EXTRA AUTOMORPHISMS AND AUTOMORPHISM GROUPS OF ORBIFOLD VERTEX OPERATOR ALGEBRAS ASSOCIATED WITH LATTICES

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Abstract. In this article, we study the automorphism groups for orbifold VOA $V_L^{\hat{g}}$ for some $g \in O(L)$. In particular, we give a sufficient condition for which $\mathrm{Aut}\,(V_L^{\hat{g}})$ contains an extra automorphism. Some examples are also discussed.

1. Introduction

Let L be a any positive definite even lattice. Then one can associate a vertex operator algebra (VOA) V_L with L. For any isometry g of L, one can also lift g to an automorphism $\hat{g} \in \mathrm{Aut}\,(V_L)$ of the same order. The fixed point subspace $V^{\langle \hat{g}\rangle}$ is a subVOA and is often called an orbifold subVOA. In this article, we will study the automorphism groups of the orbifold VOA $V_L^{\hat{g}}$. For simplicity, we assume that g is fixed point free on L. In this case, the lift $\hat{g}$ is unique, up to conjugation. By abuse of notation, We often denote $\hat{g}$ by g.

It is clear that for any $h \in N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g}\rangle)$ and $x \in V_L^{\hat{g}}$, we have $hx \in V_L^{\hat{g}}$. Therefore, $N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g}\rangle)$ acts on $V_L^{\hat{g}}$ and there is a group homomorphism

$$f : N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g}\rangle)/\langle \hat{g}\rangle \longrightarrow \mathrm{Aut}\,(V_L^{\hat{g}}).$$

The main idea is to study the homomorphism f and to determine if f is injective and/or surjective. In particular, it is important to determine if there exist automorphisms in $\mathrm{Aut}\,(V_L^{\hat{g}})$ which are not induced from $N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g}\rangle)$. We call such an automorphism an extra automorphism. For generic cases, $\mathrm{Aut}\,(V_L^{\hat{g}})$ is often isomorphic to $N_{\mathrm{Aut}\,(V_L)}(\langle \hat{g}\rangle)/\langle \hat{g}\rangle$; however, $\mathrm{Aut}\,(V_L^{\hat{g}})$ can also be strictly larger. One of the main purpose of this article is to provide a sufficient condition which guarantees the existence of certain extra automorphisms in $\mathrm{Aut}\,(V_L^{\hat{g}})$. Some examples will also be discussed.

2. Lattice VOA V_L

First, we recall the construction of a lattice VOA V_L associated with a rank n positive definite even lattice L with an inner product $\langle\ ,\ \rangle$ from [FLM].

Consider $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ as an abelian Lie algebra and extend the inner product $\langle\ ,\ \rangle$ $\mathbb{C}$-linearly. Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$ be its affine Lie algebra with the Lie bracket

$$[a \otimes t^n, b \otimes t^m] = \delta_{n+m,0} n \langle a, b\rangle.$$

Then

$$M(1) = \mathbb{C}[\alpha(n) \mid \alpha \in \mathfrak{h}, n < 0] \cdot \mathbf{1}$$

is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)\mathbf{1} = 0$ for $\alpha \in \mathfrak{h}$, $n \geq 0$, where $\alpha(n) = \alpha \otimes t^n$.

Let $\hat{L} = \{e^\alpha, \kappa e^\alpha \mid \alpha \in L\}$ be a central extension of L by $\langle \kappa \mid \kappa^2 = 1\rangle$ such that $e^\alpha e^\beta = \kappa^{\langle \alpha,\beta\rangle} e^\beta e^\alpha$. Let

$$\begin{aligned} \mathbb{C}\{L\} &= \mathrm{Ind}_{<\kappa>}^{\hat{L}} \mathbb{C} = \mathbb{C}[\hat{L}]/ < \kappa + 1 > \\ &\cong \mathrm{span}_{\mathbb{C}}\{e^\alpha \mid \alpha \in L\} \quad \text{ as a vector space.} \end{aligned}$$

