_\alpha\left(x, \frac{\tau^\alpha}{\alpha}\right). \tag{1.8}$$

We can show that the large exponent limit U_∞ exists and can be determined by

$$U_\infty(x) = \lim_{\alpha \to \infty} U_\alpha(x,1) \quad \text{for } x \in \mathbb{R}.$$

Letting $\alpha \to 0$ in (1.7)–(1.8) formally gives rise to the following fully nonlinear parabolic equation

$$U_t = F(t, U_{xx}) \quad \text{in } \mathbb{R} \times (\tau, \infty) \tag{1.9}$$

with

$$U(\cdot, \tau) = u_0 \quad \text{in } \mathbb{R}, \tag{1.10}$$

where $F : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is defined to be

$$F(t,z) = \begin{cases} 0 & \text{if } |z| < 1/t, \\ \operatorname{sgn}(z)/t & \text{if } |z| = 1/t, \\ \infty & \text{if } z > 1/t, \\ -\infty & \text{if } z < -1/t. \end{cases} \tag{1.11}$$

When u_0 is of class $C^{1,1}$, it turns out that, as $\alpha \to \infty$, U_α does converge to the unique continuous solution U of (1.9)–(1.10) locally uniformly in $\mathbb{R} \times [\tau, 1]$ and $u_\alpha(x,t)$ converges to $U_\infty(x) = U(x,1)$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$

The main difficulty lies at the uniqueness issue for the limit problem (1.9)–(1.10), since the operator F is highly singular and does not fit into the classical viscosity solution theory [7]. We use the notion of envelope viscosity solutions and establish a comparison principle for this problem. However, we need to assume that the second derivative of u_0

is small near space infinity so that u_α hardly evolves outside a compact set. It essentially gives us an extra boundary condition for the limit of U_α and facilitates our analysis. Once the comparison principle for (1.9) is established, the convergence of u_α then follows the standard Barles-Perthame procedure [4].

We remark that our results can be used as an alternative approach to study a model describing collapsing sandpiles [3, 8]. Also, large exponent asymptotics for a more general class of equations than (1.1), especially the power curvature flow, has applications in image processing [2, 1, 13, 5]. We refer the reader to [12, 10] for more details on these applications. It is also an important topic to investigate the limit behavior as $\alpha \to 0$, which was discussed in [6, 11].

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2 The Power-type Diffusion in One Dimension

Let us quickly review the wellposedness of (PH). Using the theory of viscosity solutions [7], one can show that for every fixed $\alpha > 0$ there exists a unique viscosity solution $u_\alpha \in C(\mathbb{R} \times [0, \infty))$ of (PH) satisfying the following growth condition: for any $T > 0$ there exists $C_T > 0$ such that

$$|u_\alpha(x, t)| \leq C_T(1 + |x|) \quad \text{for all } (x, t) \in \mathbb{R} \times [0, T].$$

In particular, the following comparison result holds.

Theorem 2.1 (Comparison principle for power-type heat equation)**.** *Fix $\alpha \geq 1$. Suppose that $u \in USC(\mathbb{R} \times [0, \infty))$ and $v \in LSC(\mathbb{R} \times [0, \infty))$ are respectively a locally bounded subsolution and a locally bounded supersolution of* (1.1)*. Assume that either $u(\cdot, 0)$ or $v(\cdot, 0)$ is Lipschitz in $\mathbb{R}$. Assume in addition that for any $T > 0$ there exists $C_T > 0$ such that*

$$|u(x, t)| + |v(x, t)| \leq C_T(1 + |x|) \quad \textit{for all } (x, t) \in \mathbb{R} \times [0, T].$$

If $u(x, 0) \leq v(x, 0)$ in $\mathbb{R}$, then

$$u(x, t) \leq v(x, t) \quad \textit{for all } (x, t) \in \mathbb{R} \times [0, \infty).$$

We refer the reader to [10, Appendix A] for a complete proof of this result and detailed discussion on the existence of solutions of (PH) by Perron's method.

Moreover, we have the following result on the Lipschitz preserving property.

