Complex Hadamard matrices coming from association schemes

Takuya IKUTA
(joint work with Akihiro MUNEMASA)

Kobe Gakuin University

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X = a finite set, $n = |X|$.

complex Hadamard matrix

W: a square matrix of order n, $|W_{i,j}| = 1$ for $\forall i, j \in X$.
W: a complex Hadamard matrix
$$\overset{\text{def}}{\Longleftrightarrow} W\overline{W}^{\top} = nI.$$

(real: ± 1) Hadamard matrix $\subset$ complex Hadamard matrix.
A Hadamard matrix of order n exists for

$$n = 1, 2, 4, 8, 12, 16, \ldots, \text{(multiples of } 4), \ldots, 428(2004). \quad \underline{668}?$$

Conjecture

A Hadamard matrix of order n exists for any $n \equiv 0 \pmod 4$.

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix}.$$

J. Wallis, *Complex Hadamard matrices*,
Linear and Multilinear Algebra, 1 (3), (1973), 257–272.
entries: ± 1, $\pm i$,
In this talk, we allow entries in $\{\xi \in \mathbb{C} \mid |\xi| = 1\}$.
A complex Hadamard matrix is said to be Butson-type, if all of its entries are roots of unity.

$\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$: commutative d-class association scheme.
A_j: adjacency matrix $\longleftrightarrow R_j$.

Bose–Mesner algebra

- $\mathfrak{A} = \langle A_j \mid j = 0, \ldots, d\rangle \subset M_n(\mathbb{C})$, Bose–Mesner algebra, semi-simple,
  $\mathfrak{A} = \langle A_j \mid j = 0, \ldots, d\rangle = \langle E_j \mid j = 0, \ldots, d\rangle$,
  $\{E_j\}_{j=0}^d$: the set of the primitive idempotents, $E_0 = \frac{1}{n}J$,
- $(A_0, A_1, \cdots, A_d) = (E_0, E_1, \cdots, E_d)P$,

$$P = \left(\begin{array}{c|ccc} 1 & k_1 & \cdots & k_d \\ \hline 1 & & & \\ \vdots & & P_0 & \\ 1 & & & \end{array}\right),$$

P: the first eigenmatrix of $\mathfrak{X}$.

Our aim is to find both of

- a complex Hadamard matrix W
- an association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$

such that

$$\begin{aligned} W =& A_0 + w_1 A_1 + \cdots + w_d A_d \in \mathfrak{A}, \\ |w_1| &= \cdots = |w_d| = 1. \end{aligned}$$

Set

$$W = \sum_{j=0}^{d} w_j A_j \in \mathfrak{A}, \tag{1}$$

where $w_0 = 1$ and $|w_j| = 1$ for $\forall j \in \{1, \ldots, d\}$. Then

$$\begin{aligned} W &= \sum_{k=0}^{d} (\sum_{j=0}^{d} w_j P_{k,j}) E_k, \\ \overline{W}^{\top} &= \sum_{j=0}^{d} \frac{1}{w_j} A_{j'} \qquad (A_{j'} = A_j^{\top}) \\ &= \sum_{k=0}^{d} (\sum_{j=0}^{d} \frac{1}{w_j} P_{k,j'}) E_k. \end{aligned}$$

$$W = \sum_{k=0}^{d} (\sum_{j=0}^{d} w_j P_{k,j}) E_k, \quad \overline{W}^{\top} = \sum_{k=0}^{d} (\sum_{j=0}^{d} \frac{1}{w_j} P_{k,j'}) E_k.$$

$$W\overline{W}^{\top} = \sum_{k=0}^{d} \left((\sum_{j=0}^{d} P_{k,j} w_j)(\sum_{j=0}^{d} \frac{P_{k,j'}}{w_j}) \right) E_k.$$

Let $X_0 = 1$ and let X_j $(1 \leq j \leq d)$ be indeterminates. For $k = 0, 1, \ldots, d$, let e_k be the polynomial in $X_1, \ldots, X_d$ defined by

$$e_k = X_1 \cdots X_d \left(\left(\sum_{j=0}^{d} P_{k,j} X_j \right) \left(\sum_{j=0}^{d} \frac{P_{k,j'}}{X_j} \right) - n \right).$$

$$W = \sum_{j=0}^{d} w_j A_j \in \mathfrak{A}, \qquad (1)$$

Lemma 1

The following statements are equivalent.