The lattice VOA V_L is given by $V_L = M(1) \otimes \mathbb{C}\{L\}$. The VOA V_L has a natural $\mathbb{N}$-grading such that $V_L = \bigoplus_{n=0}^\infty (V_L)_n$, where the weight of an element is defined by

$$\mathrm{wt}(\alpha_1(-n_1)\cdots\alpha_r(-n_r)e^\beta) = n_1 + \cdots + n_r + \frac{\langle \beta, \beta\rangle}{2}.$$

It is easy to show that $(V_L)_0 = \mathbb{C}\mathbf{1}$ and

$$(V_L)_1 = \sum_{\alpha \in L} \mathbb{C}\alpha(-1)\mathbf{1} + \sum_{\alpha \in \Phi(L)} \mathbb{C}e^\alpha,$$

where $\Phi(L) = L(2) = \{\alpha \in L \mid \langle \alpha, \alpha\rangle = 2\}$ (see [FLM] for the detail).

For any VOA $V = \oplus_{n=0}^\infty V_n$ with $\dim V_0 = 1$, it is well known [FLM] that the weight one space V_1 is a Lie algebra with the bracket $[u, v] = u_{(0)}v$ and with an invariant bilinear form given by $(v, u)\mathbf{1} = v_{(1)}u$. In particular, if L is a root lattice of type A_n, D_n or E_n, then $(V_L)_1$ is a simple Lie algebra, where $[M(1)]_1$ is a Cartan subalgebra and $\{e^\alpha \mid \alpha \in L(2)\}$ is the set of root vectors.

For any VOA V and $v \in V_1$, $\exp(v_{(0)})$ always defines an automorphism of V_L for any $v \in (V_L)_1$. As an application, we can induce any inner automorphism of Lie algebra V_1 to an automorphism of V.

2.1. **Automorphism groups of lattice VOA.** Next we will review some facts about the automorphism groups of lattice VOA.

Let $^{-} : \hat{L} \to L$ be the natural projection of $\hat{L}$ to L and let $\iota : a \in L \to e^a \in \hat{L}$ be a section, i.e. $^{-} \circ \iota = id_L$. For any $g \in \mathrm{Aut}\,(\hat{L})$, define $\bar{g} = {}^{-} \circ g \circ \iota \in O(L)$, the isometry group of L. Set

$$O(\hat{L}) = \mathrm{Aut}(\hat{L}, \langle\ ,\ \rangle) = \{g \in \mathrm{Aut}\,\hat{L} \,|\, \langle \bar{g}\alpha, \bar{g}\beta\rangle = \langle \alpha, \beta\rangle \text{ for all } \alpha, \beta \in L\}.$$

The following lemma can be proved easily from the construction (cf. [FLM]).

Lemma 2.1. *For any $\mu \in O(\hat{L})$, we can define an automorphism $\tilde{\mu}$ of V_L naturally by*

$$\tilde{\mu}(\alpha_1(-n_1)\cdots\alpha_k(-n_k)\otimes e^a) = (\bar{\mu}\alpha_1)(-n_1)\cdots(\bar{\mu}\alpha_k)(-n_k)\otimes \mu(e^a),$$

where $\alpha_1, \ldots, \alpha_k \in L$ and $e^a \in \hat{L}$. On the other hand, if $\tau \in \mathrm{Aut}(V_L)$ that keeps $M(1)_1$ invariant, then there exist $\mu \in \mathrm{Aut}(\hat{L})$ and $b \in M(1)_1 = \mathbb{C}\otimes_{\mathbb{Z}} L$ such that $\tau = \tilde{\mu}\cdot\exp(b_{(0)})$.