Lemma 2.2 (Lipschitz regularity preserving property)**.** *Assume that $u_0 \in C^{0,1}(\mathbb{R})$. Let u_α be the unique viscosity solution of (PH) for every $\alpha > 0$. Then $u_\alpha(\cdot, t)$ is Lipschitz in $\mathbb{R}$ for any $t \geq 0$. Moreover,*

$$\|(u_\alpha)_x(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|(u_0)_x\|_{L^\infty(\mathbb{R})}$$

holds for any $t \geq 0$.

The proof is based on comparison between u_α and a translation of u_α in space.

3 Large exponent behavior

We next discuss the large exponent behavior for (PH) with $u_0 \in C_b^{0,1}(\mathbb{R})$ defined in (1.3). Our method is based on a rescaling technique described in Section 1.

3.1 Rescaled evolution

We assume that $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$ and recall L_0, τ as given in (1.4) and (1.5). We also recall the function U_α given by (1.6), which solves (1.7)–(1.8).

Let us rigorously study the behavior of U_α, which is supposed to be related to (1.9)–(1.10), where F is given in (1.11). We start with a definition of viscosity solutions to the limit equation. We recall in [7] the notion of semijets $P^{2,\pm}u(x_0, t_0)$ and $\overline{P}^{2,\pm}u(x_0, t_0)$ at a given point (x_0, t_0).

Definition 3.1 (Definition of viscosity solutions)**.** Let $\mathcal{O}$ be an open subset of $\mathbb{R} \times (0, \infty)$. A locally bounded upper semicontinuous (resp., lower semicontinuous) function $U : \mathcal{O} \to \mathbb{R}$ is called a subsolution (resp., supersolution) of

$$U_t = F(t, U_{xx}) \quad \text{in } \mathcal{O} \tag{3.1}$$

with F given by (1.11) if

$$z \geq -1/t_0, \quad \text{and} \quad \eta \leq 0 \text{ if } -1/t_0 \leq z < 1/t_0.$$

$$(\text{resp.,} \quad z \leq 1/t_0, \quad \text{and} \quad \eta \geq 0 \text{ if } -1/t_0 < z \leq 1/t_0.)$$

holds for any $(\eta, p, z) \in \overline{P}^{2,+}u(x_0, t_0)$ (resp., $(\tau, p, z) \in \overline{P}^{2,-}u(x_0, t_0)$). A locally bounded function u is a solution if it is both a subsolution and a supersolution.

It is equivalent to use the test functions instead of the semijets to define solutions as above. We next give a comparison principle for the Cauchy problem (1.9)–(1.10). A comparison principle for more general equations of the same type can be found in [10, Theorem 4.1].

Theorem 3.2 (Comparison principle)**.** *Fix $\tau > 0$ and $T > \tau$. Suppose that $u \in USC(\mathbb{R} \times [\tau, T])$ and $v \in LSC(\mathbb{R} \times [\tau, T])$ are bounded and respectively a subsolution and a supersolution of (3.1) with F given by (1.11) and $\mathcal{O} = \mathbb{R} \times (\tau, T)$. Assume that $u(\cdot, t)$ and $v(\cdot, t)$ are Lipschitz continuous in $\mathbb{R}$ uniformly for all $\tau \geq 0$. Assume in addition that there exists an open bounded subset $\mathcal{I} \subset \mathbb{R}$ such that $u \leq v$ holds in $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, T)$. If $u \leq v$ in $\mathbb{R} \times \{\tau\}$, then $u \leq v$ in $\mathbb{R} \times [\tau, T)$.*

Proof. Assume by contradiction that $u - v > 0$ somewhere in $\mathbb{R} \times [\tau, \infty)$. Then there exist $\sigma > 0$ small such that

$$(x, t) \mapsto u(x, t) - v(x, t) - \frac{\sigma}{T - t}$$

attains a positive value in $\mathbb{R} \times [\tau, T)$, which implies that

$$\sup_{(x,y,t) \in \mathbb{R}^2 \times [0,T)} u(x, t) - v(x, t) - \frac{\sigma}{T - t} \geq \mu$$

for some $\mu > 0$, since $u \leq v$ in $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, T)$. For any $\varepsilon > 0$, set

$$\Phi(x, y, t) = u(x, t) - v(y, t) - \frac{(x-y)^2}{2\varepsilon} - \frac{\sigma}{T-t}.$$

