Construction and stability analysis of one-peak symmetric stationary solutions to the Schnakenberg model with heterogeneity

首都大学東京理学研究科　石井　裕太
Yuta Ishii
Department of Mathematical Sciences,
Tokyo Metropolitan University
首都大学東京理学研究科　倉田和浩
Kazuhiro Kurata
Department of Mathematical Sciences,
Tokyo Metropolitan University

1 Introduction and Main Results

In this paper, based on a recent work [5], we present our study on the existence and the linear stability of stationary solutions for the following Schnakenberg model:

$$u_t = \varepsilon^2 u_{xx} + d\varepsilon - u + g(x)u^2 v, \quad x \in (-1,1),\ t>0, \tag{1}$$

$$\varepsilon v_t = D v_{xx} + \tfrac{1}{2} - \tfrac{c}{\varepsilon} g(x)u^2 v, \quad x \in (-1,1),\ t>0, \tag{2}$$

$$u_x(\pm 1) = v_x(\pm 1) = 0, \tag{3}$$

where d and c are positive constants, ε^2 and D are positive diffusion coefficients. $u(x,t)$ and $v(x,t)$ represent the density of two chemical substances. Here, $g(x)$ is a positive function, which represents the reaction speed of the chemical reaction at $x \in (-1,1)$ and may vary on the location x, for example by the effect of temperature.

Our system (1)-(3) is obtained from the original Schnakenberg model:

$$U_t = D_1 U_{xx} + a - U + g(x)U^2 V, \quad x \in (-1,1),\ t>0,$$

$$V_t = D_2 V_{xx} + b - g(x)U^2 V, \quad x \in (-1,1),\ t>0,$$

$$U_x(\pm 1) = V_x(\pm 1) = 0$$

by using the spacial scaling: $c = \frac{1}{4b^2}, d = ac^{-\frac{1}{2}} = 2ab$, and

$$U = \frac{1}{2b\varepsilon}u, \quad V = 2b\varepsilon v, \quad D_1 = \varepsilon^2, \quad D_2 = \frac{D}{\varepsilon}.$$

Especially, we treat sufficiently small ε and a fixed D, i.e. the ratio of diffusion coefficients $\frac{D}{\varepsilon^2}$ is large (cf. Turing's diffusion-driven instability). Moreover, (2) means that v reacts very rapidly than u in our model.

Inspired by the work of Iron, Wei and Winter [3] which studied in the case $d = 0$ and $g(x) = 1$, the purpose of our study is to investigate the effect of symmetric heterogeneity $g(x)$, namely $g(x) = g(-x)$, on the linear stability of stationary solutions for (1)-(3) rigorously. To state our main results, we prepare some notations. Let w_0 be the unique solution of

$$w_0'' - w_0 + w_0^2 = 0, \quad x \in \mathbb{R},$$
$$w_0 > 0,\ w_0(0) = \max_{\mathbb{R}} w_0,\ \lim_{|y|\to\infty} w_0(y) = 0.$$

It is known that w_0 is unique and can be written explicitly $w_0(y) = \frac{3}{2}(\cosh \frac{y}{2})^{-2}$.
Let w be the unique solution of the following problem:

$$w'' - w + g(0)w^2 = 0, \quad x \in \mathbb{R},$$
$$w > 0,\ w(0) = \max_{\mathbb{R}} w,\ \lim_{|y|\to\infty} w(y) = 0.$$

Then it is easy to see $w(y) = g(0)^{-1}w_0(y)$. Let χ be a cut-off function:

$$\chi \in C_0^\infty(\mathbb{R})\ ,\ 0 \le \chi \le 1\ ,\ \chi(x) = \begin{cases} 1, & |x| < \frac{1}{4}, \\ 0, & |x| > \frac{1}{2}. \end{cases}$$

Define symmetric function spaces: for each $a \in (0, \infty)$,

$$L_s^2(-a,a) := \big\{ u \in L^2(-a,a) \;\big|\; u(x) = u(-x) \big\},$$

$$H_s^2(-a,a) := \big\{ u \in H^2(-a,a) \;\big|\; u(x) = u(-x), u'(\pm a) = 0 \big\}.$$

Let $I := (-1,1)$ and $I_\varepsilon := \big(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\big)$ for $\varepsilon > 0$. We also use the following notation for the rescaling: for a function $u : I \to \mathbb{R}$, define $\overline{u}(y) := u(\varepsilon y)$ $(y \in I_\varepsilon)$.

The steady-state problem for (1)$\sim$(3) is the following:

$$0 \ = \varepsilon^2 u'' + d\varepsilon - u + g(x)u^2 v, \quad x \in (-1,1), \tag{4}$$
$$0 \ = Dv'' + \tfrac{1}{2} - \tfrac{c}{\varepsilon} g(x)u^2 v, \quad x \in (-1,1), \tag{5}$$
$$u'(\pm 1) = v'(\pm 1) = 0. \tag{6}$$

First, we state the existence of a one-peak solution.

Theorem 1 *Fix $D < +\infty$ arbitrarily. Assume that $g(x)$ is positive, Lipschitz continuous and satisfies $g(x) = g(-x)$. Then, there exists a sufficiently small $\varepsilon_1 > 0$ such that, for $0 < \varepsilon < \varepsilon_1$, (4)$\sim$ (6) admits a symmetric one-peak solution $(u_\varepsilon(x), v_\varepsilon(x)) \in H_s^2(I_\varepsilon)\times H_s^2(I_\varepsilon)$, where $u_\varepsilon(x)$ concentrates at $x = 0$. Moreover, $u_\varepsilon(x)$ takes the following asymptotic form:*

$$u_\varepsilon(x) = w_\varepsilon(x) + \phi_\varepsilon(x), \tag{7}$$

where

$$w_\varepsilon(x) := \frac{1}{\xi_0} w\Big(\frac{x}{\varepsilon}\Big)\chi(x), \quad \xi_0 := cg(0)\int_{\mathbb{R}} w^2(y)dy$$

and $\overline{\phi_\varepsilon} \in H^2_s(I_\varepsilon)$ *such that*

$$||\overline{\phi_\varepsilon}||_{H^2(I_\varepsilon)} \le C\sqrt{\varepsilon} \tag{8}$$

holds for some constant $C > 0$ *independent of* ε*. Also,* $v_\varepsilon(x)$ *satisfies*

$$v_\varepsilon(0) = \xi_0 + O(\sqrt{\varepsilon}) \text{ as } \varepsilon \to 0. \tag{9}$$

