A SYMMETRIC CRITICALITY PRINCIPLE FOR O'HARA'S ENERGIES

ALEXANDRA GILSBACH
INSTITUT FÜR MATHEMATIK, RWTH AACHEN UNIVERSITY, GERMANY

ABSTRACT. In geometric knot theory, one tries to examine knots, i.e. embedded curves in $\mathbb{R}^3$, by using energy functionals. These so-called knot energies shall determine whether and to what extent a curve is knotted. We focus on the O'Hara energies, which are a family of smooth knot energies, and try to evaluate their energy landscape. Using a modified version of Palais' symmetric criticality principle, we show the existence of at least two distinct critical knots for O'Hara's energies. These critical knots are smooth.

Note: The results presented in the following have already been published in joint work with Heiko von der Mosel in [GvdM18].

1. Introduction

To examine knotted objects, we define a knot as a set $\Gamma \subset \mathbb{R}^3$, which is the image of a closed, injective curve. Precisely, there exists a closed, injective, rectifiable curve γ, such that $\gamma(\mathbb{R}/\mathbb{Z}) = \Gamma$. We will restrict to curves which are at least Lipschitz-continuous.

When dealing with knots, one of the major questions is, whether two given knots are the same. And by "the same", we mean that one knot may be deformed into the other knot without cutting or glueing arcs of the knot. This is modelled by using the notion of ambient isotopy, and two knots Γ_1 and Γ_2 will be equivalent, if for suitable parametrisations γ_1 and γ_2 there exists an ambient isotopy mapping γ_1 to γ_2. Knots which are related in that way are said to be of the same knot type and they lie in the same *knot class*. For an embedded (knotted) curve $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, we define its knot class as

$$[\gamma] = \left\{ \eta \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) \,\middle|\, \eta \text{ embedded and ambient isotopic to } \gamma \right\}.$$

To either find such an ambient isotopy or to show that no such isotopy exists can be a tedious task. Therefore, in knot theory, one considers different methods to gain information about a given knot.

The tool of geometric knot theory is to assign certain energy values to such a knot, depending on its entangledness. The idea is to punish situations, where two curve points are close to each other in space but distant along the curve, by a high energy value. Like this, unneccessary entanglements are avoided by demanding a low energy. Additionally, we require these functionals to be charge or self-repulsive, that is, they shall provide infinitely high energy barriers between knot classes. A functional which is bounded from below and charge will be called *knot energy.* Various knot energies have been constructed, among them being the O'Hara energies, introduced by Jun O'Hara in [O'H92],

$$\text{(1)}\quad \mathcal{E}_\alpha(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1}{|\gamma(s+t) - \gamma(s)|^\alpha} - \frac{1}{d_\gamma(s+t,s)^\alpha} \right) |\gamma'(s+t)||\gamma'(s)| \, dt ds, \quad \alpha \in [2,3).$$

Here,

$$d_\gamma(s, s+t) := \min \left\{ \mathscr{L}\left(\gamma|_{[s,s+t]}\right), \mathscr{L}(\gamma) - \mathscr{L}\left(\gamma|_{[s,s+t]}\right) \right\} \quad \text{for } |t| \le 1/2$$

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denotes the intrinsic distance between the points $\gamma(s)$ and $\gamma(t)$ on the curve γ. $\mathscr{L}$ denotes the length functional.

To be able to use a knot energy to gain information about the knot class of an arbitrary given knot, one would like to proceed in the following way: Firstly, we apply a gradient flow to the parametrisation of the given knot, arriving at the minimiser of the (unknown) knot class. Then, we use the energy value and shape of the limit knot to narrow down the possible knot classes the knot might belong to. The self-repulsiveness is used to ensure that the minimiser of a knot energy in a given knot class is a nice representative of that knot class.

However, this method requires a lot of prerequisites. We need a high regularity of the energy functional as well as the existence of a gradient flow. In addition, in each knot class, we have to ensure that the gradient flow stays within that knot class. Finally, the existence of a minimiser is required.

But even in the case where all these conditions hold, the gradient flow might get stuck in local minima. In fact, numerical experiments suggest for several knot energies that local minima or saddle points might be present: Constructing a topological invariant, Moffatt proposed in [Mof90], that there might be several local minima present in the case of torus knot classes, showing this in the case of the knot class of the trefoil. In [Kau12], Kauffman conducted experiments with the knot energies in Scharein's KnotPlot [Sch17], with the result that distinct local minima might be present.

In view of such experiments, our aim is to examine these situations analytically. We will prove that, in certain knot classes, there exist at least two different critical points:

Theorem 1.1. *Let $a, b \in \mathbb{Z} \setminus \{0, \pm 1\}$ be relatively prime and let $\alpha \in (2,3)$. Then there exist at least two different arclength parametrised, embedded curves $\gamma_1, \gamma_2 \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ that are critical for the energy*

$$\mathcal{E}^*_\alpha(\gamma) := \mathscr{L}^{\alpha-2}(\gamma)\mathcal{E}_\alpha(\gamma) \tag{2}$$

in the torus knot class $\mathcal{T}(a, b)$.

Remark 1.2.

(i) *Different* means here that there is no isometry of $\mathbb{R}^3$ or reparametrisation transforming γ_1 into γ_2. This is a natural nomenclature since knot energies are invariant with respect to isometries of the $\mathbb{R}^3$ and reparametrisations.

(ii) Note that, due to technical reasons, we use the *scaling invariant* version of O'Hara's energies, $\mathcal{E}^*_\alpha$.

(iii) Even though we defined O'Hara's energies for $\alpha = 2$ as well, we do not deal with this case here. We will discuss the difficulties of this case in Remark 2.8.

By showing the existence of several critical points, Theorem 1.1 supports the hypothesis that the gradient flow for the scaling invariant $\mathcal{E}^*_\alpha$ could get stuck in one of these critical points. However, it is important to emphasise that we are not able to deduce the nature of the gained critical points, it remains to be shown whether more than one critical point might be a local minimum.

To show Theorem 1.1, we will, similarly to Cantarella et al. in [CFMER14] for the Ropelength functional, use the principle of symmetric criticality introduced by Palais [Pal79]. This principle uses symmetry properties of a given system. Under suitable restrictions on a symmetry group G, a manifold $\mathscr{M}$ and a function $f \colon \mathscr{M} \to \mathbb{R}$, the principle states that any critical point of f in the set of fixed points with respect to G is already a critical point of f in $\mathscr{M}$.

In contrast to the method of Cantarella et al., we are able to apply the classic version of Palais' principle, since the functional $\mathcal{E}^*_\alpha$ is sufficiently smooth. Furthermore, the critical points we obtain are smooth as well.

The same method we use here may also be applied to other sufficiently smooth knot energies such as the integral Menger curvature or the tangent-point energies, see [Gil18]. Those energies were thoroughly investigated in [SSvdM10, SvdM12, SSvdM13].

We proceed as follows. In the preliminaries we present the principle of symmetric criticality of Palais and its transformation into the knot theoretic setting. We will therefore present some properties of the O'Hara energies and of curves of finite O'Hara energy. Then we will go into detail about the symmetry of the system we consider: Since knot energies are invariant with respect to isometries in $\mathbb{R}^3$, we detect possible symmetry groups for knots, which are subgroups of $\mathrm{Isom}(\mathbb{R}^3)$. We will focus on rotational symmetries for knots and their rotational axes, see Theorem 2.15.
In Section 3, we use the principle of symmetric criticality to show Theorem 1.1. Therefore, we firstly need to prove the existence of critical points in the set of fixed points with respect to the group considered. This will be done by showing the existence of a minimiser in that set of G-fixed points, see Theorem 3.9. For the knot classes of the torus knots, we are able to show the existence of two disjoint sets of fixed points for two different groups G_1 and G_2, using Theorem 2.15. Applying the principle of symmetric criticality then yields the central result Theorem 1.1.