The Lipschitz continuity of $u(\cdot, t)$ and $v(\cdot, t)$ implies that there exists $L > 0$ such that

$$|u(x, t) - u(y, t)| \leq L|x - y|, \quad |v(x, t) - v(y, t)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}$. Using again the fact that $u \leq v$ in $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, T)$, we have

$$u(x, t) - v(y, t) \leq L|x - y| + \sup_{\mathcal{I} \times [\tau, T]} (|u| + |v|) \tag{3.2}$$

for all $x, y \in \mathbb{R}$ and $t \in [\tau, T]$. It follows that Φ is bounded from above; suppose that

$$\sup_{\mathbb{R}^2 \times [\tau, T)} \Phi = \mu_\varepsilon.$$

It is clear that $\mu_\varepsilon > \mu$. The supremum can actually be attained. Indeed, for any fixed $\varepsilon > 0$, suppose that there exists a maximizing sequence $(x_m, y_m, t_m) \in \mathbb{R}^2 \times [\tau, T)$ with $m > 0$ large such that

$$\Phi(x_m, y_m, t_m) \geq \mu_\varepsilon - \frac{1}{m}. \tag{3.3}$$

In light of the boundedness of $u - v$, we may use the relation

$$u(x_m, t_m) - v(y_m, t_m) - \frac{2\sigma}{T - t_m} - \frac{1}{m} \geq \frac{(x_m - y_m)^2}{2\varepsilon}$$

together with (3.2) to deduce that

$$|x_m - y_m| \leq C\varepsilon^{\frac{1}{2}}$$

for some $C > 0$ independent of ε and m. Suppose that $x_m, y_m \in \mathbb{R} \setminus \mathcal{I}$. Then we have

$$\Phi(x_m, y_m, t_m) \leq u(x_m, t_m) - v(y_m, t_m) \leq u(x_m, t_m) - u(x_m, t_m) \leq CL\varepsilon^{\frac{1}{2}}.$$

This contradicts (3.3) when $\varepsilon > 0$ is taken small enough to satisfy $CL\varepsilon^{\frac{1}{2}} < \mu$. The above argument also implies that x_m, y_m are bounded in $\mathbb{R}$ uniformly for all ε, m. Hence, by the upper semicontinuity of Φ, the maximum of Φ can be attained at the limit of a convergent subsequence of (x_m, y_m, t_m).

Let $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^2 \times [\tau, T)$ be a maximizer of Φ. Taking a subsequence such that $(\hat{x}, \hat{y}, \hat{t}) \to (x^*, x^*, t^*)$ for some $(x^*, t^*) \in \mathbb{R} \times [\tau, \infty)$ as $\varepsilon \to 0$, we have

$$u(x^*, t^*) - v(x^*, t^*) \geq \mu.$$

Since $u(\cdot, \tau) \leq v(\cdot, \tau)$, we deduce that $t^* \neq \tau$, which implies that $\hat{t} > \tau$ when $\varepsilon > 0$ is sufficiently small. Take

$$\Phi_b(x, y, t) = u(x, t) - v(y, t) - \frac{(x-y)^2}{2\varepsilon} - b(t - \hat{t})^2 - \frac{\sigma}{T-t},$$

where we choose $b \in \mathbb{R}$ satisfying

$$0 < b < \frac{\sigma}{2T^3} \tag{3.4}$$

for our later use. Then it is clear that Φ_b attains the same maximum value μ_ε at $(\hat{x}, \hat{y}, \hat{t})$ as well. Moreover, any maximizer (x, y, t) of Φ_b clearly fulfills that $t = \hat{t}$; in other words, any maximizer of Φ_b is a maximizer of Φ.

Hence, setting

$$K = \{(x, y) \in \mathbb{R}^2 : \Phi_b(x, y, \hat{t}) = \mu_\varepsilon\},$$

we can express the set of maximizers of Φ_b as $K \times \{\hat{t}\}$.