Moreover, there exists $v_0 \in H^1(I)$ *such that* $v_\varepsilon \rightharpoonup v_0$ *weakly in* $H^1(I)$*, where* v_0 *satisfies*

$$\begin{aligned} -Dv_0''(x) &= \tfrac{1}{2} - \delta_0(x), \quad x \in (-1,1), \\ v_0(0) &= \xi_0, \quad v_0'(\pm 1) = 0 \end{aligned}$$

and $\delta_0(x)$ *is the Dirac's delta function.*

Next, we study the linear stability of the solutions $(u_\varepsilon, v_\varepsilon)$ constructed in Theorem 1. We linearize the system (1)$\sim$ (3) at $(u_\varepsilon, v_\varepsilon)$ and obtain the following eigenvalue problem:

$$\varepsilon^2\varphi_\varepsilon'' - \varphi_\varepsilon + 2gu_\varepsilon v_\varepsilon\varphi_\varepsilon + gu_\varepsilon^2\psi_\varepsilon = \lambda_\varepsilon\varphi_\varepsilon, \quad x \in (-1,1), \tag{10}$$

$$D\psi_\varepsilon'' - \frac{2c}{\varepsilon}gu_\varepsilon v_\varepsilon\varphi_\varepsilon - \frac{c}{\varepsilon}gu_\varepsilon^2\psi_\varepsilon = \varepsilon\lambda_\varepsilon\psi_\varepsilon, \quad x \in (-1,1), \tag{11}$$

$$\varphi_\varepsilon'(\pm 1) = \psi_\varepsilon'(\pm 1) = 0,$$

where, λ_ε is an eigenvalue, and $(\varphi_\varepsilon, \psi_\varepsilon) \neq (0,0)$ is an eigenfunction. We say that the solution $(u_\varepsilon, v_\varepsilon)$ is stable if $\mathrm{Re}\lambda_\varepsilon < 0$ holds for all eigenvalues and unstable if there exists an eigenvalue satisfying $\mathrm{Re}\lambda_\varepsilon > 0$. We have the following result on the stability.

Theorem 2 *Fix* $D < +\infty$*. Let* $\varepsilon > 0$ *be sufficiently small. Let* $(u_\varepsilon, v_\varepsilon)$ *be the solution given in Theorem 1 . Then, we have the following* **for large eigenvalues***, namely* $\lambda_\varepsilon \to \lambda_0 \neq 0$*:*

(1) $(u_\varepsilon, v_\varepsilon)$ *is stable for any* $D < +\infty$*, namely* $\mathrm{Re}(\lambda_\varepsilon) < 0$ *holds.*

Furthermore, let $g \in C^3(-1,1)$*. Then, we have the following* **for small eigenvalues***, namely* $\lambda_\varepsilon \to 0$*:*

(2) If $g''(0) \le 0$*, then* $(u_\varepsilon, v_\varepsilon)$ *is stable for any* $D < +\infty$*.*

(3) If $g''(0) > 0$*,* $(u_\varepsilon, v_\varepsilon)$ *is stable for* $D < D_1$*,* $(u_\varepsilon, v_\varepsilon)$ *is unstable for* $D > D_1$*, where,* $D_1 > 0$ *is*

$$D_1 := \frac{1}{2c\int_{\mathbb{R}} w_0^2} \cdot \frac{g^2(0)}{g''(0)} = \frac{1}{12c} \cdot \frac{g^2(0)}{g''(0)}.$$

In fact, we have the following asymptotic behavior of λ_ε *as* $\varepsilon \to 0$*:*

$$\lambda_\varepsilon = \varepsilon^2 \frac{\int_{\mathbb{R}} w^3}{\int_{\mathbb{R}} (w')^2} \Big(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3} \Big) + O(\varepsilon^{\frac{5}{2}}). \tag{12}$$

Remark 1 *Note* $\int_{\mathbb{R}} w_0^2 = 6$ *and*

$$\xi_0 = cg(0) \int_{\mathbb{R}} w^2(y)dy = cg(0)^{-1} \int_{\mathbb{R}} w_0^2(y)dy = 6cg(0)^{-1}.$$

Hence, for the case $g''(0) > 0$ *the condition* $D < D_1$ *is equivalent to*

$$\Big(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3} \Big) < 0.$$

Remark 2 *Since we are concerned with the existence of unstable eigenvalues, we can assume that* $\mathrm{Re}\lambda_\varepsilon \geq -\frac{1}{4}$ *for example. We can show that eigenvalues* λ_ε *are uniformly bounded under the assumption* $\mathrm{Re}\lambda_\varepsilon \geq -\frac{1}{4}$*. Therefore, we can assume that there exists a* λ_0 *such that* $\lambda_\varepsilon \to \lambda_0$ *as* $\varepsilon \to 0$*, taking a subsequence if necessary.*

Remark 3 *In the case* $g(x) = 1$ *and* $d = 0$*, by the result of Iron, Wei, and Winter[3], the one-peak symmetric solution is stable for any* $D > 0$*. Compared with the case* $g(x) = 1$*, Theorem 2 reveals the strong influence of the heterogeneity* $g(x)$ *on the stability of the one-peak symmetric solution. We should mention that a similar destabilization effect of the heterogeneity has been studied for the Gierer-Meinhardt system(see [8]). We note that our results also cover the case* $d = 0$*. We emphasize that, even in the case* $g(x) = 1$*, the remainder estimate* $O(\varepsilon^{\frac{5}{2}})$ *for small eigenvalues is more precise than the result of Iron, Wei and Winter. We also emphasize that for the case* $d > 0$ *we need to take care of the remainder terms more carefully, compared with the case* $d = 0$*.*

Even for non-symmetric heterogeneity $g(x)$, we can expect similar results. However, we need more computations and left to future works. For the related works with some heterogeneity in other Turing systems, see for example [2], [6], [7] , [8] and the references therein. Recently, for a given $N \in 2, N \in \mathbb{N}$ and a given symmetric $\frac{2}{N}$-periodic function $g(x)$ in the interval $I = (-1, 1)$, one of the authors studied the existence of multi-peak symmetric solutions and its stability in details (see [4]). We also mention that Ao and Liu[1] studied recently another heterogeneity effect on the existence and its stability for the Schnakenberg model with precursors.