(i) *the matrix W defined by (1) is a complex Hadamard matrix,*

(ii) *the sequence $(w_j)_{1 \leq j \leq d}$ is a common zero of e_k $(k = 1, \ldots, d)$.*

- $R = \mathbb{C}[X_1, \ldots, X_d]$,
- $\mathcal{I} = \text{ideal}\langle R \mid e_k(k = 1, \ldots, d)\rangle \Longrightarrow$ basis ?
- Magma(computations in algebra)

E. R. van Dam, *Three-class association schemes*,
J. Algebraic Combin. 10 (1999), 69–107,
$\sharp$ examples of symmetric 3 class association schemes $= 103$.

Appendix B

Four integral eigenvalues; excluded here are association schemes generated by SRG$\otimes J_n$, and the rectangular schemes $R(m,n)$, except the 6-cycle C_6 and the Cube.

v	spectrum	L_1	L_2	L_3	#	
6	$\{2, 1^2, -1^2, -2^1\}$	0 1 0	1 0 1	0 1 0	1	$C_6 \simeq R(3,2)$
	$\{2, -1, -1, 2\}$	1 0 1	0 1 0	1 0 0		DRG
	$\{1, -1, 1, -1\}$	0 2 0	2 0 0	0 0 0		Q-123

v	spectrum	L_1	L_2	L_3	#	
8	$\{3, 1^3, -1^3, -3^1\}$	0 2 0	2 0 1	0 1 0	1	Cube $\simeq R(4,2)$
	$\{3, -1, -1, 3\}$	2 0 1	0 2 0	1 0 0		DRG
	$\{1, -1, 1, -1\}$	0 3 0	3 0 0	0 0 0		Q-123
15	$\{4, 2^5, -1^4, -2^5\}$	1 2 0	2 4 2	0 2 0	1	L(Petersen)
	$\{8, -2, -2, 2\}$	1 2 1	2 4 1	1 1 0		DRG, R_2 SRG
	$\{2, -1, 2, -1\}$	0 4 0	4 4 0	0 0 1		
20	$\{9, 3^5, -1^9, -3^5\}$	4 4 0	4 4 1	0 1 0	1	$J(6,3)$
	$\{9, -3, -1, 3\}$	4 4 1	4 4 0	1 0 0		$R_1 \simeq R_2$ DRG
	$\{1, -1, 1, -1\}$	0 9 0	9 0 0	0 0 0		Q-123, Q-321
27	$\{6, 3^6, 0^{12}, -3^8\}$	1 4 0	4 4 4	0 4 4	1	$H(3,3)$
	$\{12, 0, -3, 3\}$	2 2 2	2 5 4	2 4 2		DRG
	$\{8, -4, 2, -1\}$	0 3 3	3 6 3	3 3 1		Q-123
27	$\{8, 2^{12}, -1^8, -4^6\}$	1 6 0	6 8 2	0 2 0	2	GQ(2,4)\spread
	$\{16, -2, -2, 4\}$	3 4 1	4 10 1	1 1 0		R_1 DRG, R_2 SRG
	$\{2, -1, 2, -1\}$	0 8 0	8 8 0	0 0 1		

	(symmetric) construction
$d = 3$	*Complex Hadamard matrices contained in a Bose–Mesner algebra*, Spec. Matrices, 3 (2015), 91–110.
$d = 4$	*Complex Hadamard matrices attached to even orthogonal scheme of class 4*, (2016), submitted.