2. Preliminaries

In this section, we will explain how to adapt Palais' principle [Pal79] to the knot setting. Therefore, we will briefly recall needed properties of O'Hara's energies. Furthermore, we will deduce valid symmetry groups for knots.

2.1. **The principle of symmetric criticality.** In his paper [Pal79], Palais formulated the principle of symmetric criticality as follows.

Theorem ([Pal79, Thm. 5.4])**.** *Let G be a compact Lie group and $\mathscr{M}$ be a G-Banach manifold. Let $f\colon \mathscr{M} \to \mathbb{R}$ be a smooth, G-invariant function. Then the set Σ of G-invariant points in $\mathscr{M}$ is a smooth submanifold of $\mathscr{M}$ and every critical point of f in Σ is a critical point of f in $\mathscr{M}$.*

Indeed, it suffices if the function f is only of C^1-regularity, as may be seen in the proofs in [Pal79]. Furthermore, the theorem still holds for G being a finite group, since it may be interpreted as zero-dimensional compact Lie group, see [Coh57, p. 48, Exp. 5]. Thus, we will deal with a slightly modified version of the principle of symmetric criticality:

Theorem 2.1. *Let G be a finite group and $\mathscr{M}$ be a G-Banach manifold. Let $f \in C^1(\mathscr{M}, \mathbb{R})$ be a G-invariant function. Then the set Σ of G-invariant points in $\mathscr{M}$ is a smooth submanifold of $\mathscr{M}$ and every critical point of f in Σ is a critical point of f in $\mathscr{M}$.*

To understand how to make use of that principle in our knot theoretic setting, we briefly recall the definition of a G-Banach manifold. Note that a Banach manifold is a topological space that is locally diffeomorphic to a Banach space.

Definition 2.2 (***G***-Banach manifold)**.** Let $\mathscr{M}$ be a Banach manifold and let G be a topological group acting on $\mathscr{M}$ via the representation $\tau\colon G \times \mathscr{M} \to \mathscr{M}$, i.e. $\mathscr{M}$ is a G-space. $\mathscr{M}$ is said to be a G-Banach manifold, if the representation $\tau(g, \cdot)$ is a diffeomorphism on $\mathscr{M}$ for every $g \in G$. If G is a Lie group, we additionally assume $\tau \in C^1$ with respect to the product topology of $G \times \mathscr{M}$.

To adapt the principle of symmetric criticality, our aim is to choose a knot class as a G-Banach manifold. Therefore, we will need to specify how a group G shall act on elements of a knot class. To do this, we will need to find a suitable group G. Since $\mathcal{E}_\alpha^*$ will be the G-invariant function considered, that will be a restriction to the choice of G, as we will see later on.
But before, we need to verify whether $\mathcal{E}_\alpha^*$ satisfies the requirements of Palais' principle. In lieu of that, we recall some properties of O'Hara's energies.

2.2. **Properties of O'Hara's energies.** We merely present the results needed; the proofs may be found in the respective sources or as well in our self-contained paper [GvdM18].

We begin with some basic properties.

Lemma 2.3 (Properties of $\mathcal{E}_\alpha$)**.**

(i) *For $\alpha = 2$, $\mathcal{E}_2$ is invariant with respect to Möbius transformations. Therefore, it is also called* Möbius energy*. For $\alpha \in (2,3)$, $\mathcal{E}_\alpha$ is invariant with respect to isometries of the $\mathbb{R}^3$ and reparametrisations.[FHW94, Thm. 2.1]*

(ii) *$\mathcal{E}_2$ is minimisable in any tame prime knot class, see [FHW94, Thm. 4.3].*

(iii) *For $\alpha \in (2,3)$, $\mathcal{E}_\alpha$ is minimisable in tame knot classes, see [O'H94, Thm. 3.2].*

(iv) *For $\alpha \in [2,3)$, over all knot classes, the circle uniquely minimises $\mathcal{E}_\alpha$, see [ACF⁺03].*

Recall that a tame knot class is a knot class which contains a polygonal representative. Any knot class containing C^1-representatives is tame, see [CF77, App.I], and any tame knot class contains smooth representatives. We will only deal with tame knot classes here.

We now present regularity properties of curves of finite energy $\mathcal{E}_\alpha$. We begin with a bi-Lipschitz estimate for such curves, which was shown by O'Hara in [O'H92, Theorem 2.3].

Lemma 2.4. *Let $\alpha \in [2,3)$ and let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\gamma'(t)| > 0$ for a.e. $t \in \mathbb{R}/\mathbb{Z}$. Then for all $b \geq 0$ there exists a constant $C = C(b) \geq 0$ such that $\mathcal{E}_\alpha(\gamma) \leq b$ implies the bi-Lipschitz estimate*

$$|\gamma(s) - \gamma(t)| \geq C d_\gamma(s,t) \quad \text{for all } s,t \in \mathbb{R}/\mathbb{Z}. \tag{3}$$

Hence, any $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $|\gamma'| > 0$ a.e. and $\mathcal{E}_\alpha(\gamma) < \infty$ is injective.

Due to the results of Blatt in [Bla12], we may even give a precise characterisation of curves of finite energy $\mathcal{E}_\alpha$. We present a slightly refined version of his theorem, the proof may be found in [GvdM18].

Theorem 2.5 (Characterisation of curves of finite energy)**.** *We have*

(i) *Let $\alpha \in [2,3)$ and let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with length $0 < L := \mathscr{L}(\gamma)$ such that $|\gamma'(t)| > 0$ for a.e. $t \in \mathbb{R}/\mathbb{Z}$. If $\mathcal{E}_\alpha(\gamma) < \infty$, then $\gamma|_{[0,1)}$ is injective and for its arclength parametrisation $\tilde\gamma \in C^{0,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ we have $\tilde\gamma \in W^{(\alpha+1)/2,2}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$. Furthermore, we then have*

$$[\tilde\gamma']^2_{(\alpha-1)/2,2} \leq 4^4 \cdot 2^{2-2\alpha} \mathcal{E}_\alpha(\gamma). \tag{4}$$

(ii) *Let $\alpha \in (2,3)$. Now, assume that $\gamma \in W^{(\alpha+1)/2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, $|\gamma'| > 0$ a.e., and further $\gamma|_{[0,1)}$ is injective. Then we have $\mathcal{E}_\alpha(\gamma) < \infty$.*

In [Bla12], Blatt proved part (ii) only for arclength parametrised curves, but for the full two-parameter family of O'Hara's energies which also includes the case $\alpha = 2$. In [GvdM18], we generalised (ii) to the setting above.
The characterisation theorem is crucial in Section 3, amongst others to identify the correct Banach manifold (Corollary 3.2), on which Palais' principle of symmetric criticality is applicable.

For a better understanding of the characterisation theorem above, we briefly recall the definition of the Sobolev-Slobodetckij spaces $W^{s,p}$. We restrict to those spaces which consist of functions with domain $\mathbb{R}/L\mathbb{Z}$, with arbitrary $L > 0$, and whose images lie in $\mathbb{R}^3$.