2 Outline of the Proof of Theorem 1

2.1 Heuristic explanation of the choice of ξ_0

Before giving the outline of the proof of Theorem 1, we explain briefly why we choose ξ_0 as follows in Theorem 1:

$$\xi_0 := cg(0) \int_{\mathbb{R}} w^2(y)dy.$$

Suppose $\overline{u_\varepsilon}$ and $\overline{v_\varepsilon}$ are uniformly bounded. Then by the equation for v:

$$-D\overline{v_\varepsilon}'' = \frac{\varepsilon^2}{2} - \varepsilon c \overline{g}\overline{u_\varepsilon}^2 \overline{v_\varepsilon}, \tag{13}$$

we have $|D\overline{v_\varepsilon}''(y)| \le C\varepsilon$. Since $\overline{v_\varepsilon}$ is symmetric, we have $\overline{v_\varepsilon}'(0) = 0$. Therefore, for fixed $R > 0$ we have $|\overline{v_\varepsilon}'(y)| \le CR\varepsilon$ $(|y| \le R)$. This implies $\overline{v_\varepsilon}(y) \sim C_0$ $(|y| \le R)$ for some positive constant C_0. On the other hand $\overline{u_\varepsilon}$ satisfies

$$-\overline{u_\varepsilon}'' = d\varepsilon - \overline{u_\varepsilon} + \overline{g}\overline{u_\varepsilon}^2 \overline{v_\varepsilon}, \;\; y \in I_\varepsilon = (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}).$$

Now we expect $\overline{u_\varepsilon}(y) \sim A_0 w(y) := u_\infty(y)$. Then we have

$$-u_\infty''(y) + u_\infty(y) = g(0) u_\infty(y)^2 C_0, \;\; y \in \mathbb{R}.$$

So if we take $w(y)$ to be a solution to $-w'' + w = g(0)w^2$, we must have $A_0 C_0 = 1$. Now integrating (13), we have

$$0 = 1 - c\int_{I_\varepsilon} \overline{g}\overline{u_\varepsilon}^2 \overline{v_\varepsilon}\, dy.$$

So taking the limit, we would have

$$1 = c\int_{\mathbb{R}} g(0) A_0^2 w(y)^2 C_0 \, dy.$$

Therefore, since $A_0 C_0 = 1$ we should have

$$A_0 = \frac{1}{cg(0)\int_{\mathbb{R}} w^2\, dy}.$$

Thus if we define $\xi_0 := cg(0)\int_{\mathbb{R}} w^2(y)dy$, then we have

$$u_\varepsilon(x) \sim \frac{1}{\xi_0} w(\frac{x}{\varepsilon}) \quad \text{and} \quad v_\varepsilon(0) \sim \xi_0.$$

2.2 Outline of the construction by using the contraction mapping principle.

Let $u = w_\varepsilon + \phi$ with $\overline{\phi}(y) \in B(C_0)$, where

$$w_\varepsilon(x) := \frac{1}{\xi_0} w\Big(\frac{x}{\varepsilon}\Big)\chi(x), \quad \xi_0 := cg(0)\int_{\mathbb{R}} w^2(y)dy$$

and

$$B(C_0) := \left\{ \overline{\phi} \in H_s^2(I_\varepsilon) \;\middle|\; ||\overline{\phi}||_{H^2(I_\varepsilon)} \le C_0\sqrt{\varepsilon}\,,\; \overline{\phi}'(\pm\frac{1}{\varepsilon}) = 0 \right\}, \tag{14}$$

where the constant C_0 is independent of ε, which will be chosen suitably later. Then, we can find a unique solution $v := T[u] = T[w_\varepsilon + \phi]$ of the second equation (5):

$$-Dv'' + \frac{c}{\varepsilon} g(x) u^2 v = \frac{1}{2}, \;\; x \in (-1,1), \quad v'(\pm 1) = 0.$$

We seek a unique $\phi(x) \in H_s^2(I)$ such that $(u(x), v(x)) = (w_\varepsilon + \phi, T[w_\varepsilon + \phi])$ satisfies the first equation (4). Substituting $u = w_\varepsilon + \phi, v = T[w_\varepsilon + \phi]$ into the first equation (4):

$$-\varepsilon^2 u'' = d\varepsilon - u + g(x)u^2 T[u], \ x \in (-1,1), \quad u'(\pm 1) = 0,$$

we have

$$\varepsilon^2 \phi'' - \phi + 2gw_\varepsilon \phi T[w_\varepsilon + \phi] + gw_\varepsilon^2 T[w_\varepsilon + \phi] + \varepsilon^2 w_\varepsilon'' - w_\varepsilon + d\varepsilon + g\phi^2 T[w_\varepsilon + \phi] = 0. \quad (15)$$

Using the Fréchet derivative $R_\varepsilon[\phi] = \langle T'[w_\varepsilon], \phi \rangle$, we have

$$S_\varepsilon[\phi] + gw_\varepsilon^2 T[w_\varepsilon] + \varepsilon^2 w_\varepsilon'' - w_\varepsilon + d\varepsilon + N_1[\phi] = 0,$$

where

$$S_\varepsilon[\phi] := \varepsilon^2 \phi'' - \phi + 2gT[w_\varepsilon] w_\varepsilon \phi + R_\varepsilon[\phi] g w_\varepsilon^2 \quad (16)$$

and $N_1[\phi]$ is the higher order term. Here, in the y-variable, using $\overline{w_\varepsilon}'' - \overline{w_\varepsilon} = -\xi_0 g(0) \overline{w_\varepsilon}^2 + O(e^{-\frac{1}{4\varepsilon}})$ we rewrite as follows:

$$\overline{S_\varepsilon}[\overline{\phi}] + \overline{g}\ \overline{w_\varepsilon}^2 \overline{T[w_\varepsilon]} - g(0)\xi_0 \overline{w_\varepsilon}^2 + d\varepsilon + O(e^{-\frac{1}{4\varepsilon}}) + \overline{N_1[\phi]} = 0, \quad (17)$$

where

$$\overline{S_\varepsilon}[\overline{\phi}] := \overline{S_\varepsilon[\phi]} = \overline{\phi}'' - \overline{\phi} + 2\overline{g}\ \overline{T[w_\varepsilon]}\ \overline{w_\varepsilon}\ \overline{\phi} + \overline{R_\varepsilon[\phi]}\ \overline{g}\ \overline{w_\varepsilon}^2.$$