By Proposition 5.4.1 of [FLM], we also have an exact sequence

$$1 \to \mathrm{hom}(L, \mathbb{Z}/2\mathbb{Z}) \to O(\hat{L}) \to O(L) \to 1.$$

Let $N(V_L) = \left\langle \exp(a_{(0)}) \mid a \in (V_L)_1 \right\rangle$ be the subgroup generated by the linear automorphisms $\exp(a_{(0)})$. Since

$$\sigma \exp(a_{(0)})\sigma^{-1} = \exp((\sigma a)_{(0)})$$

for any $\sigma \in \mathrm{Aut}\,(V_L)$, $N(V_L)$ is a normal subgroup of $\mathrm{Aut}\,(V_L)$.

Theorem 2.2 ([DN99]). *Let L be a positive definite even lattice. Then*

$$\mathrm{Aut}\,(V_L) = N(V_L)\,O(\hat{L})$$

Moreover, the intersection $N(V_L)\cap O(\hat{L})$ contains a subgroup $\mathrm{hom}(L, \mathbb{Z}/2\mathbb{Z})$ and the quotient $\mathrm{Aut}\,(V_L)/N(V_L)$ is isomorphic to a quotient group of $O(L)$.

Remark 2.3. *If $L(2)=\emptyset$, then $(V_L)_1 = span\{\alpha(-1)\mathbf{1} \mid \alpha \in L\}$. In this case, the normal subgroup $N(V_L)=\{\exp(\lambda\alpha(0)) \mid \alpha \in L,\ \lambda \in \mathbb{C}\}$ is abelian and we have $N(V_L)\cap O(\hat{L}) = \hom(L,\mathbb{Z}/2\mathbb{Z})$ and*

$$\mathrm{Aut}\,(V_L)/N(V_L) \cong O(L).$$

In particular, we have an exact sequence

$$1 \to N(V_L) \to \mathrm{Aut}\,(V_L) \overset{\varphi}{\to} O(L) \to 1. \tag{1}$$

Note also that $\exp(\lambda\alpha_{(0)})$ acts trivially on $M(1)$ and $\exp(\lambda\alpha(0))e^\beta = \exp(\lambda\langle\alpha,\beta\rangle)e^\beta$ for any $\lambda\in\mathbb{C}$ and $\alpha,\beta\in L$.

The following theorem can also be proved by the same argument as in [LY14, Theorem 5.15].

Theorem 2.4. *Let L be an even positive definite lattice with $L(2)=\emptyset$. Let g be a fixed point free isometry of L of prime order p and $\hat{g}$ a lift of g in $O(\hat{L})$. Then we have the following exact sequences.*

$$1 \longrightarrow \hom(L/(1-g)L,\mathbb{Z}_p) \longrightarrow N_{\mathrm{Aut}\,(V_L)}(\langle\hat{g}\rangle) \overset{\varphi}{\longrightarrow} N_{O(L)}(\langle g\rangle) \longrightarrow 1;$$
$$1 \longrightarrow \hom(L/(1-g)L,\mathbb{Z}_p) \longrightarrow C_{\mathrm{Aut}\,(V_L)}(\hat{g}) \overset{\varphi}{\longrightarrow} C_{O(L)}(g) \longrightarrow 1.$$

3. Weight one space $(V_L)_1$

3.1. **Root lattice of type A.** We shall review some basic properties of the root lattices of type A_n. We use the *standard model* for A_n, i.e.,

$$A_n = \left\{ (x_1,x_2,\dots,x_{n+1}) \in \mathbb{Z}^{n+1} \,\middle|\, \sum_{i=1}^{n+1} x_i = 0 \right\}.$$

Then the roots of A_n are given by

$$\{\pm(v_i - v_j) \mid 1 \le i < j \le n+1\},$$

where $\{v_1=(1,0,\dots,0),\dots,v_{n+1}=(0,0,\dots,1)\}$ is the standard basis of $\mathbb{Z}^{n+1}$.