Definition 2.6 (Sobolev-Slobodeckji-Space)**.** Let $L > 0$, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $s \in (0,1)$. For a function $f \colon \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ we define the following seminorm,

$$\left[f^{(k)}\right]_{W^{s,p}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^3)} = \int_{\mathbb{R}/L\mathbb{Z}} \int_{\mathbb{R}/L\mathbb{Z}} \frac{|f^{(k)}(x) - f^{(k)}(y)|^p}{|x-y|^{1+sp}}.$$

The set

$$W^{k+s,p}\left(\mathbb{R}/L\mathbb{Z},\mathbb{R}^3\right) := \left\{f \in W^{k,p}\left(\mathbb{R}/L\mathbb{Z},\mathbb{R}^3\right) \,\Big|\, ||f||_{W^{k+s,p}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^3)} < \infty\right\},$$

with

$$||f||_{W^{k+s,p}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^3)} := ||f||_{W^{k,p}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^3)} + \left[f^{(k)}\right]_{W^{s,p}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^3)},$$

is called Sobolev-Slobodeckji-Space or fractional Sobolev space, with fractional differentiability $k+s$ and integrability p.

Remark 2.7. It is well-known that Sobolev-Slobodetckij are Banach spaces. Furthermore, one has the following embedding into Hölder spaces, see [DNPV12, Thm. 8.2].

$$W^{k+s,p}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^n) \hookrightarrow C^{k,s-1/p}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^n) \text{ for } p \in (1,\infty), s \in (1/p,1) \tag{5}$$

For $\alpha \in (2,3)$, $s = \alpha - 1/2 \in (1/2,1)$ and $p = 2$ we obtain

$$W^{\alpha+1/2,2}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^n) \hookrightarrow C^{1,\alpha/2-1}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^n), \tag{6}$$

which is a continuous embedding. This implies the existence of a constant $C = C(L,n)$ such that

$$\|f\|_{C^{1,\alpha/2-1}} \le C\|f\|_{W^{\alpha+1/2,2}} \quad \text{for all } f \in W^{\alpha+1/2,2}(\mathbb{R}/L\mathbb{Z},\mathbb{R}^n). \tag{7}$$

We will use this estimate to show that we will stay in a fixed knot class when proceeding to the limit of a minimal sequence, see Theorem 3.9.

Remark 2.8. For $\alpha = 2$, we see that the Characterisation Theorem 2.5 in combination with the embedding (5) only yields curves of Lipschitz-regularity, implying that for curves with uniform energy bound, we merely obtain convergence in the C^0-topology. This might lead to phenomena like the pull-tight-phenomenon, see [O'H03], resulting in limit curves not necessarily lying in the same knot class. This is why we restrict to the case $\alpha \in (2,3)$.

Lower semicontinuity of $\mathcal{E}_2$ was shown by Freedman, He, and Wang in [FHW94, Lemma 4.2]. Their argument works for any $\alpha \in [2,3)$, for the proof see [GvdM18, Lem. 3.5].

Lemma 2.9 (Lower semicontinuity)**.** *Let $\alpha \in [2,3)$ and consider curves $\gamma, \gamma_i \in C^{0,1}(\mathbb{R}/\mathbb{Z} \to \mathbb{R}^3)$, $i \in \mathbb{N}$, with $|\gamma'| > 0$, $|\gamma_i'| > 0$ a.e. on $\mathbb{R}/\mathbb{Z}$ for all $i \in \mathbb{N}$. Assume that $\gamma_i \to \gamma$ pointwise everywhere on $\mathbb{R}/\mathbb{Z}$ as $i \to \infty$. Then we have*

$$\mathcal{E}_\alpha(\gamma) \le \liminf_{i\to\infty} \mathcal{E}_\alpha(\gamma_i).$$

To apply Theorem 2.1, we will need the functional to be sufficiently smooth. This is why the following regularity results for $\mathcal{E}_\alpha$ and its critical points, by Blatt and Reiter in [BR13], are crucial for us.

Theorem 2.10 (Regularity of $\boldsymbol{\mathcal{E}_\alpha}$ and its critical points [BR13])**.**

(i) The functional $\mathcal{E}_\alpha$ is C^1-differentiable on the space of all embedded and regular curves $\gamma \in W^{\frac{\alpha+1}{2},2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$.

(ii) If an arclength parametrised, embedded curve $\gamma \in W^{\frac{\alpha+1}{2},2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ is a critical point of the linear combination $\mathcal{E}_\alpha + \lambda\mathscr{L}$, then it is C^∞-smooth. Here, $\lambda \in \mathbb{R}$ is an arbitrary parameter.

Remark 2.11. When minimising $\mathcal{E}_\alpha$ under a fixed length constraint, the parameter λ in Theorem 2.10 appears as Lagrange parameter.
Alternatively, that parameter appears if one considers the scaling invariant version $\mathcal{E}^*_\alpha$ as in (2) in the introduction. If we compute the differential of $\mathcal{E}^*_\alpha$ at an injective regular curve $\gamma \in W^{(\alpha+1)/2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$, we arrive at

$$d(\mathcal{E}^*_\alpha)_\gamma = \mathscr{L}(\gamma)^{\alpha-2} d(\mathcal{E}_\alpha)_\gamma + ((\alpha-2)\mathscr{L}(\gamma)^{\alpha-1}\mathcal{E}_\alpha(\gamma))d\mathscr{L}_\gamma.$$

Therefore, Theorem 2.10 applies as well to any arclength parametrised critical point γ of $\mathcal{E}_\alpha^*$, implying that γ is smooth.

Since the functionals $\mathcal{E}_\alpha$ are invariant with respect to isometries, we will assume that the group G which is considered for the principle of symmetric criticality is a subgroup of $\mathrm{Isom}(\mathbb{R}^3)$. In fact, we are able to narrow it down further.

2.3. **Symmetry groups of knots.** To put the principle of symmetric criticality to use, we want the set of symmetric points Σ in Theorem 2.1 to be non-empty. Thus, we will consider the parametrisation $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z})$ of an arbitrary knot Γ and ask for which subroups $G \subset \mathrm{Isom}(\mathbb{R}^3)$ the set of fixed points is non-empty. Since Γ is a compact set, we require G to be compact as well, i.e. we search for compact subgroups $G \subset O(3)$.
If the group was compact but not finite, a copy of the $SO(2)$ would be contained in G. But then, the only possible knot in Σ would be the circle. Since we want to consider arbitrary knot classes, that would imply for any knot class but the class of the unknot, that Σ was empty. Hence, we assume G to be finite. In fact, in [GS85], the possible symmetry groups of knots are narrowed down even further, ruling out the point groups of the $O(3)$ as well.

For the sake of simplicity, in this paper, we only consider finite cyclic groups, i.e. $G = C_n$ with $n \in \mathbb{N}$ being the order of the cyclic group. Recall that $C_n \sim \mathbb{Z}/n\mathbb{Z}$. This group will act via rotations about a fixed axis v on $\mathbb{R}^3$. Since cyclic groups are generated by one element, the group action is sufficiently described when explaining the representative of the generator. The generator of $\mathbb{Z}/n\mathbb{Z}$ may be represented by the rotation with an angle $\frac{2\pi}{n}$ about the axis v. If a knot is invariant with respect to that group action (i.e. rotation), we also say that it is of n-fold symmetry about the axis v.

However, we do not consider a knot as a set of points in $\mathbb{R}^3$, but a parametrisation of that knot. Therefore, we need to add an interior group action on the parameter space to adjust the rotations. Thus, we define

Definition 2.12. Let K be an arbitrary knot class and let $G := \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, be a cyclic group of order n. Then G acts on K via

$$T^k \colon G \times K \to K,\ (g, \gamma) \mapsto \mathrm{rot}\left(\frac{2\pi i(g)}{n}, e_3\right)\gamma\left(t + \frac{ki(g)}{n}\right) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}, \tag{8}$$

with $k \in \mathbb{N}$ being a fixed parameter depending on K, and $i(g) \in 0, \ldots, n-1$ is the representative of $g \in G$.

Here, $\mathrm{rot}(\alpha, v)$ denotes the rotation with an angle α about the axis v. The well-definedness of the group action is discussed in [GvdM18, Rem. 4.5].

Remark 2.13.