Now we have the following invertibility of the operator $\overline{S_\varepsilon}: \ H_s^2(I_\varepsilon) \to L_s^2(I_\varepsilon)$.

Lemma 1 ([5, Lemma 3.2]) *There exist $\varepsilon_0 > 0$ and $\lambda > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the following inequality holds:*

$$\left|\left|\overline{S_\varepsilon}[\overline{\phi}]\right|\right|_{L^2(I_\varepsilon)} \geq \lambda \left|\left|\overline{\phi}\right|\right|_{H^2(I_\varepsilon)}, \ \overline{\phi} \in H_s^2(I_\varepsilon). \quad (18)$$

Furthermore, $\mathrm{Ran}(\overline{S_\varepsilon}) = L_s^2(I_\varepsilon)$ *holds.*

Thus we have

$$\overline{\phi} = -\overline{S_\varepsilon}^{-1}\left[\overline{I[w_\varepsilon]}\right] - \overline{S_\varepsilon}^{-1}\left[\overline{N_1[\phi]}\right] =: M_\varepsilon[\overline{\phi}], \quad (19)$$

where

$$\overline{I[w_\varepsilon]} := \overline{g}\ \overline{w_\varepsilon}^2 \overline{T[w_\varepsilon]} - g(0)\xi_0 \overline{w_\varepsilon}^2 + d\varepsilon + O(e^{-\frac{1}{4\varepsilon}}).$$

If we choose $C_0 > 0$ large enough such that

$$\|\overline{S_\varepsilon}^{-1}\left[\overline{I[w_\varepsilon]}\right]\|_{H^2(I_\varepsilon)} \leq \frac{C_0}{2}\sqrt{\varepsilon} \quad (20)$$

we can show that M_ε is a contraction mapping on $B(C_0)$ for small $\varepsilon > 0$. Actually, we can choose C_0 so that $C_0 > \frac{4(C_1+d)}{\lambda}$, where C_1 is the constant will appear in Corollary 1 later. (Note that the constants $\lambda > 0$ and C_1 depend only on $w(x), g(x)$ and the fixed parameters $c > 0, D > 0$.) Thus, there exists a unique $\overline{\phi} \in B(C_0)$ which satisfies the desired equation.

2.3 Basic estimates, including the estimates for $T[w_\varepsilon + \phi]$.

We note the following estimates, which play key roles throughout this work.

Lemma 2 ([5, Lemma 2.8, 2.9]) *Fix $C_0 > 0$. For each $\overline{\phi} \in B(C_0)$, let $\eta(x) \in H^2(-1,1)$ satisfy*

$$-D\eta'' + \frac{cg(w_\varepsilon + \phi)^2}{\varepsilon}\eta = \frac{h}{\varepsilon},\ x \in (-1,1), \qquad \eta'(\pm 1) = 0, \tag{21}$$

where $h(x)$ is a given function on $L^2(-1,1)$. Then, the following estimates hold:

(1) $||\overline{\eta}||_{L^\infty(I_\varepsilon)} \leq C||\overline{h}||_{L^1(I_\varepsilon)}$.

(2) $||\overline{\eta}'||_{L^2(I_\varepsilon)} \leq C\sqrt{\varepsilon}||\overline{h}||_{L^1(I_\varepsilon)}$.

(3) $|\overline{\eta}(y)\overline{g}(y) - \overline{\eta}(0)\overline{g}(0)| \leq C\sqrt{\varepsilon|y|}||\overline{h}||_{L^1(I_\varepsilon)},\ y \in I_\varepsilon$.

Here, the constant C is independent of ε. Furthermore, if we have a uniform bound $\|\overline{h}\|_{L^1(I_\varepsilon)} \leq M$, then we have

$$\eta(0) = \xi_0 \int_{I_\varepsilon} \overline{h}\, dy + O(\sqrt{\varepsilon}) \quad as \quad \varepsilon \to 0. \tag{22}$$

By using Lemma 2, we can obtain the following estimate which allows the estimate (20).

Corollary 1 *There exists a constant C_1 such that the following estimates hold*

$$\|\overline{T[w_\varepsilon]}\|_{L^\infty(I_\varepsilon)} \leq C_1, \qquad |T[w_\varepsilon](0) - \xi_0| \leq C_1\sqrt{\varepsilon},$$

$$\|\overline{g}\ \overline{w_\varepsilon}^2\overline{T[w_\varepsilon]} - g(0)\xi_0\overline{w_\varepsilon}^2\|_{L^2(I_\varepsilon)} \leq C_1\sqrt{\varepsilon}.$$

(Proof.) By (1) , (3) of Lemma 2 and (22) as $\phi = 0$ and $h(x) = \frac{\varepsilon}{2}$, there exists a constant C_1, independent of ε and C_0, such that the following estimates hold:

$$\|\overline{T[w_\varepsilon]}\|_{L^\infty(I_\varepsilon)} \leq C_1, \quad |T[w_\varepsilon](0) - \xi_0| \leq C_1\sqrt{\varepsilon},$$

$$|\overline{T[w_\varepsilon]}(y)\overline{g}(y) - \overline{T[w_\varepsilon]}(0)\overline{g}(0)| \leq C_1\sqrt{\varepsilon|y|},\ y \in I_\varepsilon.$$

Thus we have

$$\begin{aligned}
&\overline{w_\varepsilon}^2(y)|\overline{T[w_\varepsilon]}(y)\overline{g}(y) - \xi_0\overline{g}(0)| \\
&\leq \overline{w_\varepsilon}^2(y)\Big(|\overline{T[w_\varepsilon]}(y)\overline{g}(y) - \overline{T[w_\varepsilon]}(0)\overline{g}(0)| + \overline{g}(0)|T[w_\varepsilon](0) - \xi_0|\Big) \\
&\leq g(0)C_1\overline{w_\varepsilon}^2(y)(\sqrt{\varepsilon|y|} + \sqrt{\varepsilon}) \leq g(0)\frac{C_1}{\xi_0^2}w^2(y)(\sqrt{\varepsilon|y|} + \sqrt{\varepsilon}),\ y \in I_\varepsilon.
\end{aligned}$$

This implies the desired estimate.

