Note on Symmetric Group and Classical Invariant Theory

Nur Hamid, Masashi Kosuda, Manabu Oura

Abstract

In this paper, we construct the analogue theory of Eisenstein series in classical invariant theory. The groups appearing from the construction are also investigated.

1 Introduction

Eisenstein series can give very concrete example of modular forms. By corresponding combinatorics and modular forms, we introduced the concept of E-polynomials.

On the other hand, classical invariant theory plays important roles in many branches of mathematics. Here we show a construction of the analogue theory of Eisenstein series. We give an investigation of graded and centralizer ring that appear.

2 Classical Invariant Theory

We begin the discussion by a ground form of degree m

$$f = \sum_{i=0}^{m} u_i \binom{m}{i} \xi_1^{m-i} \xi_2^{i}$$

for a positive number m. While $\xi_1,\ \xi_2$ are transformed according to

$$(\xi_1\ \xi_2) = (\xi_1'\ \xi_2')A \text{ (”contragrediently”)},$$

f changes into a form of the new variables $\xi_1',\ \xi_2'$ with the coefficients $u_0',\ u_1',\ \ldots,\ u_m'$ where

$$\begin{pmatrix} u_0' \\ u_1' \\ \vdots \\ u_m' \end{pmatrix} = \big(A\big)_m \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

To shorten, we write $u' = \big(A\big)_m u$.

We operate $SL(2, \mathbf{C})$ on $\mathbf{C}[u] = \mathbf{C}[u_0, u_1, \ldots, u_m]$ by the above representation and consider the invariant subring $S(2, m)$ defined by:

$$S(2, m) := \{J \in \mathbf{C}[u] :\ J(u') = J(u),\ {}^{\forall}A \in SL(2, \mathbf{C})\}.$$

It is known that $S(2,m)$ is of finite type over $\mathbf{C}$ and here we consider only invariants of even degree and denote it by $S(2,m)^e$.

In order to obtain the useful construction of invariants, we shall interpret the ground form as

$$f = u_0 \prod_{i=1}^{m} (\xi_1 - \varepsilon_i \xi_2)\,.$$

The fundamental theorem of symmetric functions gives the explicit relations between u_is and ε_is. At any rate, the following lemma is a construction of invariants we expected (*cf.* [2]).

Lemma 1 *An expression of the form*

$$u_0^r \sum (\varepsilon_i - \varepsilon_j)(\varepsilon_k - \varepsilon_l)\ldots,$$

in which every ε_i appears r times in each product and which is symmetric in $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ can be considered as an invariant of degree r.

3 Preliminaries

Let g be a positive integer. We start with a ground form of degree $2g+2$

$$\begin{aligned} f &= \sum_{i=0}^{2g+2} u_i \binom{2g+2}{i} \xi_1^{(2g+2)-i} \xi_2^i \\ &= u_0 \prod_{i=1}^{2g+2} (\xi_1 - \varepsilon_i \xi_2)\,. \end{aligned}$$

We would like to concentrate on one type of invariants we shall define now. We fix the following polynomial

$$\varphi_{2n} = u_0^{2n} (\varepsilon_1 - \varepsilon_2)^{2n} (\varepsilon_3 - \varepsilon_4)^{2n} \ldots (\varepsilon_{2g+1} - \varepsilon_{2g+2})^{2n}.$$

We denote by G the symmetric group of degree $2g+2$. The group G acts on the polynomial ring $\mathbf{C}[\varepsilon_1, \ldots, \varepsilon_{2g+2}]$ as $F(\ldots, \varepsilon_i, \ldots)^\sigma = F(\ldots, \varepsilon_{i^\sigma}, \ldots)$. The stabilizer $G_{\varphi_{2n}}$ of φ_{2n} is defined by the elements of G that do not move φ_{2n}.

Proposition 2 *The group $G_{\varphi_{2n}}$ can be generated by the $(g+1)+2$ elements*

$$\begin{gathered} (1\ 2), (3\ 4), \ldots, (2g+1\ 2g+2), \\ (1\ 3)(2\ 4), (1\ 3\ 5\ \ldots\ 2g+1)(2\ 4\ \ldots\ 2g+2) \end{gathered}$$

and is isomorphic to $C_2^{g+1} \rtimes S_{g+1}$. In particular, $G_{\varphi_{2n}}$ does not depend on n.

We denote by K the group $G_{\varphi_{2n}}$ and by κ the cardinality of $K \backslash G$.

4 Result

In this section, we investigate the subring of $S(2, 2g+2)$.

We set

$$\psi_{2n} = \sum_{K\backslash G \ni \sigma} \varphi_{2n}^{\sigma},$$

which is actually an element of degree $2n$ in $S(2, 2g+2)$ by Lemma 1. We shall denote by A_g the ring generated by ψ_{2n} $(n = 1, 2, \dots)$ over $\mathbf{C}$. The ring A_g is a subring of the invariant ring $S(2, 2g+2)$.