(i) Since k will be fixed depending on the knot class, we will omit this parameter in the notation, unless we want to emphasise on how to calculate it (see e.g. Lemma 3.6).

(ii) Fixing the rotational axis to be $v = e_3$ is not too strong of a restriction, since $\mathcal{E}_\alpha$ and $\mathcal{E}_\alpha^*$ are invariant with respect to isometries. Hence, a knot which is invariant to an action as described above, but with a rotation about another axis, may be isometrically mapped such that the rotation is about the e_3-axis.

The group action we defined above is orientation preserving. This implies that for any knotted curve being invariant with respect to the group action of G, the e_3-axis must *not* intersect the knot. This leads to a certain kind of rotational symmetry for knots, the periodicity.

We define an adjusted notion of periodicity for knots based on the definition of Burde and Zieschang, see [BZ85, p. 256] and also [Liv95, Defintition 8.3]. This property does not depend on the parametrisation γ but only on its image $\Gamma = \gamma(\mathbb{R}/\mathbb{Z})$.

Definition 2.14 (Periodicity)**.** We consider a knot Γ and an axis v. If we have

(i) $\Gamma \cap v = \emptyset$ and

(ii) $\operatorname{rot}\left(\frac{2\pi}{q}, v\right)\Gamma = \Gamma$,

then Γ has period q, or is q-periodic.

To show the central result, Theorem 1.1, we will exploit the following result on possible rotational symmetries of non-trivial knots. Most of these facts can also be extracted from Grünbaum and Shephard's classification of possible symmetry groups of knots [GS85] in combination with their characterisation of finite subgroups of $O(3)$ in [GS81]. In [GvdM18], we added information about possible periods of knots. Furthermore, we presented a purely geometrical approach to show the results. We will reproduce that proof here. Note that this theorem solely deals with a knot Γ as an object in $\mathbb{R}^3$, i.e. we do not consider its parametrisation.

Theorem 2.15 (Rotational symmetries of knots, [GvdM18, Lem. 4.12])**.** *If a non-trivially knotted curve Γ is rotationally symmetric about an axis v with angle $\varphi \in (-\pi, \pi]$ and $\Gamma \cap v \neq \emptyset$, then $\varphi = \pi$.*
If Γ has two axes v_1 and v_2 of rotational symmetry with respect to rotation angles $\varphi_1 = \frac{2\pi}{a_1}$, $\varphi_2 = \frac{2\pi}{a_2}$, where $a_1, a_2 \geq 2$, then $v_1 \cap v_2 \neq \emptyset$.
Furthermore, if $v_1 \cap v_2 = \{p\}$ for some $p \in \mathbb{R}^3$, the following holds.

1) For $\varphi_1 \neq \varphi_2$ we have

(i) $v_1 \perp v_2$;

(ii) W.l.o.g. $a_1 \geq 3$ and $a_2 = 2$;

(iii) $v_1 \cap \Gamma = \emptyset$ and $v_2 \cap \Gamma \neq \emptyset$.

2) If $\varphi_1 = \varphi_2$, then we have $\varphi_1 = \varphi_2 = \pi$.

To show this theorem, we will need the subsequent short lemma, which is a slight generalisation of Grünbaum and Shephard [GS85, Lemma 1]. The proof may be found in [GvdM18].

Lemma 2.16 ([GvdM18, Lem. 4.13])**.** *A knot may not have more than one axis of n-fold symmetry with $n \geq 3$. Let $a \in \mathbb{N}$ such that $a \geq 3$. Then a knot cannot have more than one axis of rotational symmetry with rotational angle $2\pi/a$.*

Proof of Thm. 2.15. For the first claim, assume that we had $v \cap \Gamma \neq \emptyset$, where v is the axis about which Γ is rotationally symmetric about angle $\varphi \neq \pi$. But then we have $2\pi/|\varphi| > 2$ arcs entering $x \in v \cap \Gamma$, implying that Γ is not embedded. Hence, if $\varphi \neq \pi$, we need to have $v \cap \Gamma = \emptyset$.

Now we assume that Γ has two different rotational symmetry axes v_1, v_2 with rotational angles $\varphi_1 = 2\pi/a_1$ and $\varphi_2 = 2\pi/a_2$ for some integers $a_1, a_2 \geq 2$. We want to show that $v_1 \cap v_2 \neq \emptyset$ and assume the contrary to arrive at a contradiction. Therefore, we split up the cases depending on a_1 and a_2.
We assume $v_1 \cap v_2 = \emptyset$. If $a_1 = a_2 = 2$, we argue as follows. Consider the two parallel affine planes $H_1, H_2 \subset \mathbb{R}^3$ such that $v_1 \subset H_1$ and $v_2 \subset H_2$, and $d := \operatorname{dist}(H_1, H_2) > 0$. Both planes separate $\mathbb{R}^3$ into half-spaces, which we denote by H_i^- and H_i^+, $i = 1, 2$, in such a way that the set

$$S := H_1^- \cap H_2^+ = \left\{x \in \mathbb{R}^3 \mid \operatorname{dist}(x, H_i) < d,\ i = 1, 2\right\}$$

is not empty. This implies $H_1^+ \cap H_2^- = \emptyset$. Due to the rotational symmetry about both axes v_1 and v_2, we have $\Gamma \cap S \neq \emptyset$. We consider $x_0 \in \Gamma \cap S$ as well as its rotational symmetric copies with respect to the axes v_1 and v_2

$$x_1^i = \operatorname{rot}(\pi, v_i)\, x_0, \quad i = 1, 2.$$

For those we have $x_1^1 \in H_1^+$ and $x_1^2 \in H_2^-$, hence, $x_1^1 \neq x_1^2$. However, if we consider the lengths of the arcs connecting x_0 with x_1^1 and x_1^2, respectively, we have

$$\mathscr{L}\left(a(x_0, x_1^1)\right) = \mathscr{L}(\Gamma)/2 = \mathscr{L}\left(a(x_0, x_1^2)\right), \tag{9}$$

where $a(x, y)$ denotes the arc of Γ linking x to y along the parametrisation. But this implies that $x_1^1 = x_1^2$, which is a contradiction.
Now we consider w.l.o.g. $a_1 \geq 3$ and $a_2 \geq 2$. We may consider the rotation about v_2 with respect to the rotational angle $2\pi/a_2$ as an isometry I in $\mathbb{R}^3$ and therefore have $I(\Gamma) = \Gamma$. Then, Lemma A.1 in the appendix allows us to find new symmetry axes for Γ by rotating v_1 about v_2. That is, all axes

$$v_1^i = \operatorname{rot}\left(\tfrac{2\pi \cdot i}{a_2}, v_2\right) v_1, \quad i = 0, ..., a_2 - 1$$

are axes of rotational symmetry for Γ with rotational angle $\varphi_1 = 2\pi/a_1$. Since $a_2 \geq 2$ and $v_1 \cap v_2 = \emptyset$, there are now at least two different axes of rotational symmetry with angle $\varphi_1 = 2\pi/a_1$, contradicting Lemma 2.16. Thus we have shown that $v_1 \cap v_2 \neq \emptyset$.