3 Global pointwise estimates for solutions

Since $\overline{u_\varepsilon}(y) = \overline{w_\varepsilon}(y) + \overline{\phi_\varepsilon}(y)$ with $\|\overline{\phi_\varepsilon}\|_{H^2(I_\varepsilon)} \le C\sqrt{\varepsilon}$, we easily have

$$|\overline{u_\varepsilon}(y)| \le C\sqrt{\varepsilon} + Ce^{-\frac{|y|}{\sqrt{2}}},\ y \in I_\varepsilon$$

by using the Sobolev's embedding theorem. However, this estimate is not enough to treat several error terms in the stability analysis. We need the following pointwise estimates for the solution $(u_\varepsilon, v_\varepsilon)$ in our stability analysis.

Lemma 3 ([5, Proposition 4.2]) *There exists a constant C, which is independent of ε, such that the following estimates hold:*
(i)

$$\|\overline{v_\varepsilon}'\|_{L^\infty(I_\varepsilon)} \le C\varepsilon,$$

(ii)

$$|\overline{u_\varepsilon}(y)| \le C(d\varepsilon + e^{-\frac{1}{\sqrt{2}\varepsilon}} + e^{-\frac{|y|}{\sqrt{2}}}),\ y \in I_\varepsilon,$$

(iii)

$$|\overline{u_\varepsilon}'(y)| \le C(d^2\varepsilon^2 + e^{-\frac{\sqrt{2}}{\varepsilon}} + e^{-\frac{|y|}{2}}),\ y \in I_\varepsilon.$$

These estimates can be obtained by using comparison arguments. In particular, $\overline{u_\varepsilon}(y)$ and $\overline{u_\varepsilon}'(y)$ are exponentially small near the boundary of I_ε if $d = 0$. We also have the following uniform bounds:

$$\|\overline{u_\varepsilon}\|_{L^\infty(I_\varepsilon)} \le C, \quad \|\overline{v_\varepsilon}\|_{L^\infty(I_\varepsilon)} \le C, \quad \|\overline{u_\varepsilon}\|_{L^2(I_\varepsilon)} \le C, \quad \|\overline{u_\varepsilon}'\|_{L^2(I_\varepsilon)} \le C.$$

4 Outline of the Proof of Theorem 2

We may assume that $\|\overline{\varphi_\varepsilon}\|_{H^2(I_\varepsilon)} = 1$. By the extension theorem we have $\|\overline{\varphi_\varepsilon}\|_{H^2(\mathbf{R})} \le C$. So, there exists a subsequence and $\overline{\varphi} \in H^2(\mathbf{R})$ such that $\overline{\varphi_\varepsilon}$ converges to $\overline{\varphi}$ weakly in $H^2(\mathbf{R})$ and strongly in $C^1_{loc}(\mathbf{R})$.

Lemma 4 (Boundedness of unstable eigenvalues, [5, Proposition 4.2]) *Assume* $\mathrm{Re}(\lambda_\varepsilon) \ge -\frac{1}{4}$. *Then, we have the following:*
(1) $\overline{\varphi} \ne 0$.
(2) There exists a constant C, independent of ε, such that $|\lambda_\varepsilon| \le C$.

By this Lemma, we may assume $\lambda_\varepsilon \to \lambda_0$ for some constant λ_0. We consider two cases:
(a) **large eigenvalue**: i.e. $\lambda_\varepsilon \to \lambda_0 \ne 0$.
(b) **small eigenvalue**: i.e. $\lambda_\varepsilon \to 0$.

4.1 Stability analysis for large eigenvalues

Lemma 5 *Assume $\lambda_\varepsilon \to \lambda_0 \neq 0$. Then, we have*

$$\overline{\varphi}''(y) - \overline{\varphi}(y) + 2w_0(y)\overline{\varphi}(y) - 2\frac{\int w_0\overline{\varphi}\,dz}{\int w_0^2\,dz}w_0(y)^2 = \lambda_0\overline{\varphi}(y).$$

Then, by the well-known lemma of Wei and Winter (see Lemma 2.2 in [3], or [8]) for nonlocal eigenvalue problem above, we can conclude $\mathrm{Re}\lambda_0 < 0$. So for sufficiently small $\varepsilon > 0$ we have $\mathrm{Re}\lambda_\varepsilon < 0$, namely λ_ε is a stable eigenvalue.