Let $\alpha_i = v_i - v_{i+1}$, $i=1,\dots,n$. Then

$$\underset{\alpha_1}{\circ}\!\!-\!\!\!-\!\!\!-\underset{\alpha_2}{\circ}\!\!-\!\!\!-\!\!\!- \cdots -\!\!\!-\!\!\!-\underset{\alpha_{n-1}}{\circ}\!\!-\!\!\!-\!\!\!-\underset{\alpha_n}{\circ}$$

is a fundamental root system of type A_n

Recall that $(A_n^*)/A_n \cong \mathbb{Z}_{n+1}$. Let $\gamma_{A_n}(0) = 0$ and

$$\gamma_{A_n}(j) = \frac{1}{n+1}\left(-(n+1-j)\sum_{i=1}^{j} v_i + j\sum_{i=j+1}^{n+1} v_i\right), \text{ for } j = 1, \dots, n.$$

Then $\gamma_{A_n}(j) \in A_n^*$. In fact, $\{\gamma_{A_n}(0), \gamma_{A_n}(1), \dots, \gamma_{A_n}(n)\}$ forms a transversal of A_n in A_n^* [CS, Chapter 4]. We also note that the norm of $\gamma_{A_n}(j)$ is equal to $j(n+1-j)/(n+1)$ for all $j = 0, \dots, n$.

Let h_{A_n} be an $(n+1)$-cycle in $Weyl(A_n) \cong Sym_{n+1}$. Note that h_{A_n} is a fixed point free isometry of A_n. Next we recall few well-known facts about h_{A_n} (cf. [GL12]).

Lemma 3.1. *For $j = 1, \dots, n$, $(h_{A_n} - 1)(\gamma_{A_n}(j))$ is a root.*

Lemma 3.2. *$(h_{A_n} - 1)A_n$ is rootless, i.e, it contains no elements of norm 2.*

Lemma 3.3. *Let A_n^* be the dual lattice of A_n. Then $(h_{A_n} - 1)A_n^* = A_n$*

Now consider the lattice VOA V_{A_n}. Then the weight one subspace $(V_{A_n})_1$ is a simple Lie algebra of type A_n. As it is well known, a Lie algebra $\mathcal{G}_{A_n}$ of type A_n is isomorphic to

$$sl(n+1, \mathbb{C}) = \{F \in M_{n+1,n+1}(\mathbb{C}) \mid \mathrm{tr}F = 0\}$$

and the set T of all diagonal matrices with trace 0 is a Cartan subalgebra. Under this identification, we have an isomorphism $\phi : sl(n+1, \mathbb{C}) \to (V_L)_1$ given by $\phi(E_{ii} - E_{jj}) = (v_i(-1)\mathbf{1} - v_j(1)\mathbf{1})$ and $\phi(E_{ij}) = e^{v_i - v_j}$, where E_{ij} denotes a matrix with 1 at (i,j)-entry and zero elsewhere. Note that $\phi(T) = M(1)_1$.

Let $\omega = e^{2\pi\sqrt{-1}/(n+1)}$. Set $D = \mathrm{diag}(\omega, \omega^2, ..., 1)$,

$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{n+1}} \begin{pmatrix} \omega & \omega^2 & \cdots & \omega^n & 1 \\ \omega^2 & \omega^4 & \cdots & \omega^{2n} & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \omega^n & \omega^{2n} & \ddots & \omega^{n^2} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Then the action of h_{A_n} on $\mathcal{G}$ is given by the conjugation of P, that is,

$$h_{A_n} : A \to P^{-1}AP \qquad \text{for } A \in sl(n+1, \mathbb{C})$$

and

$$B^{-1}PB = \mathrm{diag}(\omega, \omega^2, ..., 1)$$

is a diagonal matrix. Define a map $\sigma_{A_n} : sl(n+1, \mathbb{C}) \to sl(n+1, \mathbb{C})$ by $\sigma_{A_n}(A) = B^{-1}AB$. Since $\mathcal{C} = \mathcal{G}^{<\xi_{A_n}>} =< P, P^2, ..., P^n >$ and $\sigma_{A_n}(\mathcal{C}) = T$, $\mathcal{C}$ is another Cartan subalgebra. By a direct calculation, it follows that

$$\sigma_{A_n}\xi_{A_n}\sigma_{A_n}^{-1}(E_{ij}) = B^{-1}P^{-1}BE_{st}B^{-1}PB = \omega^{j-i}E_{ij}.$$

Let $\rho_{A_n} = \frac{1}{2}(n-1, n-2, \dots, -(n-2), -(n-1))$ and let $\eta_{A_n} = \exp(\frac{1}{n+1}(2\pi i \rho_{A_n}(0))$. Then the action of η_{A_n} on $\mathcal{G}$ is given by $\eta_{A_n} : A \mapsto DAD^{-1}$.