To show the third assertion, we assume $v_1 \cap v_2 = \{p\}$ for some $p \in \mathbb{R}^3$. W.l.o.g. we may set $p = 0$. The corresponding rotational angles for the axes v_i, $i = 1, 2$ are $\varphi_i = 2\pi/a_i$ for some integers $a_i \geq 2$, $i = 1, 2$.
To prove Part 1) we take $a_1 \neq a_2$ and assume firstly that $a_1, a_2 \geq 3$ to arrive at a contradiction, proving Part 1)(ii).
We have $\varphi_i \in (0, \pi)$, $i = 1, 2$. The first assertion of the theorem implies that Γ is disjoint from both axes v_1 and v_2. As for the proof of the second assertion of the theorem, we may construct copies of v_1 by rotating it around v_2. Then Γ is rotationally symmetric with respect to the axes

$$v_1^i = \operatorname{rot}\left(\tfrac{2\pi i}{a_2}, v_2\right) v_1, \quad i = 0, ..., a_2 - 1,$$

where $v_1^0 = v_1$. Hence, there are at least two axes with rotational angle $\varphi_1 = 2\pi/a_1 \in (0, \pi)$. Since $\varphi \in (0, \pi)$, those axes do not coincide. But this contradicts Lemma 2.16. Thus, we have either $a_1 = 2$ and $a_2 \geq 3$, or $a_2 = 2$ and $a_1 \geq 3$, which proves Part 1)(ii). Furthermore, the presented argument implies Part 2).

To continue, we assume w.l.o.g. that $a_1 \geq 3$, $a_2 = 2$.
In view of proving Part 1)(i), we take into account the angle $\sphericalangle(v_1, v_2) =: \alpha \in (0, \pi/2]$ and assume that $0 < \alpha < \pi/2$. Then we may construct a second rotational symmetry axis for Γ with rotational angle $\varphi_1 = 2\pi/a_1$,

$$v_1^1 = \operatorname{rot}(\pi, v_2)\, v_1$$

and we have $v_1^1 \neq v_1$. This implies the existence of two distinct axes of rotational symmetry for Γ with rotational angle $\varphi = 2\pi/a_1 \in (0, \pi)$, contradicting Lemma 2.16 again. Therefore, we have $v_1 \perp v_2$, which is Part 1)(i).

The first assertion of the theorem already implies that $\Gamma \cap v_1 = \emptyset$ because $\varphi_1 \in (0, \pi)$. Thus, it suffices to show $v_2 \cap \Gamma \neq \emptyset$ to finally establish Part 1)(iii).
We assume $v_2 \cap \Gamma = \emptyset$. This implies that both v_1 and v_2 are disjoint from Γ, hence both are periodicity axes, see Definition 2.14.
We define the plane $H := v_1^{\perp}$. Due to Part 1)(i), it contains v_2, and we deduce $H \cap \Gamma \neq \emptyset$ because of the periodicity about v_2. Now we fix a point $x_0 \in H \cap \Gamma$ and consider its orbit under the action of the cyclic group C_{a_1}, induced by the rotation $\operatorname{rot}(\varphi_1, v_1)$, with $\varphi_1 = 2\pi/a_1$:

$$O_{v_1} := \{x_0, ..., x_{a_1 - 1}\} \subset H \cap \Gamma$$

The x_i are labelled according to the arclength parameters, i.e. their index is increasing along the parametrisation on Γ. That is, $x_i = \Gamma(s_i)$ for $i = 0, ..., a_1 - 1$ such that $0 \leq s_0 < s_1 < ... < s_{a_1 - 1} < L = \mathscr{L}(\Gamma)$. Furthermore, there exists $k \in \mathbb{N}$ with $\gcd(k, a_1) = 1$ and unique modulo a_1, such that

$$x_i = \operatorname{rot}\left(\tfrac{2\pi k i}{a_1}, v_1\right) x_0, \quad i = 0, ..., a_1 - 1. \tag{10}$$

To justify this equation, observe firstly that the periodicity of Γ implies that the subarcs on Γ connecting consecutive x_i have equal length, i.e.

$$\mathscr{L}(a(x_i, x_{i+1})) = s_{i+1} - s_i = \frac{L}{a_1} \quad \text{for all } i = 0, ..., a_1 - 1. \tag{11}$$

For the points in O_{v_1}, we know as well that they lie on a circle $C = \partial B_r(0)$, $r = \text{dist}(x_0, 0)$, which is contained in H. We reorder the points in O_{v_1} according to their position on C, proceeding counterclockwise. We set $y_0 = x_0$ and arrive at a set $Y = \{y_0, \dots, y_{a_1-1}\}$ with indices labelled such that

$$y_j = \text{rot}\left(\tfrac{2\pi}{a_1}, v_1\right) y_{j-1} \quad \text{for all } j \in \{0, \dots, a_1 - 1\}.$$

Since the sets Y and O_{v_1} coincide, there is an integer $m \in \{1, \dots, a_1 - 1\}$ such that $y_1 = \Gamma(s_m) = x_m$. Thus, using (11), the oriented subarc of Γ starting at $x_0 = y_0$ and ending at $y_1 = x_m$ has length $s_m - s_0 = mL/a_1$. The same holds true for every oriented subarc $a(y_j, y_{j+1})$, $j = 1, \dots, a_1 - 1$, so that we arrive at

$$x_{jm} = \Gamma(s_{jm}) = y_j = \text{rot}\left(\tfrac{2\pi j}{a_1}, v_1\right) y_0 = \text{rot}\left(\tfrac{2\pi j}{a_1}, v_1\right) x_0, \quad j = 1, \dots, a_1 - 1. \tag{12}$$

Note that for the indices, we use the notation modulo a_1, i.e. if $jm \notin \{0, \dots a_1 - 1\}$, we replace it by $[jm \mod a_1]$. Such an $m \in \mathbb{Z}/a_1\mathbb{Z}$ does exist and it holds $\gcd(m, a_1) = 1$. If these numbers were not relatively prime, then their least common multiple $\text{lcm}(m, a_1)$ could be written as $\text{lcm}(m, a_1) = \frac{ma_1}{\gcd(m,a_1)} =: l \cdot m$, where $1 < l < a_1 - 1$ is a positive integer. This would imply $l \cdot m = 0 \mod a_1$, and therefore $x_{l \cdot m} = \Gamma(s_0) = y_l$ in (12). But then the remaining points $y_{l+1}, \dots, y_{a_1-1}$ would not be contained in the orbit O_{v_1}, which is a contradiction to $Y = O_{v_1}$. Hence, we have $\gcd(m, a_1) = 1$, implying the existence of an inverse k of m in $\mathbb{Z}a_1\mathbb{Z}$, i.e. $km \equiv_{a_1}= 1$. Using (12), we obtain

$$x_{[j \cdot m]} = \text{rot}\left(\tfrac{2\pi j \cdot m \cdot k}{a_1}, v_1\right) x_0 \quad \text{for all } j = 1, \dots, a_1 - 1.$$

Given any $i \in \{1, \dots, a_1 - 1\}$ we choose $j := i \cdot k$ to finally obtain (10).

Next, we use the 2-periodicity of Γ around v_2 to construct further points lying on the circle C. There exist $\overline{x}_i = \Gamma(\bar{s}_i) \in \Gamma \cap H$ such that

$$\overline{x}_i = \text{rot}\,(\pi, v_2)\, x_i, \quad i = 0, ..., a_1 - 1.$$

We have

$$\mathscr{L}(a(x_i, \overline{x_i})) = \frac{L}{2} \quad \text{for all } i = 0, \dots, a_1 - 1.$$

By a short calculation, we arrive at

$$\overline{x}_i = \text{rot}\left(\frac{2\pi \cdot k(-i)}{a_1}, v_1\right) \overline{x}_0, \quad \text{for all } i = 0, ..., a_1 - 1. \tag{13}$$