(Sketch of the proof of Lemma 5.) From the equation (11) for ψ_ε, we can show $\|\psi_\varepsilon\|_{H^1(I)} \leq C$ and apply Lemma 2 to obtain

$$\overline{\psi_\varepsilon}(0) = \psi_\epsilon(0) = \xi_0\int_{I_\epsilon}(-2c\overline{gu_\varepsilon v_\varepsilon\varphi_\varepsilon} - \varepsilon^2\lambda_\varepsilon\overline{\psi_\varepsilon})\,dy + O(\sqrt{\varepsilon})$$

$$\to \psi(0) = -2cg(0)\xi_0\int_{\mathbb{R}} w\overline{\varphi}\,dy = -2\xi_0^2\frac{\int_{\mathbb{R}} w\overline{\varphi}\,dy}{\int_{\mathbb{R}} w^2\,dy} \text{ as } \varepsilon \to 0. \tag{23}$$

Here we used $\xi_0 = cg(0)\int_{\mathbb{R}} w^2\,dy$. On the other hand, from the equation (10), we have

$$\overline{\varphi_\varepsilon}'' - \overline{\varphi_\varepsilon} + 2\overline{gu_\varepsilon v_\varepsilon\varphi_\varepsilon} + \overline{gu_\varepsilon}^2\overline{\psi_\varepsilon} = \lambda_\varepsilon\overline{\varphi_\varepsilon} \text{ on } I_\varepsilon.$$

Then , for any $\zeta \in C_0^\infty(\mathbb{R})$, we have

$$\int_{I_\varepsilon}\Big(\overline{\varphi_\varepsilon}'' - \overline{\varphi_\varepsilon} + 2\overline{gu_\varepsilon v_\varepsilon\varphi_\varepsilon} + \overline{gu_\varepsilon}^2\overline{\psi_\varepsilon}\Big)\zeta(y)\,dy = \int_{\mathbb{R}}\lambda_\varepsilon\overline{\varphi_\varepsilon}\zeta(y)\,dy.$$

Taking $\varepsilon \to 0$ and using Lemma 2, we obtain

$$\int_{\mathbb{R}}\Big(\overline{\varphi}''(y) - \overline{\varphi}(y) - +2g(0)w(y)\overline{\varphi}(y) + \frac{g(0)}{\xi_0^2}w^2(y)\psi(0)\Big)\zeta(y)\,dy = \lambda_0\int_{\mathbb{R}}\overline{\varphi}(y)\zeta(y)\,dy. \tag{24}$$

Since $w(y) = g(0)^{-1}w_0(y)$, (23) and (24) yield Lemma 5.

4.2 Stability analysis for small eigenvalues

We have the following key precise asymptotic for small eigenvalues λ_ε which yields the proof of Theorem 2.

Proposition 1 *Assume . $\lambda_\varepsilon \to 0$. Then, as $\varepsilon \to 0$, the asymptotic form of λ_ε is given as follows:*

$$\lambda_\varepsilon = \varepsilon^2\frac{\int_{\mathbb{R}} w^3}{\int_{\mathbb{R}}(w')^2}\Big(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3}\Big) + O(\varepsilon^{\frac{5}{2}}). \tag{25}$$

(Sketch of the proof of Proposition 1.) We denote $u_{\epsilon,1}(x) = u_\varepsilon(x)\chi(x)$. To show Proposition 1, let us decompose

$$\varphi_\varepsilon(x) = \varepsilon a_\varepsilon u'_{\varepsilon,1}(x) + \varphi_\varepsilon^\perp(x), \tag{26}$$

where a_ε is some complex number and $\varphi_\varepsilon^\perp$ satisfies

$$\overline{\varphi_\varepsilon^\perp} \perp \mathcal{K}_\varepsilon \text{ in } L^2(I_\varepsilon), \quad \mathcal{K}_\varepsilon := \operatorname{span}\{\overline{u_{\varepsilon,1}}'\} \subset H^2(I_\varepsilon).$$

In y-variable, we have

$$\overline{\varphi_\varepsilon}(y) = a_\varepsilon \overline{u_{\varepsilon,1}}'(y) + \overline{\varphi_\varepsilon^\perp}(y).$$

Similarly, we decompose

$$\psi_\varepsilon(x) = \varepsilon a_\varepsilon \psi_{\varepsilon,1}(x) + \psi_\varepsilon^\perp(x). \tag{27}$$

Here, $\psi_{\varepsilon,1}$ is a unique solution of

$$D\psi''_{\varepsilon,1} - \frac{c}{\varepsilon} g(x) u_\varepsilon^2 \psi_{\varepsilon,1} - \frac{2c}{\varepsilon} g(x) v_\varepsilon u'_{\varepsilon,1} = \varepsilon\lambda_\varepsilon \psi_{\varepsilon,1},\ x \in (-1,1), \quad \psi'_{\varepsilon,1}(\pm 1) = 0, \tag{28}$$

and $\psi_\varepsilon^\perp$ is defined by $\psi_\varepsilon^\perp := \psi_\varepsilon - \varepsilon a_\varepsilon \psi_{\varepsilon,1}$. We have the following formula for λ_ε:

Lemma 6 ([5, Lemma 6.1])

$$J_1 + J_2 + J_3 + J_4 + O(|a_\varepsilon|\varepsilon^3) = \lambda_\varepsilon a_\varepsilon \xi_0^{-2} \int_{\mathbb{R}} (w')^2 dy + O(\sqrt{\varepsilon}\lambda_\varepsilon |a_\varepsilon|). \tag{29}$$

where J_i $(i = 1, 2, 3, 4)$ are defined as follows:

$$J_1 := a_\varepsilon \int_{I_\varepsilon} \left(\varepsilon\overline{\psi_{\varepsilon,1}} - \overline{v_\varepsilon}'\right) \overline{u_{\varepsilon,1}}^2 \overline{u_{\varepsilon,1}}' dy - a_\varepsilon \int_{I_\varepsilon} \overline{g}' \overline{v_\varepsilon}\, \overline{u_{\varepsilon,1}}^2 \overline{u_{\varepsilon,1}}' dy,$$

$$J_2 := -\int_{I_\varepsilon} \left(\overline{g}'\overline{v_\varepsilon} + \overline{g}\ \overline{v_\varepsilon}'\right) \overline{u_{\varepsilon,1}}^2\ \overline{\varphi_\varepsilon^\perp} dy,$$

$$J_3 := \int_{I_\varepsilon} \overline{g}\ \overline{u_\varepsilon}^2 \overline{\psi_\varepsilon^\perp}\ \overline{u_{\varepsilon,1}}' dy, \quad J_4 := \int_{I_\varepsilon} \overline{r_\varepsilon}\ \overline{\varphi_\varepsilon^\perp} dy.$$

Here, $\overline{r_\varepsilon}$ is a function satisfying $\overline{r_\varepsilon}(y) = 0$ on $|y| \le \frac{1}{4\varepsilon}$ and $\overline{r_\varepsilon}(y) = O(\varepsilon^2)$ on $\frac{1}{4\varepsilon} \le |y| \le \frac{1}{\varepsilon}$.