Theorem 3 *The ring A_g is finitely generated over $\mathbf{C}$ and generated by $\psi_2, \psi_4, \dots, \psi_{2\kappa}$.*

Theorem 4 *(1) A_1 is generated by ψ_2, ψ_6 and coincides with $S(2,4)^e$.*

(2) A_2 is generated by $\psi_2, \psi_4, \psi_6, \psi_{10}$ and coincides with $S(2,6)^e$.

(3) A_3 is strictly smaller than $S(2,8)^e$.

Now we explore combinatorial properties of the permutation group arising from the action of G on $\Omega = K\backslash G$. Define a permutation group $\mathcal{G}$ as a representation of G on Ω. Let G_1 be the stabilizer of a point 1. For each orbit Δ, we define an adjacency matrix $\mathfrak{P}(\Delta) = (v)^{\Delta}_{\alpha,\beta}$ by

$$v^{\Delta}_{\alpha,\beta} = \begin{cases} 1 & \exists g \text{ such that } 1^g = \beta \text{ and } \alpha^{g^{-1}} \in \Delta \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Denote the matrices $\mathfrak{P}(\Delta)$ by $A_0 = I, A_1, ..., A_d$ where d is class of association scheme. It is known that the matrices $A_0 = I, A_1, ..., A_d$ generate an algebra, called *bose-Mesner* algebra $\mathfrak{A}$ of the association scheme $\mathfrak{X}$.

Denote $\mathfrak{A}^{(k)}$ be the centralizer of algebra of the $k-$th tensor representation of G. We have the following theorem.

Theorem 5 *For $g = 2$, we have that*

$$\mathfrak{A}^{(k)} \cong \begin{cases} 3\mathcal{M}_1 & k = 1 \\ \bigoplus_{i\in I} \mathcal{M}_i & k \geq 2 \end{cases}$$

$$\begin{aligned} \overrightarrow{d^{(k)}} &= \overrightarrow{d^{(k-1)}} A \\ &= (a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, i_k, j_k, k_k), \end{aligned}$$

where $I = \{a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, i_k, j_k, l_k\}$ and

$$a_k = \frac{1}{48}(15^{k-1} - 7^k + 7.3^{k-2} - 8) \qquad b_k = \frac{1}{48}(5.15^{k-1} + 7^k - 24.3^{k-2} - 4)$$
$$c_k = \frac{1}{16}(3.15^{k-1} - 7^k + 4) \qquad d_k = \frac{1}{48}(5.15^{k-1} + 3.7^k + 66.3^{k-2})$$
$$e_k = \frac{1}{24}(5.15^{k-1} + 7^k - 24.3^{k-2} - 4) \qquad f_k = \frac{1}{3}(15^{k-1} - 3^{k-1})$$
$$g_k = \frac{1}{48}(5.15^{k-1} - 3.7^k + 84.3^{k-2} - 12 \qquad h_k = \frac{1}{24}(5.15^{k-1} - 7^k - 6.3^{k-2} + 4)$$
$$i_k = \frac{1}{16}(3.15^{k-1} + 7^k + 2.3^k) \qquad j_k = \frac{1}{48}(5.15^{k-1} - 7^k + 30.3^{k-2} - 8)$$
$$k_k = \frac{1}{48}(15^{k-1} + 7^k + 60.3^{k-2} + 20)$$

Corollary 6 *We have that*

1. $\mathfrak{A}^{(k)}$ *is commutative if and only if* $k = 1$.
2. *The dimension of* $\mathfrak{A}^{(k)}$ *can be obtained by*

$$\dim \mathfrak{A}^{(k)} = \frac{1}{720}(15^{2k} + 15.7^{2k} + 100.3^{2k} + 300).$$

We apply the Corollary 6 for $k = 1, .., 5$. The following table is the result.

k	1	2	3	4	5
$\dim \mathfrak{A}^{(k)}$	3	132	18373	3680582	806796423

References

[1] Brauer, R., On algebras which are connected with the semisimple continuous groups, Ann. of Math. (2) 38 (1937), no. 4, 857-872.

[2] Igusa, J., Arithmetic variety of moduli for genus two, Ann. of Math. (2) 72 1960 612-649.

[3] Igusa, J., On Siegel modular forms of genus two, Amer. J. Math. 84 1962 175-200.

[4] Igusa, J., Modular forms and projective invariants, Amer. J. Math. 89 1967 817-855.

[5] Schur, I., Vorlesungen über Invariantentheorie, Bearbeitet und herausgegeben von Helmut Grunsky. Die Grundlehren der mathematischen Wissenschaften, Band 143 Springer-Verlag, Berlin-New York 1968.

[6] Shioda, T., On the graded ring of invariants of binary octavics, Amer. J. Math. 89 1967 1022-1046.

[7] Weyl, H., The classical groups. Their invariants and representations. Princeton University Press, Princeton, N.J., 1939.