Let $0 < a < \min\{1, \mu_\varepsilon\}$. We consider

$$\Psi_a(x, y, t) := u(x, t) - v(y, t) - \frac{(x-y)^2}{2\varepsilon} + ax^2 - b(t-\hat{t})^2 - \frac{\sigma}{T-t}.$$

Since Φ_b has a maximum attained strictly at a compact subset $K \times \{\hat{t}\}$ of $\mathbb{R}^2 \times (\tau, T)$ and $\Psi_a(x, y, t) = \Phi_b(x, y, t) + aQ(x)$ with Q bounded, a standard argument [9, Lemma 2.2.5] yields that $\Psi_a(x_y, t)$ attains a local maximum $(x_a, y_a, t_a) \in \mathbb{R}^2 \times (\tau, T)$ close to $K \times \{\hat{t}\}$ for all $a > 0$ small. We therefore can take a convergent subsequence, still indexed by a for simplicity, such that $(x_a, y_a, t_a) \to (x_0, y_0, \hat{t})$ as $a \to 0$ for some $(x_0, y_0) \in K$. Below we turn back to use $(\hat{x}, \hat{y})$ instead of (x_0, y_0) to denote the limit of (x_a, y_a) for our convenience.

We next apply the Crandall-Ishii lemma (cf. [7, Theorem 8.3]) for Ψ_a at (x_a, y_a, t_a, s_a) to obtain

$$(\eta_1, p_1, z_1) \in \overline{P}^{2,+} u(x_a, t_a), \qquad (\eta_2, p_2, z_2) \in \overline{P}^{2,-} v(y_a, t_a)$$

satisfying

$$\eta_1 - \eta_2 = 2b(t_a - \hat{t}) + \frac{\sigma}{(T-t_a)^2}, \tag{3.5}$$

$$p_1 = -2ax_a + \frac{x_a - y_a}{\varepsilon}, \quad p_2 = \frac{x_a - y_a}{\varepsilon}$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} z_1 + 2a & 0 \\ 0 & -z_2 \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{3.6}$$

Note that (3.5) implies that

$$\eta_1 - \eta_2 > 0 \tag{3.7}$$

due to the requirement on b as given in (3.4).

By (3.6), we deduce that

$$z_1 - z_2 \le -2a. \tag{3.8}$$

Using the maximality of Ψ_a at (x_a, y_a, t_a), we now adopt a part of the definition of subsolutions and supersolutions to obtain

$$z_1 \ge -\frac{1}{t_a}, \qquad z_2 \le \frac{1}{t_a}. \tag{3.9}$$

By (3.8), we therefore have

$$z_1 \le \frac{1}{t_a} + z_1 - z_2 \le \frac{1}{t_a} - 2a < \frac{1}{t_a},$$
$$z_2 \ge -\frac{1}{t_a} + z_2 - z_1 \ge -\frac{1}{t_a} + 2a > \frac{1}{t_a}.$$

It thus follows from the remaining part of definition of sub- and supersolutions that

$$\eta_1 \le 0 \le \eta_2$$

which is obviously a contradiction to (3.7). □

3.2 Large exponent limit

We first consider the behavior of U_α near space infinity. Since $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$, we may regularize u_0 to obtain existence of $0 < \lambda_0 < 1$, an open bounded subset $\mathcal{I} \subset \mathbb{R}$ and two family of functions $\psi_\alpha^\pm \in C^\infty(\mathbb{R})$ $(\alpha \ge 1)$ such that

$$\psi_\alpha^- \le u_0 \le \psi_\alpha^+ \quad \text{in } \mathbb{R},$$

$$\sup_{x\in\mathbb{R}\setminus\mathcal{I}} \{|\psi_\alpha^+(x) - u_0(x)| + |\psi_\alpha^-(x) - u_0(x)|\} \le \frac{1}{\alpha}$$

and

$$\max\left\{\sup_{\mathbb{R}} |(\psi_\alpha^+)_{xx}|,\ \sup_{\mathbb{R}} |(\psi_\alpha^-)_{xx}|\right\} \le \lambda_0. \tag{3.10}$$