This is obtained by multiplying $\overline{u_{\varepsilon,1}}'$ and integration by parts. Among them, J_1 is the leading term to determine the precise asymptotic for λ_ε. The following Proposition 2 is important and decide the asymptotic behavior of J_1.

Proposition 2 ([5, Proposition 6.2]) *The following estimates hold:*

(1) $\left|\left|\left(\varepsilon\overline{\psi_{\varepsilon,1}} - \overline{v_\varepsilon}'\right)\overline{g}\ \overline{u_\varepsilon}^2\right|\right|_{L^1(I_\varepsilon)} \le C\varepsilon^2.$

(2) $|\varepsilon\overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon}'(y)| \le C\varepsilon^2 |y|.$

(3) $\varepsilon\overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon}'(y) = \varepsilon^2 y \cdot \frac{cg(0)\xi_0^{-1}}{2D}\int_{\mathbb{R}} w^2 dt + O(\varepsilon^{\frac{5}{2}}|y|).$

Here, the constant C *is independent of* $\varepsilon > 0$.

These are obtained by the representation of $\overline{v_\varepsilon}'(y)$ and $\overline{\psi_{\varepsilon,1}}(y)$ by using Dirichlet and Neumann Green functions, respectively. By Proposition 2, we have the following.

Proposition 3 *It holds that*

$$J_1 = a_\varepsilon \varepsilon^2 \Big(- \frac{g(0)\int_{\mathbb{R}} w^3}{6D\xi_0^3} + \frac{g''(0)\int_{\mathbb{R}} w^3}{3\xi_0^2}\Big) + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}),$$

where the constant C *is independent of* $\varepsilon > 0$.

(Sketch of the proof of Proposition 3.) By (3) of Proposition 2, we have

$$a_\varepsilon \int_{I_\varepsilon} (\varepsilon\overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon}'(y))\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'\, dy$$

$$= a_\varepsilon\varepsilon^2 \frac{cg(0)\xi_0^{-1}}{2D}\left(\int_{\mathbb{R}} w^2\, dx\right)\int_{\mathbb{R}} y\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'\, dy + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$$

$$= -a_\varepsilon\varepsilon^2 \frac{cg(0)^2\int_{\mathbb{R}} w^2\, dx}{6D\xi_0^4}\int_{\mathbb{R}} w^3\, dx + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}) = -a_\varepsilon\varepsilon^2\frac{g(0)}{6D\xi_0^3}\int_{\mathbb{R}} w^3\, dx + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}).$$

Here, we used

$$cg(0)\int_{\mathbb{R}} w(y)^2\, dy = \xi_0, \qquad \int_{\mathbb{R}} yw(y)^2 w'(y)\, dy = -\frac{1}{3}\int_{\mathbb{R}} w(y)^3\, dy.$$

Since by using $\overline{g}'(y) = \varepsilon^2 y g''(0) + O(\varepsilon^3|y|^2)$ we also have

$$\int_{I_\varepsilon} \overline{g}'\overline{v_\varepsilon}\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}'\, dy = -\varepsilon^2\frac{g''(0)\xi_0^{-2}}{3}\int_{\mathbb{R}} w^3\, dx + O(\varepsilon^{\frac{5}{2}}), \tag{30}$$

we can conclude that

$$J_1 = a_\varepsilon\int_{I_\varepsilon}(\varepsilon\overline{\psi_{\varepsilon,1}} - \overline{v_\varepsilon}')\overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}' dy - a_\varepsilon\int_{I_\varepsilon}\overline{g}'\overline{v_\varepsilon}\ \overline{u_{\varepsilon,1}}^2\overline{u_{\varepsilon,1}}' dy$$

$$= a_\varepsilon\varepsilon^2\Big(- \frac{g(0)\int_{\mathbb{R}} w^3}{6D\xi_0^3} + \frac{g''(0)\int_{\mathbb{R}} w^3}{3\xi_0^2}\Big) + O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}).$$

To estimate the remainder terms J_2, J_3, and J_4, we need the following several estimates.

Lemma 7 ([5, Lemma 6.6]) *For* $\psi_\varepsilon^\perp$, *it holds that:*

(1) $||\overline{\psi_\varepsilon^\perp}||_{L^\infty(I_\varepsilon)} \leq C||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

(2) $||\overline{\psi_\varepsilon^\perp}'||_{L^2(I_\varepsilon)} \leq C\sqrt{\varepsilon}||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

Basically, these estimates can be obtained by applying Lemma 2 for $\psi_\varepsilon^\perp$. Lemma 7 implies the following estimates.

Lemma 8 ([5, Lemma 6.8]) *For J_i $(i = 2, 3, 4)$, it holds that:*

(1) $|J_2| \leq C\varepsilon ||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

(2) $|J_3| \leq C\sqrt{\varepsilon} ||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

(3) $|J_4| \leq C\varepsilon^{\frac{3}{2}} ||\overline{\varphi_\varepsilon^\perp}||_{L^2(I_\varepsilon)}$.

Here, the constant C is independent of $\varepsilon > 0$.

By using Proposition 2 again and the inveritibility of some operator $\tilde{L}_\varepsilon$, which is close to the operator S_ε, we can show the following estimates.

Lemma 9 ([5, Lemma 6.9]) *The following hold:*

(1) $|a_\varepsilon| \neq 0$.

(2) $||\overline{\varphi_\varepsilon^\perp}||_{L^2} \leq C|a_\varepsilon|\varepsilon^{\frac{3}{2}}$.

Combing these estimates, we arrive at the remainder estimates for J_2, J_3 and J_4.

Lemma 10 $J_2 = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$, $J_3 = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$, $J_4 = O(|a_\varepsilon|\varepsilon^3)$.