The following is easy to verify.

Lemma 3.4. *We have $\sigma_{A_n}h_{A_n}\sigma_{A_n}^{-1} = \eta_{A_n}$ and $\sigma_{A_n}\eta_{A_n}\sigma_{A_n}^{-1} = h_{A_n}^{-1}$ on $\mathcal{G}$.*

Remark 3.5. *Since $\eta_{A_n} = \exp(\frac{1}{n+1}(2\pi i \rho_{A_n}(0))$ and $\sigma_{A_n}h_{A_n}\sigma_{A_n}^{-1} = \eta_{A_n}$, we also have $h_{A_n} = \exp(\frac{1}{n+1}(2\pi i \sigma_{A_n}(\rho_{A_n})(0))$.*

4. Extra automorphisms

Let $R = A_{k_1} \oplus \cdots \oplus A_{k_j}$ be an orthogonal sum of simple root lattices of type A. Let L be an even overlattice of R. Let $\hat{\rho} = \sum_{i=1}^{j} \frac{1}{(k_i+1)}\rho_{A_{k_i}}$ and set

$$X = L(\hat{\rho}) = \{\alpha \in L \mid \langle \alpha, \hat{\rho} \rangle \in \mathbb{Z}\}.$$

Then $L = \mathrm{Span}_{\mathbb{Z}}(X \cup R)$.

Recall that the automorphisms h_{A_n}, η_{A_n} and σ_{A_n} of $sl_{n+1}(\mathbb{C})$ are inner. We can extend $h = h_{A_{k_1}} \otimes \cdots \otimes h_{A_{k_j}}$, $\eta = \eta_{A_{k_1}} \otimes \cdots \otimes \eta_{A_{k_j}}$ and $\sigma = \sigma_{A_{k_1}} \otimes \cdots \otimes \sigma_{A_{k_j}}$ to V_L by using the same exponential expressions. By Remark 3.5, Lemma 3.4 still holds in $\mathrm{Aut}\,(V_L)$.

Theorem 4.1. *We have $\sigma(V_X^h) = V_X^h$ and σ induces an automorphism of V_X^h. Moreover, σ is an extra automorphism in $\mathrm{Aut}\,(V_X^h)$.*

Proof. By definition, $\eta(e^\alpha) = e^\alpha$ for any $\alpha \in X$. Moreover, $h\eta h^{-1} = \eta$ on V_R because $(1-h)\rho_{A_n} < A_n^*$. Hence, h commutes with η on V_L. Then $V_X^h = V_L^{\langle h,\eta \rangle}$. Since $\sigma \langle h, \eta \rangle \sigma^{-1} = \langle h, \eta \rangle$ by the discussion above, we have $\sigma(V_X^h) = \sigma(V_L^{\langle h,\eta \rangle}) = V_L^{\langle h,\eta \rangle} = V_X^h$.

Finally, we note that $\sigma(V_X) = V_L^h \neq V_X$; hence, σ is not an automorphism of V_X. □

5. Explicit Examples

In this section, we will discuss several explicit examples. We are particularly interested in examples that can be realized as certain coinvariant sublattices of the Leech lattice.

Definition 5.1. *Let L be an integral lattice and $g \in O(L)$. We denote the fixed point sublattice of g by $L^g = \{x \in L \mid gx = x\}$. The coinvariant lattice of g is defined to be*

$$L_g = Ann_L(L^g) = \{x \in L \mid \langle x, y\rangle = 0 \textit{ for all } y \in L^g\}.$$

Next we recall the so-called Holy construction for the Leech lattice.