It follows that

$$w_\alpha^+(x,t) = \psi_\alpha^+(x) + \lambda_0^\alpha t, \quad w^-(x,t) = \psi_\alpha^-(x) - \lambda_0^\alpha t$$

are respectively a super- and subsolution of (1.1), which by Theorem 2.1 yields that $w_\alpha^- \le u_\alpha \le w_\alpha^+$ in $\mathbb{R}\times[0,\infty)$. Therefore, we have

$$u_0(x) - \frac{1}{\alpha} - \frac{(\lambda_0 t)^\alpha}{\alpha} \le U_\alpha(x,t) \le u_0(x) + \frac{1}{\alpha} + \frac{(\lambda_0 t)^\alpha}{\alpha}$$

for any $(x,t) \in (\mathbb{R}\setminus\mathcal{I}) \times [\tau,\infty)$ and thus we obtain the following result.

Lemma 3.3 (Convergence outside a compact set)**.** *Assume that $u_0 \in C_c^{1,1}(\mathcal{T})$. Let u_α be the solution of (GP) and U_α be as in* (1.6)*. Let τ be given as in* (1.5)*. Then there exist an open bounded subset $I \subset \mathbb{R}$ and $\lambda_0 \in (0,1)$ such that $U_\alpha \to 0$ locally uniformly in $(\mathbb{R}\setminus\mathcal{I})\times[\tau, 1/\lambda_0)$ as $\alpha\to\infty$, .*

Our first main result is as follows.

Theorem 3.4 (Large exponent convergence for the rescaled equation)**.** *Assume that $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$. Let $\tau > 0$ be given in* (1.5)*. Let u_α be the unique solution of (PH) and U_α be as in* (1.6)*. Let $\lambda_0 \in (0,1)$ be given by* (3.10)*. Then $U_\alpha \to U$ locally uniformly in $\mathbb{R}\times[\tau, 1/\lambda_0)$ as $\alpha\to\infty$, where U is the unique continuous solution of* (3.1) *and* (1.10) *with $\mathcal{O} = \mathbb{R}\times(\tau, 1/\lambda_0)$ and F given by* (1.11)*.*

We use the so-called Barles-Perthame procedure [4] to prove the convergence. Let us take the half relaxed limits of U_α in $\mathbb{R}\times[\tau, 1/\lambda_0)$:

$$\overline{U} = \limsup_{\alpha\to\infty}{}^{*}\, U_\alpha, \qquad \underline{U} = \liminf_{\alpha\to\infty}{}_{*}\, U_\alpha. \tag{3.11}$$

Proposition 3.5 (Sub- and supersolution properties of relaxed limits)**.** *Let $\tau > 0$. Assume that U_α is a solution of* (1.7)*. Then $\overline{U}$ and $\underline{U}$ as in* (3.11) *are respectively a subsolution and a supersolution of* (1.9)*.*

Proof. We only prove that $\overline{U}$ is a subsolution. The supersolution part can be proved using a symmetric argument. Let $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$.

Suppose that there exist $(x_0, t_0) \in \mathcal{O}$ and $\phi \in C^2(\mathcal{O})$ such that $\overline{U} - \phi$ attains a strict maximum at (x_0, t_0). By [9, Lemma 2.2.5], $U_\alpha - \phi$ has a local maximum at (x_α, t_α) with $(x_\alpha, t_\alpha) \to (x_0, t_0)$ as $\alpha \to \infty$. Since U_α is a subsolution of (1.7), we have

$$\phi_t(x_\alpha, t_\alpha) \le t_\alpha^{\alpha-1} |\phi_{xx}(x_\alpha, t_\alpha)|^{\alpha-1} \phi_{xx}(x_\alpha, t_\alpha). \tag{3.12}$$

It follows that

$$\liminf_{\alpha\to\infty} \left(t_\alpha \phi_{xx}(x_\alpha, t_\alpha)\right) \ge -1,$$

which implies that $\phi_{xx}(x_0, t_0)) \ge -1/t_0$. Moreover, if

$$-1/t_0 \le \phi_{xx}(x_0, t_0) < 1/t_0,$$

then sending $\alpha \to \infty$ in (3.12) yields $\phi_t(x_0, t_0) \le 0$. □

Proposition 3.6 (Verification of initial value)**.** *Assume that $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$. Let $\tau > 0$. Let u_α be the solution of (PH). Let U_α be as in* (1.6)*. Then $\overline{U}$ and $\underline{U}$ in* (3.11) *satisfy*

$$\overline{U}(x, \tau) \le u_0 \le \underline{U}(x, \tau) \quad \textit{for all } x \in \mathbb{R}.$$