not *Butson!*

$e \geq 3$: an *odd* positive integer

$$\mathrm{GF}(2^e) = \frac{\mathrm{GF}(2)[x]}{(\varphi(x))},$$

$\varphi(x)$: a primitive polynomial of degree e over $\mathrm{GF}(2)$

$$\mathrm{GF}(2^e)^\times = \langle \zeta \rangle$$

$$\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$$

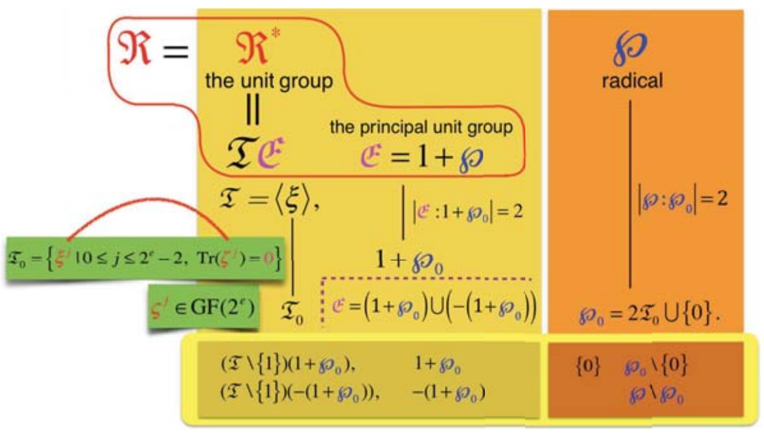
$\exists \Phi(x)$: a monic polynomial of degree e over $\mathbb{Z}_4$ s.t.

$$\begin{cases} \Phi(x) \equiv \varphi(x) \mod 2\mathbb{Z}_4[x], \\ \Phi(x) \mid x^{2^e-1} - 1 \text{ in } \mathbb{Z}_4[x]. \end{cases}$$

$$\mathfrak{R} = \frac{\mathbb{Z}_4[x]}{(\Phi(x))}$$: *Galois ring*

$\wp = 2\mathfrak{R}$. radical

$$|\mathfrak{R}| = 4^e \qquad |\wp| = 2^e$$



$\mathfrak{R}^*$ $\qquad$ $\wp$

$S_1 = (\mathfrak{T} \setminus \{1\})(1 + \wp_0)$, $\quad S_3 = 1 + \wp_0$,

$S_2 = (\mathfrak{T} \setminus \{1\})(-(1 + \wp_0))$, $\quad S_4 = -(1 + \wp_0)$.

$S_0 = \{0\}$, $S_5 = \wp_0 \setminus \{0\}$,

$S_6 = \wp \setminus \wp_0$.

We've constructed a commutative nonsymmetric association scheme $\mathfrak{X}$ of class 6 on Galois rings of characteristic 4, whose first eigenmatrix is given by

$$(p_{i,j})_{\substack{0\le i\le 6\\ 0\le j\le 6}} = \begin{pmatrix} 1 & 2b(b-1) & 2b(b-1) & b & b & b-1 & b \\ 1 & bi & -bi & 0 & 0 & -1 & 0 \\ 1 & -bi & bi & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & bi & -bi & b-1 & -b \\ 1 & 0 & 0 & -bi & bi & b-1 & -b \\ 1 & -2b & -2b & b & b & b-1 & b \\ 1 & 0 & 0 & -b & -b & b-1 & b \end{pmatrix},$$

where b is a power of 4.

Theorem 2 (A. Munemasa and T. I.)

Let $w_0 = 1$ and w_j $(1 \le j \le 6)$ be complex numbers of absolute value 1. Set

$$W = \sum_{j=0}^{6} w_j A_j \in \mathfrak{A},$$

and assume that W is hermitian.
Then, W is a complex Hadamard matrix $\overset{\text{iff}}{\Longleftrightarrow}$

$$W = A_0 + \epsilon_1 i(A_1 - A_2) + \epsilon_2 i(A_3 - A_4) + A_5 + A_6, \quad \textit{or} \qquad (2)$$

$$W = A_0 + \epsilon_1 i(A_1 - A_2) + \epsilon_2 (A_3 + A_4) + A_5 - A_6, \qquad (3)$$

for some $\epsilon_1, \epsilon_2 \in \{\pm 1\}$.

Let $\mathfrak{X}$ be an association scheme given in Theorem 2. Fusion schemes of $\mathfrak{X}$ with at least three classes are listed in Table 1.

	fused relations	class	nonsymmetirc or symmetric
$\mathfrak{X}_1$	$\{1,2\}$	5	nonsymmetric
$\mathfrak{X}_2$	$\{3,4\}$	5	nonsymmetric
$\mathfrak{X}_3$	$\{1,2\},\{3,4\}$	4	symmetric
$\mathfrak{X}_4$	$\{3,4,6\}$	4	nonsymmetric
$\mathfrak{X}_5$	$\{1,2\},\{3,4\},\{5,6\}$	3	symmetric
$\mathfrak{X}_6$	$\{1,2,3,4\}$	3	symmetric
$\mathfrak{X}_7$	$\{1,3\},\{2,4\},\{5,6\}$	3	nonsymmetric
$\mathfrak{X}_8$	$\{1,4\},\{2,3\},\{5,6\}$	3	nonsymmetric