Let N be a Niemeier lattice with the root lattice $R = R_1 \oplus \cdots \oplus R_j$, where R_i's are simple root lattices of type A, D or E. Let ρ_i be a Weyl vector of R_i, that is the half sum of all positive roots. Set $\rho = \frac{1}{h}\sum_{i=1}^{j} \rho_i$ and define

$$N(\rho) = \{x \in N \mid \langle x, \rho\rangle \in \mathbb{Z}\},$$

where h is the Coxeter number of R_i. Let $\alpha \in \rho + N$ such that $\langle \alpha, \alpha\rangle \in 2\mathbb{Z}$. Then the lattice $\tilde{N}_\rho = \mathrm{Span}_{\mathbb{Z}} N \cup \{\alpha\}$ is isomorphic to the Leech lattice [CS, Chapter 24]. In particular, the Leech lattice contains a sublattice isometric to $R(\rho) = \{x \in R \mid \langle x, \rho\rangle \in \mathbb{Z}\}$.

Example 1: Let g be a $5B$-element of $O(\Lambda)$. Then the fixed point sublattice Λ^g has rank 8 and both Λ^g and Λ_g have discriminant 5^4. Moreover, Λ_g can be realized as a sublattice of $N(A_4^6)$ as follows:

Recall that the glue code for $N(A_4^6)/A_4^6$ is given by [1(01441)] and it has order 5^3. It contains a codeword [013024].

Let $L = \mathrm{Span}_{\mathbb{Z}}\{A_4^4, (\gamma(1), \gamma(3), \gamma(2), \gamma(4))\}$ and set $X = L(\hat{\rho})$. Then $L^*/L \cong 5^4$ and L can be regraded as a sublattice of Λ . In fact, $L \cong \Lambda_g$. The isometry group $O(L)$ is isomorphic to an index 2 subgroup of $Frob_{20} \times O_4^+(5)$ [GL11].

Theorem 5.2 ([La19])**.** *The automorphism group* $\mathrm{Aut}\,(V_L^h)$ *is isomorphic to an index* 2 *subgroup of* $O_6^+(5)$.

Example 2: Let g be a $7B$-element of $O(\Lambda)$. Then the fixed point sublattice Λ^g has rank 6 and both Λ^g and Λ_g have discriminant 7^3. We note that both lattices Λ^g and Λ_g can be realized as sublattices of the Niemeier lattice $N(A_6^4)$ with the root lattice A_6^4. Recall that the glue code for $N(A_6^4)/A_6^4$ is given by $[1(216)]$ and it has order 7^2. Therefore, (0124) is in the glue code.

Notice that the lattice $(1-h_{A_6})A_6$ has discriminant 7^3 since $[A_6 : (1-h_{A_6})A_6] = 7$. Indeed, $\Lambda^g \cong (1-h_{A_6})A_6$.

Let $L = \mathrm{Span}_{\mathbb{Z}}\{A_6^3, (\gamma(1), \gamma(2), \gamma(4))\}$ and set $X = L(\hat{\rho})$. Then X also has discriminant 7^3 and is contained in Λ and orthogonal to a sublattice isometric to $(1-h_{A_6})A_6$.

By Magma, it can be verified that $O(\Lambda_g)$ has the shape $7.3.(2.L_2(7).2)$.

Lemma 5.3. *Let g be a $7B$-element of $O(\Lambda)$. Then $N_{O(\Lambda)}(\langle g\rangle) \cong H.O_3(7)$, where H has order 21 and is a subgroup of $PSL_2(7)$.*

Theorem 5.4 ([La19a])**.** *Let g be a $7B$-element of $O(\Lambda)$. The group $\mathrm{Aut}\,(V_{\Lambda_g}^{\hat{g}})$ is isomorphic to an index 2 subgroup of the full orthogonal $O_5(7)$.*

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