Proof. In order to show $\overline{U}(\cdot, \tau) \le u_0$ in $\mathbb{R}$, we construct supersolutions of (1.1) and (1.7). Set

$$w(x, t) = u_0(x) + L_0^\alpha t \quad \text{for } (x, t) \in \mathbb{R} \times [0, \infty)$$

and

$$W(x, t) = U_\alpha(x, \tau) + \frac{L_0^\alpha}{\alpha}(t^\alpha - \tau^\alpha) \quad \text{for } (x, t) \in \mathbb{R} \times [\tau, \infty),$$

where L_0 is as in (1.4).

We claim that w and W are respectively supersolution of (1.1) and (1.7) for all $\alpha > 0$. To prove this, we may use a standard mollification to approximate u_0 with $u_0^\varepsilon \in C^\infty(\mathbb{R})$ so that for any $\delta > 0$, we get

$$|(u_0^\varepsilon)_{xx}| \le L_0 + \delta \quad \text{in } \mathbb{R}$$

when $\varepsilon > 0$ is sufficiently small. It is then easily verified that

$$w^\varepsilon(x, t) = u_0^\varepsilon(x) + \frac{(L_0 + \delta)^\alpha}{\alpha} t$$

is a supersolution of (1.1). Adopting the standard stability of viscosity solutions, we deduce that w is also a supersolution of (1.1) by letting $\varepsilon \to 0$ and then $\delta \to 0$. The claim for W can be analogously proved.

Moreover, since $w(\cdot, 0) = u_0$ in $\mathbb{R}$, we have

$$u_\alpha \le w \quad \text{in } \mathbb{R} \times [0, \infty)$$

for all $\alpha > 0$ by Theorem 2.1. A similar argument, combined with a comparison result for (1.7), yields that

$$U_\alpha \le W \quad \text{in } \mathbb{R} \times [\tau, \infty).$$

Hence, we can use the condition (1.5) on τ to deduce that

$$U_\alpha(x,\tau) = u_\alpha\left(x, \frac{\tau_\alpha}{\alpha}\right) \le w\left(x, \frac{\tau_\alpha}{\alpha}\right) \le u_0(x) + \frac{(L_0\tau)^\alpha}{\alpha} \le u_0(x) + \frac{1}{\alpha}$$

for all $x \in \mathbb{R}$ and all $\alpha \ge 1$, which in turn implies that

$$U_\alpha(x,t) \le W(x,t) \le u_0(x) + \frac{1}{\alpha} + \frac{L_0^\alpha}{\alpha}(t^\alpha - \tau^\alpha).$$

In view of (1.5) again, letting $\alpha \to \infty$, we end up with

$$\overline{U}(x,\tau) \le u_0(x) \quad \text{ for all } x \in \mathbb{R}.$$

We can similarly show that

$$\underline{U}(x,\tau) \ge u_0(x) \quad \text{ for all } x \in \mathbb{R}.$$

□

We are now in a position to prove Theorem 3.4.

Proof. As shown in Proposition 3.5, the half relaxed limits $\overline{U}$ and $\underline{U}$ are respectively a subsolution and a supersolution of (3.1) with $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$. In view of Proposition 3.6, we have $\overline{U}(\cdot, 0) = \underline{U}(\cdot, 0) = u_0$ in $\mathbb{R}$.

Since $u_\alpha(x,t)$ preserves the Lipschitz continuity of u_0 in x for all $t \in [0,\infty)$, we easily see that U_α, $\overline{U}$ and $\underline{U}$ are all Lipschitz in space with the same Lipschitz constant as u_0. In addition, by Lemma 3.3, we have $\overline{U} = \underline{U}$ in $(\mathbb{R} \setminus \mathcal{I}) \times [\tau, 1/\lambda_0)$ for some bounded open set $\mathcal{I} \subset \mathbb{R}$.