We have $x_i, \overline{x}_i \in C$ for all $i = 0, ..., a_1 - 1$. We are going to determine the order of these points on C, and consider first only the x_i. Due to the a_1-periodicity, there is a unique successor x_{i_k} of x_0 (counterclockwise) on C which has a distance of $2\pi r/a_1$ to x_0 on C and is defined by (10):

$$x_{i_k} = \text{rot}\left(\tfrac{2\pi \cdot k i_k}{a_1}, v_1\right) x_0 = \text{rot}\left(\tfrac{2\pi \cdot 1}{a_1}, v_1\right) x_0$$

which is equivalent to $ki_k \equiv_{a_1} 1$. Thus i_k is the unique inverse of k in $\mathbb{Z}/a_1\mathbb{Z}$ which exists, since $\gcd(k, a_1) = 1$. Repeating this argument for the other successors, we arrive at the order

$$x_0 - x_{i_k} - x_{2i_k} - \cdots - x_{(a_1-1)i_k}$$

on C counterclockwise. In an analogous way, by using (13), we arrive at the following counterclockwise order for the $\overline{x}_i$, $i = 0, ..., a_1 - 1$ on circle C:

$$\overline{x}_0 - \overline{x}_{(a_1-1)i_k} - \overline{x}_{(a_1-2)i_k} - \cdots - \overline{x}_{i_k}.$$

We further have

$$\mathscr{L}\left(a_C\left(x_i, x_{i+l i_k}\right)\right)=\frac{2\pi r l}{a_1}=\mathscr{L}\left(a_C\left(\overline{x}_i, \overline{x}_{i-l i_k}\right)\right), \tag{14}$$

where $a_C(x,y)$ is the circular subarc of C connecting x and y counterclockwise. Now we are going to determine the order of both sets of points combined on C. To this extent, we consider a pair $(x_j, \overline{x}_j)$ such that x_j minimises $\operatorname{dist}(x_k, v_2 \cap S)$ for $k=0,...,a_1-1$. W.l.o.g. let this be $j=0$ and assume further w.l.o.g. that $a_C(x_0,\overline{x}_0) \leq a_C(\overline{x}_0, x_0)$. Now we use

$$\beta := \mathscr{L}\left(a_S\left(x_0, \overline{x}_0\right)\right) < \frac{2\pi r}{a_1}, \tag{15}$$

which may be deduced by (assuming the contrary and) calculating the length between x_0 and x_{i_k} and the fact that $\gcd(k, a_1)=1$.
Combining (15) with (14) leads to the counterclockwise ordered combined chain

$$x_0 - \overline{x}_0 - x_{i_k} - \overline{x}_{(a_1-1)i_k} - x_{2i_k} - \overline{x}_{(a_1-2)i_k} - \cdots - x_{(a_1-1)i_k} - \overline{x}_{i_k}, \tag{16}$$

since there are no $x_i, \overline{x}_i$ in the circular arc $a_C(x_0, \overline{x}_0) \subset C$ because of the minimality of x_0, and the possible successors of x_0 and $\overline{x}_0$, respectively, are x_{i_k} and $\overline{x}_{(a_1-1)i_k}$. Equation (14) delivers that x_{i_k} has to appear before $\overline{x}_{(a_1-1)i_k}$. From there one can continue to form the whole chain (16).

The a_1-periodicity now gives us information on the shorter subarcs $a(p,q) \subset \Gamma$ connecting consecutive points p and q on the combined chain (16):

$$a\left(x_{l i_k}, \overline{x}_{(a_1-l)i_k}\right) = \operatorname{rot}\left(\frac{2\pi l}{a_1}, v_1\right) a\left(x_0, \overline{x}_0\right) \quad \text{for all} \quad l \in \mathbb{N}.$$

In particular, the lengths of these arcs coincide. But this leads to

$$\mathscr{L}\left(a\left(x_{i_k}, \overline{x}_{i_k}\right)\right) = |s_{k_k} - \bar{s}_{i_k}| = \frac{L}{2} = |s_0 - \bar{s}_0| = \mathscr{L}\left(a\left(x_0, \overline{x}_0\right)\right) = \mathscr{L}\left(a\left(x_{i_k}, \overline{x}_{(a_1-1)i_k}\right)\right),$$

and therefore $1 = a_1 - 1$, which is not the case as $a_1 \geq 3$. This final contradiction leads us to $v_2 \cap \Gamma \neq \emptyset$. This establishes Part 1)(iii) and concludes the whole proof. □

3. Symmetric critical points

The aim of this section is to show Theorem 1.1.

We first establish an open subset of the Banach space $W^{\alpha+1/2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ as the Banach manifold on which Palais' principle of symmetric criticality is applicable.

Lemma 3.1 ([GvdM18, Lem. 4.1]). *For any knot class K and for any $\alpha \in (2,3)$ the set*

$$\mathscr{C}_K := \{\gamma \in W^{\alpha+1/2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3) : |\gamma'| > 0, [\gamma] = K\}$$

is an open subset of $W^{\alpha+1/2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$.

Here, $[\gamma]$ denotes the knot class represented by γ. In particular, $[\gamma]=K$ implies automatically that $\gamma|_{[0,1)}$ is injective. The proof of the lemma may be found in [GvdM18, Lem. 4.1]. Since open subsets of Banach spaces are Banach manifolds, we immediately arrive at

Corollary 3.2 ([GvdM18, Cor. 4.2]). *The set $\mathscr{C}_K$ defined in Lemma 3.1 is a smooth manifold modeled over the Banach space $\mathscr{B} := W^{\alpha+1/2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$.*

Since the set $\mathscr{C}_K$ is a subset of an arbitrary knot class K, the group action defined in Lemma 2.12 is well-defined for elements of $\mathscr{C}_K$ as well. This representation meets the requirements of Definition 2.2, which gives us

Corollary 3.3 ([GvdM18, Lem. 4.4]). *Let $\mathscr{C}_K$ be defined as in Lemma 3.1. Let further $G = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, be a finite cyclic group of order n, which acts on $\mathscr{C}_K$ via the representation T*

$$T^k \colon G \times K \to K,\ (g,\gamma) \mapsto rot\left(\frac{2\pi i(g)}{n}, e_3\right)\gamma\left(t + \frac{ki(g)}{n}\right) \quad \textit{for all } t \in \mathbb{R}/\mathbb{Z},$$

with the parameters as defined in Lemma 2.12. Then $\mathscr{C}_K$ is a smooth G-Banach manifold.

The set of fixed points in a knot class K with respect to a finite group $G = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, will be referred to as *symmetric* knot class K_G and we write

$$\mathscr{C}_K^n = \left\{\gamma \in \mathscr{C}_K \,\middle|\, T^k(g,\gamma) = \gamma \text{ for all } g \in \mathbb{Z}/n\mathbb{Z}\right\}. \tag{17}$$

We therefore arrive at a knot theoretic version of the principle of symmetric criticality, see also [Gil18].

Theorem 3.4. *Let $G = \mathbb{Z}/n\mathbb{Z}$ be a finite cyclic group of order $n \in \mathbb{N}$ and let $\mathscr{C}_K$ be defined as in Lemma 3.1. Then the set $\mathscr{C}_K^n$ is a smooth submanifold of $\mathscr{C}_K$ and every critical point of $\mathcal{E}_\alpha^*$ in $\mathscr{C}_K^n$ is a critical point of $\mathcal{E}_\alpha^*$ in $\mathscr{C}_K$.*

In general, of course, not for every combination of knot class K and order of a cyclic group n, we do have a non-empty symmetric knot class K_G, $G = \mathbb{Z}/n\mathbb{Z}$. However, for the knot classes of the *torus knots*, we will show the existence of non-empty symmetric knot classes.

Definition 3.5. Let $a, b \in \mathbb{Z} \setminus \{0, \pm 1\}$ be relatively prime and consider a fixed $\rho \in (0,1)$. We define the smooth curve

$$\mathfrak{t}_\rho(t) := \mathrm{rot}\,(2\pi a t, e_3)\begin{pmatrix} 1 + \rho\cos(2\pi b t) \\ 0 \\ \rho \sin(2\pi b t)\end{pmatrix} = \begin{pmatrix} \cos(2\pi a t)(1 + \rho\cos(2\pi b t)) \\ \sin(2\pi a t)(1 + \rho\cos(2\pi b t)) \\ \rho\sin(2\pi b t)\end{pmatrix}, \quad t \in \mathbb{R}/\mathbb{Z}. \tag{18}$$

The image of $\mathfrak{t}_\rho$ is called an (a,b)-torus knot. Its knot class $\mathcal{T}(a,b) = [\mathfrak{t}_\rho]$ is called (a,b)-torus knot class.