Table: Fusion schemes of $\mathfrak{X}$

$\mathfrak{X} = (X, \{R_i\}_{i=0}^{3})$: a 3-class commutative nonsymmetric association scheme with the first eigenmatrix

$$\begin{pmatrix} 1 & \frac{k_1}{2} & \frac{k_1}{2} & k_2 \\ 1 & \frac{1}{2}(r+bi) & \frac{1}{2}(r-bi) & -(r+1) \\ 1 & \frac{1}{2}(r-bi) & \frac{1}{2}(r+bi) & -(r+1) \\ 1 & \frac{s}{2} & \frac{s}{2} & -(s+1) \end{pmatrix}, \tag{4}$$

where

- k_1 is an even positive integer, $k_2 \in \mathbb{Z}$,
- $r, s \in \mathbb{Z}$,
- $b \in \mathbb{R}$ and $b > 0$,
- $i^2 = -1$.

$\mathfrak{A} = \langle A_0, A_1, A_2, A_3 \rangle$: Bose–Mesner algebra of $\mathfrak{X}$ which is the linear span of the adjacency matrices A_0, A_1, A_2, A_3 of $\mathfrak{X}$, where $A_1^\top = A_2$, A_3 symmetric.

S. Y. Song showed the following.

Lemma 3 (S. Y. Song, 1995)

For the matrix (4), *one of the following holds.*

(i) $(r,s,b^2)=(0,-(k_2+1),\frac{k_1(k_2+1)}{k_2})$, $m_1=\frac{(k_1+k_2+1)k_2}{2(k_2+1)}$,

(ii) $(r,s,b^2)=(-(k_2+1),0,(k_2+1)(k_1+k_2+1))$, $m_1=\frac{k_1}{2(k_2+1)}$,

(iii) $(r,s,b^2)=(-1,k_1,k_1+1)$, $m_1=\frac{(k_1+k_2+1)k_1}{k_1+1}$.

In Lemma 3,

- (i) and (ii) are nonsymmetric fissions of a complete multipartite graph,
- (i) is self-dual, and (ii) is non self-dual.
- (iii) is a nonsymmetric fission of a disjoint union of complete graphs.

w_1, w_2, w_3: complex numbers of absolute value 1.
We assume that $w_1 \neq w_2$, and set

$$W = A_0 + w_1 A_1 + w_2 A_2 + w_3 A_3 \in \mathfrak{A}. \tag{5}$$

Theorem 4 (A. Munemasa and T. I.)

The matrix (5) *is a complex Hadamard matrix* $\overset{\text{iff}}{\Longleftrightarrow}$ $(k_1, k_2, r, s, b) = (2a(2a-1)c, 2a-1, 0, -2a, 2a\sqrt{c})$ *for some positive integers* a, c*, and one of the following holds.*

(i) $c = 1$, *and*

- (a) $(w_1, w_2, w_3) = (w, -w, 1)$ *with* $|w| = 1$,
- (b) $(w_1, w_2, w_3) = (w^{\pm}, w^{\mp}, w^{\pm}w^{\mp},)$, *where*

$$w^{\pm} = \frac{-(a-1) - ai \pm ((2a-1)i - 1)\zeta\sqrt{a(a-1)}}{2a^2 - 2a + 1},$$

  ζ *is a primitive* 8*-th root of unity, and* $i = \zeta^2$,
- (c) $a = 2$, $(w_1, w_2, w_3) = (\frac{3\pm 4i}{5}, -1, \frac{-3\mp 4i)}{5}), (-1, \frac{3\pm 4i}{5}, \frac{-3\mp 4i}{5})$,

(ii) $a = 1$, $c = 3$, *and*

- (d) $(w_1, w_2, w_3) = (\frac{1\pm 2\sqrt{2}i}{3}, -1, 1), (-1, \frac{1\pm 2\sqrt{2}i}{3}, 1)$,
- (e) $(w_1, w_2, w_3) = (\pm i, -1, \mp i), (-1, \pm i, \mp i)$.