Using Theorem 3.2 with $T = 1/\lambda_0$, we end up with $\overline{U} \le \underline{U}$ in $\mathbb{R} \times [\tau, 1/\lambda_0)$. Since by definition it is clear that $\overline{U} \ge \underline{U}$, this yields that $\overline{U} = \underline{U}$ in $\mathbb{R} \times [\tau, 1/\lambda_0)$. We thus obtain the locally uniform convergence of U_α as $\alpha \to \infty$. □

We have shown that there exists a unique solution of (3.1) and (1.10) with $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$. The following is then an immediate consequence of Theorem 3.4 and (1.6).

Theorem 3.7 (Large exponent convergence for $C^{1,1}$ initial data)**.** *Assume that $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$. Let τ be given by (1.5) and λ_0 be as in Lemma 3.3. Let u_α be a unique solution of (PH). Let U be the unique solution of (3.1) with (1.10) and $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$. Then $u_\alpha(x,t) \to U_\infty(x) := U(x,1)$ locally uniformly for all $(x,t) \in \mathbb{R} \times (0,\infty)$ as $\alpha \to \infty$.*

We next show that the limit of u_α above does not depend on the choice of τ.

Proposition 3.8 (Independence of choices of initial time)**.** *Assume that $u_0 \in C_b^{0,1}(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$. Let L_0 be given by (1.4). Assume that $0 < \tau_1 < \tau_2 < 1/L_0$. Let $\mathcal{O}_i = \mathbb{R} \times (\tau_i, 1/\lambda_0)$ and let U_i be the unique solution of (3.1) and (1.10) with $\mathcal{O} = \mathcal{O}_i$ for $i = 1, 2$. Then $U_1(x, \tau_2) = U_2(x, \tau_2)$ for all $x \in \mathbb{R}$. In particular, $U_1 = U_2$ in $\mathbb{R} \times [\tau_2, 1/\lambda_0)$.*

Proof. Since $\tau_1 < \tau_2 < 1/L_0$, it is easily seen that

$$t\,|(u_0)_{xx}| < 1 \quad \text{a.e. in } \mathbb{R} \tag{3.13}$$

for all $t \in (\tau_1, \tau_2)$, which is equivalent to saying that

$$-1/\tau_2 < (u_0)_{xx} < 1/\tau_2$$

holds in $\mathbb{R}$ in the viscosity sense. It follows that

$$U(x,t) = u_0(x) \tag{3.14}$$

is a solution of (3.1) with $\mathcal{O} = \mathbb{R} \times (\tau_1, \tau_2)$. As a consequence of the uniqueness result (Theorem 3.2) and the continuity of U_1, we have $U_1(x,t) = u_0(x)$ for all $(x,t) \in \mathbb{R} \times [\tau_1, \tau_2]$. Since U_1 and U_2 share the same value at $t = \tau_2$, it is obvious that $U_1 = U_2$ in $\mathbb{R} \times [\tau_2, 1/\lambda_0)$. □

We finally drop the $C^{1,1}$ regularity of u_0 and prove the more general convergence result in Theorem 1.1.

Proof of Theorem 1.1. For a general initial value $u_0 \in C_b^{0,1}(\mathbb{R})$, we can approximate it by regularization with the standard mollifier. We may assume that there exist a bounded open set $\mathcal{I} \subset \mathbb{R}$ and a sequence $u_{0,m} \in C^\infty(\mathbb{R}) \cap C_b^{0,1}(\mathbb{R})$ such that $u_{0,m}$ converges uniformly to u_0 in $\mathbb{R}$ and

$$|(u_{0,m})_{xx}(x)| \leq \lambda \quad \text{for any } x \in \mathbb{R} \setminus \mathcal{I}$$

for some $\lambda \in (0,1)$. Note that for every $m > 0$, we can obtain a large exponent limit $U_{\infty,m}$ corresponding to the solution $u_{\alpha,m}$ of (PH) with the initial condition $u_{0,m}$. Moreover, by Theorem 2.1, we have

$$\sup_{\mathbb{R}\times[0,\infty)} |u_{\alpha,m} - u_{\alpha,m'}| \leq \sup_{\mathbb{R}} |u_{0,m} - u_{0,m'}|$$

for all $m, m' > 0$ and $\alpha \geq 1$, which implies, by letting $\alpha \to \infty$, that

$$\sup_{\mathbb{R}} |U_{\infty,m} - U_{\infty,m'}| \leq \sup_{\mathbb{R}} |u_{0,m} - u_{0,m'}|.$$