Estimates for J_2 and J_4 follow directly from Lemma 8 and 9. However, for J_3, Lemma 8 and 9 yield just $J_3 = (|a_\varepsilon|\varepsilon^2)$, which is not enough. Actually, we need the following refined estimate to get the correct estimate for J_3.

$$\overline{\psi_\varepsilon^\perp}(y) - \overline{\psi_\varepsilon^\perp}(0) = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}|y|). \tag{31}$$

This is obtained by the representation of $\overline{\psi_\varepsilon^\perp}(y)$ by using the Neumann Green function. Now, by using Proposition 3, Lemma 10 and Lemma 6, we can complete the proof of Proposition 1.

5 Further Remarks

We give two remarks.

Remark 4 *Assume $g(x)$ is Lipschitz continuous and $g \in C^3((-1, 0])$ and $g \in C^3([0, 1))$, respectively. Let $g'(+0) := \lim_{x>0,x\to 0} g'(x)$ and $g'(-0) := \lim_{x<0,x\to 0} g'(x) = -g'(+0)$ by the symmetry. When $g'(+0) \neq g'(-0)$, the stability of the solution is determined by* *the sign of* $g'(+0)$. *First, note that using* $\overline{g}'(y) = \varepsilon g'(+0) + \varepsilon^2 y g''(+0) + O(\varepsilon^3|y|^2)$ *for* $y > 0$ *and* $\overline{g}'\overline{v_\varepsilon u_\varepsilon}^2\overline{u_\varepsilon}'$ *is an even function, we can compute*

$$\int_{I_\varepsilon} \overline{g}'\overline{v_\varepsilon u_\varepsilon}^2\overline{u_\varepsilon}'\,dy = 2\int_0^{\frac{1}{\varepsilon}} \overline{g}'\overline{v_\varepsilon u_\varepsilon}^2\overline{u_\varepsilon}'\,dy$$

$$= \varepsilon g'(+0)\xi_0^{-2}\int_0^{+\infty} w(y)^2 w'(y)\,dy + O(\varepsilon^{\frac{3}{2}}) = -\varepsilon\frac{g'(+0)w(0)^3}{3\xi_0^2} + O(\varepsilon^{\frac{3}{2}}).$$

Thus, in the computation of the small eigenvalue λ_ε, *the leading term of* J_1 *become as follows:*

$$J_1 = a_\varepsilon\frac{\varepsilon}{3\xi_0^2}g'(+0)w(0)^3 + O(|a_\varepsilon|\varepsilon^{\frac{3}{2}}).$$

Compare with (30) and Proposition 3 for the case $g \in C^3(-1,1)$. *This implies*

$$\lambda_\varepsilon = \varepsilon\frac{g'(+0)}{3\int_{\mathbb{R}} w'(x)^2\,dx}(w(0))^3 + O(\varepsilon^{\frac{3}{2}}).$$

Therefore, the solution is unstable if $g'(+0) > 0$ *and stable if* $g'(+0) < 0$.

Remark 5 (Boundary peal solution and its stability) *For a given Lipschitz continuous positive function* $g(x)$, *we can construct a boundary peak solution* $(u_\varepsilon, v_\varepsilon)$ *on the interval* $I := (-1,1)$. *Because, consider an extension of* $g(x)$ *on the interval* $\tilde{I} := (-1,3)$, *which is symmetric with respect to* $x = 1$. *We denote it by* $\tilde{g}(x)$. *For this function* $\tilde{g}$, *we can construct solution* $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ *to the corresponding Schnakenberg system on the interval* $\tilde{I}$, *which is symmetric with respect to* $x = 1$. *Restricting this solution on the original interval* I, *we obtain a boundary peak solution* $(u_\varepsilon, v_\varepsilon)$. *For the stability of this boundary peak solution, let us consider the linearized eigenvalue problem on* I. *We denote by* λ_ε *and* $(\varphi_\varepsilon, \psi_\varepsilon)$ *the eigenvalue and the associated eigenfunctions, respectively. Now, extending the eigenfunction* $(\varphi_\varepsilon, \psi_\varepsilon)$ *on the interval* $\tilde{I} = (-1,3)$ *to be symmetric with respect to* $x = 1$. *Then by the Neumann boundary condition at* $x = 1$, *this extended function* $(\tilde{\varphi}_\varepsilon, \tilde{\psi}_\varepsilon)$ *is an eigenfunction associated with the eigenvalue* λ_ε *on the interval* $\tilde{I}$. *Then, we can apply our theorem to study the stability of the boundary peak solution* $(u_\varepsilon, v_\varepsilon)$. *Namely, assuming* $\tilde{g}(x)$ *is* C^3 *function,* $\tilde{g}''(1)$ *determine the stability of the boundary peak solution constructed in this way.*

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参考文献

[1] W. Ao, C. Liu, The Schnakenberg model with precursors, Discrete Contin. Dyn. Syst., 39(4) (2019) 1923-1955.

[2] D. L. Benson, J. A. Sherrat and P. K. Maini, Diffusion driven instability in an inhomogeneous domain, Bull. of Math. Biology, 55 (1993), 365-384.

[3] D. Iron, J. Wei and M. Winter, Stability analysis of Turing patterns generated by the Schnakenberg model, J.Math. Biol. 49(2004), 358-390.

[4] Y. Ishii, Stability of multi-peak symmetric stationary solutions for the Schnakenberg model with periodic heterogeneity, submitted(2019).

[5] Y. Ishii, K. Kurata, Existence and stability of one-peak symmetric stationary solutions for the Schnakenberg model with heterogeneity, Discrete Contin. Dyn. Syst., 39(5) (2019) 2807-2875.

[6] T. Kolokolnikov and J. Wei, Pattern formation in a reaction-diffusion system with space-dependent feed rate, SIAM Rev., 60 (2018), 626-645.

[7] J. Wei and M. Winter, Stable spike clusters for the one-dimensional Gierer-Meinhardt system, Eur. J. Appl. Math., 28 (2017), 576-635.

[8] J. Wei, M. Winter, *Mathematical Aspects of pattern formation in Biological Systems, Applied Math. Sci. vol.189(2014), Springer.*

Department of Mathematical Sciences, Tokyo Metropolitan Univeristy
Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397
JAPAN
E-mail address:
Yuta Ishii (ishii-yuta1@ed.tmu.ac.jp)
Kazuhiro Kurata (kurata@tmu.ac.jp)