By [BZ85, Theorem 3.29], we know that $\mathcal{T}(a,b) = \mathcal{T}(b,a) = \mathcal{T}(-a,-b) = \mathcal{T}(-b,-a)$. Using the curve defined in (18), we will show that those symmetric knot classes, induced by cyclic groups of an order which divides either a or b, are non-empty.

Lemma 3.6 ([GvdM18, Lem. 4.8]). *Let $\alpha \in (2,3)$, $a, b \in \mathbb{Z}\setminus\{0,\pm1\}$ relatively prime, let $m \in \mathbb{N}$, $m > 1$, divide a or b, and let $G = \mathbb{Z}/m\mathbb{Z}$. Then the following is true: For any $k \in \mathbb{Z} \setminus \{0\}$ with*

$$\begin{cases} ak + 1 \equiv_m 0 & \text{if } m|b \\ bk + 1 \equiv_m 0 & \text{if } m|a \end{cases} \tag{19}$$

one has a nonempty G-symmetric subset $\mathscr{C}_K^m$ as in (17).

This lemma is shown by simply calculating for the smooth curve in (18), that

$$T^k(g, \mathfrak{t}_\rho)(t) = \mathfrak{t}_\rho$$

for all $t \in \mathbb{R}/\mathbb{Z}$, see [GvdM18, Lem. 4.8]. Note that a $k \in \mathbb{Z}/m\mathbb{Z}$ as in (19) does always exist since for $m|b$, a and m are relatively prime. For $m|a$, we have b and m relatively prime. Then we immediately deduce the existence of such a k. This is the same parameter k as in the definition of the group action (8), see also Remark 2.13.

Having shown that, for torus knot classes, we have at least two non-empty symmetric knot classes, we will proceed by showing the existence of critical points of $\mathcal{E}_\alpha^*$ in each of these sets. This will be done by showing the existence of a minimiser in a non-empty symmetric knot class, using the direct method of variational calculus. This existence proof is valid for arbitrary non-empty symmetric knot classes.

For technical reasons, we will have to reparametrise to arclength in the existence proof. Therefore, we need to understand what kind of symmetry the arclength parametrisation inherits from a symmetric curve.

Lemma 3.7 ([GvdM18, Lem. 4.6]). *Let $m, k \in \mathbb{Z}$, $G = \mathbb{Z}/m\mathbb{Z}$, and $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be an absolutely continuous curve with $|\gamma'| > 0$ a.e. and with length $\mathscr{L}(\gamma) = L \in (0, \infty)$, such that for $g \in G$ we have*

$$T^k\colon G \times K \to K,\ (g, \gamma) \mapsto rot\left(\frac{2\pi i(g)}{m}, e_3\right)\gamma\left(t + \frac{ki(g)}{m}\right) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z},$$

with $i(g) \in \{0, \dots, m-1\}$ depending on $g \in G$ and $k \in \mathbb{Z}/m\mathbb{Z}$ depending on the knot class K as in (8). Then the corresponding arclength parametrisation $\tilde{\gamma} \in C^{0,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^3)$ satisfies

$$rot\left(\frac{2\pi i(g)}{m}, e_3\right)\tilde{\gamma}\left(s + \frac{ki(g)L}{m}\right) = \tilde{\gamma}(s) \ \text{ for all } \ s \in [0, L). \tag{20}$$

Since arclength reparametrisations of curves in $W^{\alpha+1/2,2}$ inherit the same regularity, we immediately infer

Corollary 3.8 ([GvdM18, Cor. 4.7]). *Let $m, k \in \mathbb{Z}$ and $G = \mathbb{Z}/m\mathbb{Z}$ and let K be any knot class. Let further $\mathscr{C}_K$ be the Banach manifold defined in Lemma 3.1 with G-symmetric subset $\mathscr{C}_K^m$ with respect to the group action given by T as defined in (8). Then, if $\gamma \in \mathscr{C}_K^m$ has length $\mathscr{L}(\gamma) = 1$, its arclength parametrisation $\tilde{\gamma} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is contained in $\mathscr{C}_K^m$ as well.*

Now we are ready to prove the existence of symmetric minimizers for the scaled O'Hara energy defined in (2) in the introduction. Notice that since $\mathcal{E}_\alpha$ is continuously differentiable on the space of regular curves, so is $\mathcal{E}_\alpha^*$, since the length functional $\mathscr{L}$ is continuously differentiable, even in the class of regular curves of class $W^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$, and hence in particular on the Banach manifold $\mathscr{C}_K$ for any knot class K. Furthermore, due to that regularity of $\mathscr{L}$, all properties introduced in Section 2.2 for $\mathcal{E}_\alpha$ hold for $\mathcal{E}_\alpha^*$ as well.

Theorem 3.9 ([GvdM18, Thm. 4.9]). *Let $\alpha \in (2, 3)$, and consider a knot class K. Let $G = \mathbb{Z}/m\mathbb{Z}$, $m \in \mathbb{N}$, be a finite cyclic group acting on $\mathscr{C}_K$ via the action defined in (8).*[1] *Assume that the symmetric knot class $\mathscr{C}_K^m$ is non-empty. Then there exists an arclength parametrised curve $\gamma \in \mathscr{C}_K^m$ such that*

$$\mathcal{E}_\alpha^*(\gamma) = \inf_{\mathscr{C}_K^m} \mathcal{E}_\alpha^*. \tag{21}$$

We can convince ourselves, considering Theorem 3.4, that these symmetric minimizing torus knots are all critical for the scaled energy functional $\mathcal{E}_\alpha^*$ on all of $\mathscr{C}_K$.

Corollary 3.10. *Any of the minimizing knots $\gamma \in \mathscr{C}_K^m$ found in Theorem 3.9 are critical points of the scaled energy $\mathcal{E}_\alpha^* = \mathscr{L}^{\alpha-2}E_\alpha$ and therefore of class $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.*

In order to show that there are *at least two* $\mathcal{E}_\alpha^*$-critical knots in every non-trivial torus knot class $\mathcal{T}(a, b)$, we consider this classical result from Burde and Zieschang concerning the periods of torus knots. We already know by Lemma 3.6, that the divisors of a and b are possible periods of torus knots. The following proposition shows that these are the only possible periods.

Proposition 3.11 ([BZ85, Prop. 14.27]). *If $q \in \mathbb{N}_{\geq 2}$ is a period of a curve $\gamma \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with $[\gamma] = \mathcal{T}(a, b)$ for relatively prime integers $a, b \in \mathbb{Z} \setminus \{0, \pm 1\}$, then $q|a$ or $q|b$. Conversely, if $q \in \mathbb{N}_{\geq 2}$ divides a or b, then there is a representative $\gamma \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ such that q is a period of γ.*

This result will allow us to show the existence of at least two $\mathcal{E}_\alpha^*$-critical knots in every torus knot class $\mathcal{T}(a, b)$, which is our central result, Theorem 1.1 mentioned in the introduction.

Theorem. *Let $a, b \in \mathbb{Z} \setminus \{0, \pm 1\}$ be relatively prime and let $\alpha \in (2, 3)$. Then there exist at least two different arclength parametrised, embedded curves $\gamma_1, \gamma_2 \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ that are critical for the energy*

$$\mathcal{E}_\alpha^*(\gamma) := \mathscr{L}^{\alpha-2}(\gamma)\mathcal{E}_\alpha(\gamma)$$

in the torus knot class $\mathcal{T}(a, b)$.