In other words, $U_{\infty,m}$ is a Cauchy consequence in $C_b^{0,1}(\mathbb{R})$. Since $U_{\infty,m} = u_0$ in $\mathbb{R} \setminus \mathcal{I}$ for all $m > 0$, we can find a unique limit U_∞ of $U_{\infty,m}$ in $C_b^{0,1}(\mathbb{R})$. It is then easy to see that $u_\alpha(x,t) \to U_\infty(x)$ locally uniformly in $\mathbb{R} \times [\tau, \infty)$ as $\alpha \to \infty$. □

Let us finally discuss the large exponent behavior of solution to (PH) when u_0 is Lipschitz and convex in $\mathbb{R}$.

Suppose that u_0 is smooth, Lipschitz and convex in $\mathbb{R}$. Recall that $U_\infty = U(\cdot, 1)$, where U is the solution of (3.1) with $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$ and the initial value u_0. It turns out that U_∞ satisfies

$$-W_{xx} + 1 \geq 0 \quad \text{in } \mathbb{R} \tag{3.15}$$

in the viscosity sense. To see this, suppose that there exist $x_0 \in \mathbb{R}$ and $\phi \in C^2(\mathbb{R})$ such that $U_\infty - \phi$ attains a strict minimum over $\mathbb{R}$ at x_0. Then for any $\beta > 0$ large,

$$(x,t) \mapsto U(x,t) - \phi(x) + \beta|t-1|^2$$

attains a local minimum at $(x_\beta, t_\beta) \in \mathbb{R} \times (0, \infty)$ with $(x_\beta, t_\beta) \to (x_0, 1)$ as $\beta \to \infty$. It then follows from Definition 3.1 that

$$\phi_{xx}(x_\beta) \leq 1/t_\beta,$$

which yields that $\phi_{xx}(x_0) \leq 1$ by passing to the limit as $\beta \to \infty$.

In addition, thanks to the convexity of u_0, we can show that u_0 is a stationary subsolution of (PH), which by Theorem 2.1 implies that $u_\alpha(x,t) \geq u_0(x)$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$. In view of Theorem 3.7, passing to the limit as $\alpha \to \infty$, we deduce that

$$U_\infty \geq u_0 \quad \text{in } \mathbb{R}. \tag{3.16}$$

Moreover, we can show that $U_\infty \leq W$ for any supersolution W of (3.15) that satisfies $W \geq u_0$. To prove this, it suffices to show the same result for W that has at most a linear growth at space infinity, since one can reduce the value of W to meet this condition. It is not difficult to verify that W itself is a supersolution of (3.1) with $\mathcal{O} = \mathbb{R} \times (\tau, 1/\lambda_0)$ with $\tau, \lambda \in (0,1)$ depending on the regularity of u_0. Then by Theorem 3.2, we get

$$U(x,t) \leq W(x) \quad \text{for any } (x,t) \leq \mathbb{R} \times (\tau, 1/\lambda),$$

which implies that

$$U_\infty(x) = U(x,1) \leq W(x) \quad \text{for any } x \in \mathbb{R}. \tag{3.17}$$

Our argument above amounts to saying that U_∞ is the minimal supersolution of the obstacle problem

$$\min\{-U_{xx} + 1,\ U - u_0\} = 0 \quad \text{in } \mathbb{R}; \tag{3.18}$$

that is,

$$U_\infty(x) = \inf\{U(x) :\ U \text{ is a viscosity supersolution of } (3.18)\}. \tag{3.19}$$

For u_0 that is only Lipschitz and convex without smoothness, we can obtain the same result by approximation. This result is consistent with [12, Theorem 8.3].

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