[1]Note that the parameter k of the group action T^k is uniquely defined by the knot class.

Proof of Theorem 1.1. We consider a torus knot class $K = \mathcal{T}(a,b)$, with $a,b \in \mathbb{Z} \setminus \{0,\pm 1\}$ relatively prime, and $m \in \mathbb{N}_{\geq 2}$ dividing a or b. By Lemma 3.6, each symmetric knot class $\mathscr{C}_K^m$ is non-empty. By Theorems 3.9 and 3.4 in connection with Corollary 3.10, we arrive at at least one arclength parametrised curve

$$\gamma^m \in \mathscr{C}_K^m \cap C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$$

that is $\mathcal{E}_\alpha^*$-critical in $\mathscr{C}_K$. This curve of course satisfies

$$\operatorname{rot}\left(\frac{2\pi i(g)}{m}, e_3\right)\gamma^m\left(t + \frac{k_m i(g)}{m}\right) = \gamma^m(t) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}.$$

Note that we wrote k_m for the parameter of the group action as defined in (8), because it depends on the group and therefore in this case on the order of the cyclic group, which is m.

We will show that there are at least two different of such critical points by showing that

$$\mathscr{C}_K^e \cap \mathscr{C}_K^f = \emptyset \quad \text{for all } e|a,\ f|b.$$

To do so, assume there was an element $\eta \in \mathscr{C}_K^e \cap \mathscr{C}_K^f$. Now we consider the groups inducing the symmetric knot classes, i.e. the cyclic groups C_e and C_f. Since we fixed the rotational axis for group actions to be e_3, formally, we have assume that there is $\eta \in \mathscr{C}_K^e$ and there exists and isometry I such that $I(\eta) \in \mathscr{C}_K^f$. But this is equivalent to η having an e-fold rotational symmetry about the e_3-axis and and f-rotational symmetry about the $I^{-1}(e_3)$-axis. So now, we consider these two axes.
If the actions of C_e and C_f were induced around different rotational axes, this would imply by Theorem 2.15, that one rotation would have to be about an angle π and the corresponding axis would intersect the knot. This would imply that the orientation is reversed, which is a contradiction to the definition of the group operation of a cyclic group, which preserves orientation, see Lemma 2.12. Hence, both rotations have to be induced around the same axis $v = e_3$ and the isometry I may only change the direction of rotation. If I does not change the direction of rotation, then for any $g \in C_e$, $h \in C_f$ and for all $t \in \mathbb{R}/\mathbb{Z}$ we have

$$\eta(t) = T_h(T_g(\eta(t))) = \operatorname{rot}\left(\frac{2\pi i(g)}{e}, v\right)\operatorname{rot}\left(\frac{2\pi i(h)}{f}, v\right)\eta\left(t + \frac{k_e i(g)}{e} + \frac{k_f i(h)}{f}\right), \tag{22}$$

where $i(g) \in \{0,\ldots,e-1\}$ and $i(h) \in \{0,\ldots,f-1\}$. This implies, particularly in the case $i(g) = i(h) = 1$, that

$$\operatorname{rot}\left(2\pi\frac{f+e}{ef}, v\right)\eta\left(t + \frac{k_e f + k_f e)}{ef}\right) = \eta(t) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}. \tag{23}$$

If the rotations are induced in opposite directions of rotation, then we assume w.l.o.g.

$$\eta(t) = \operatorname{rot}\left(\frac{2\pi}{e}, v\right)\eta\left(t + \frac{k_e}{e}\right) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}.$$

Also, the isometry $I\colon \mathbb{R}^3 \to \mathbb{R}^3$ has to be such that the function $\tilde{\eta} := I \circ \eta$ fulfills

$$\tilde{\eta}(t) = \operatorname{rot}\left(\frac{2\pi}{f}, v\right)\tilde{\eta}(t) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}.$$

The isometry ϕ itself therefore has to be

$$\phi(x) = \operatorname{rot}(\pi, w)\, x + \xi,$$

with w orthogonal to v and $\xi \in v$ such that the rotational axis of C_f is mapped onto the one of C_e. Thus, we have

$$\tilde{\eta}(t) = \operatorname{rot}(\pi, w)\eta(t) + \xi \quad \text{for all } t \in \mathbb{R}/\mathbb{Z},$$

and $\tilde{\eta}$ has a rotational symmetry in the same direction of rotation as η. We arrive at

$$\eta(t) = \operatorname{rot}(\pi, w)\tilde{\eta}(t) + \xi = \operatorname{rot}(\pi, w)\operatorname{rot}\left(\frac{2\pi}{f}, v\right)\tilde{\eta}(t) + \xi$$

$$= \operatorname{rot}\left(\frac{2\pi}{f}, v\right) \operatorname{rot}(\pi, w)\tilde{\eta}(t) + \xi = \operatorname{rot}\left(\frac{2\pi}{f}, v\right) (\operatorname{rot}(\pi, w)\tilde{\eta}(t) + \xi)$$
$$= \operatorname{rot}\left(\frac{2\pi}{f}, v\right) \eta(t)$$

because we have $\xi \in v$ and due to Lemma A.1.

Hence, also in this case, we arrive at (22) and thus (23), yielding

$$\operatorname{rot}\left(\frac{2\pi(f+e)}{ef}, v\right) \eta\left(t + \frac{k_e f + k_f e}{ef}\right) = \eta(t) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}.$$

This implies that $\operatorname{rot}\left(\frac{2\pi(f+e)}{ef}, v\right) M = M$, where $M = \eta(\mathbb{R}/\mathbb{Z})$. Since $(f+e)$ and ef are relatively prime, there is some $s \in \mathbb{N}$ such that $(e+f)s \equiv 1 \mod ef$. Therefore, we obtain

$$M = \operatorname{rot}\left(\frac{2\pi(f+e)}{ef}, v\right)^s M = \operatorname{rot}\left(\frac{2\pi s(f+e)}{ef}, v\right) M = \operatorname{rot}\left(\frac{2\pi}{ef}, v\right) M.$$

That equation yields M being ef-periodic. However, by Lemma 3.11, we know that the only periods of an (a,b)-torus knot are the divisors of a and b. This implies $(ef)|a$ or $(ef)|b$, which is a contradiction to $\gcd(a,b) = 1$. Hence, $\mathscr{C}_K^e$ and $\mathscr{C}_K^f$ are disjoint and the theorem is proven. □

Appendix A.

In the proof of Theorem 2.15 we used the following simple result concerning images of rotationally symmetric sets under isometries of $\mathbb{R}^3$. A proof may be found in [GvdM18].

Lemma A.1. *Let $w \in \mathbb{S}^2$, $\alpha \in \mathbb{R}$, and $I : \mathbb{R}^3 \to \mathbb{R}^3$ be an orientation preserving isometry of $\mathbb{R}^3$ with $I(w) \neq 0$. Then for any set $M \subset \mathbb{R}^3$ with*

$$rot(\alpha, \mathbb{R}w)\, M = M \tag{24}$$

one has

$$rot(\alpha, I(\mathbb{R}w))\, I(M) = I(M), \tag{25}$$

where similarly as before $rot(\alpha, \tilde{w})$ stands for the rotation about the affine line $\tilde{w} = \mathbb{R}e_w + d \subset \mathbb{R}^3$ for some $e_w \in \mathbb{S}^2$ and $d \in \mathbb{R}^3$ with rotational angle $\alpha \in \mathbb{R}$.

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Institut für Mathematik,
RWTH Aachen University,
Templergraben 55,
D-52062 Aachen, Germany
Email address: gilsbach@instmath.rwth-aachen.de