Nonoscillation of half-linear dynamic equations with mixed derivatives

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1 Introduction

We consider the nonlinear dynamic equations with mixed derivatives

$$\big(r(t)\Phi_p(x^{\Delta}(t))\big)^{\nabla} + c(t)\Phi_p(x(t)) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a time scale (arbitrary nonempty closed subset of the real numbers) unbounded above; $r : \mathbb{T} \to \mathbb{R}$ is continuous function and $r(t) > 0$ for all $t \in \mathbb{T}$; $c : \mathbb{T} \to \mathbb{R}$ is real left-dense continuous function; p is a parameter that is greater than 1; Φ_p is the real-valued function defined by $\Phi_p(u) = |u|^{p-2}u$ for $u \neq 0$ and $\Phi_p(0) = 0$. For simplicity, let q be the conjugate exponent of p; that is, the number $1/p + 1/q = 1$. Then, the function Φ_q is the inverse function Φ_p. Here, the term mixed derivatives represents the use of Δ-derivative [3] and ∇-derivative [2], introduced in:

$$x^{\Delta}(t) := \lim_{s\to t} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s} \quad \text{and} \quad x^{\nabla}(t) := \lim_{s\to t} \frac{x(\rho(t)) - x(s)}{\rho(t) - s},$$

where $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the forward jump operator; $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ is the backward jump operator. Also, the graininess function $\mu, \nu : \mathbb{T} \to [0, \infty)$ are called *forward graininess* and *backward graininess* respectively, and are defined by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t).$$

A point $t \in \mathbb{T}$ is said to be *right-dense* if $\mu(t) = 0$, and it is said to be *right-scattered* if $\mu(t) > 0$. Similarly, a point $t \in \mathbb{T}$ is said to be *left-dense* if $\nu(t) = 0$, and it is said to be *left-scattered* if $\nu(t) > 0$. We will use abbreviations rd, rs, ld and ls respectively. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{M\}$, otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then we define $\mathbb{T}_{\kappa} = \mathbb{T} \setminus \{m\}$, otherwise $\mathbb{T}_{\kappa} = \mathbb{T}$. By these definitions, we have

$$x^{\Delta}(t) = x'(t) = x^{\nabla}(t)$$

if $\mathbb{T} = \mathbb{R}$, while

$$x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t) \quad \text{and} \quad x^{\nabla}(t) = \nabla x(t) = x(t) - x(t-1)$$

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if $\mathbb{T} = \mathbb{Z}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if it is right continuous at all rd points and the left limit at ld points exists. If f is rd-continuous, then there exists a Δ-differentiable function F such that $F^{\Delta}(t) = f(t)$. While a function $g : \mathbb{T} \to \mathbb{R}$ is said to be *ld-continuous* if it is left continuous at all ld points and the right limit at rd points exists. If g is ld-continuous, then there exists a ∇-differentiable function G such that $G^{\nabla}(t) = g(t)$. The Δ-integral and the ∇-integral are defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{and} \quad \int_a^b g(t)\nabla t = G(b) - G(a).$$

In particular, if $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt = \int_a^b f(t)\nabla t,$$

while if $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t) \quad \text{and} \quad \int_a^b f(t)\nabla t = \sum_{t=a+1}^{b} f(t).$$

For the case $p = 2$, equation (1.1) turns out to be

$$\left(r(t)x^{\Delta}(t)\right)^{\nabla} + c(t)x(t) = 0, \tag{1.2}$$

i.e. the linear dynamic equation. It is well known that the solution space of any linear dynamic equation is homogeneous and additive. In contrast, the solution space of (1.1) has just one half of the above properties, namely homogeneity (but not additivity). For this reason, equations such as (1.1) are called *half-linear*. It was shown that equation (1.1) is very convenient since it transforms to the usual half-linear differential equation

$$(r(t)\Phi_p(x'(t)))' + c(t)\Phi_p(x(t)) = 0 \tag{1.3}$$

if $\mathbb{T} = \mathbb{R}$, while it transforms to the half-linear difference equation

$$\Delta(r(t-1)\Phi_p(\Delta x(t-1))) + c(t)\Phi_p(x(t)) = 0 \tag{1.4}$$

if $\mathbb{T} = \mathbb{Z}$. We can easily find the literatures related to oscillation theory for (1.3) and (1.4) (for example, see [6, 7]).

Now we introduce the definition on oscillation and nonoscillation of (1.1).

Definition 1.1. We say that a solution x of (1.1) has a *generalized zero* at t if $x(t) = 0$ or, if t is left-scattered and $x(\rho(t))x(t) < 0$.

Definition 1.2. We say that (1.1) is *disconjugate* on an interval $[a, b]$ if the following hold:

(i) If x is a non-trivial solution of (1.1) with $x(a) = 0$, then x has no generalized zero in $(a, b]$.

(ii) If x is a non-trivial solution of (1.1) with $x(a) \neq 0$, then x has at most one generalized zero in $(a, b]$.

Definition 1.3. Let $\omega = \sup \mathbb{T}$, and if $\omega < \infty$, assume $\rho(\omega) = \omega$. Let $a \in \mathbb{T}$. We say that (1.1) is *oscillatory* on $[a, \omega)$ if every non-trivial solution has infinitely many generalized zero in $[a, \omega)$. We say (1.1) is *nonoscillatory* on $[a, \omega)$ if it is not oscillatory on $[a, \omega)$.

The use of mix derivatives such as equation (1.2) was considered by Messer [8], Anderson and Hall [1] for oscillation problem. In extension, Došlý and Marek [5] studied the half-linear equation (1.1) and its oscillatory properties. For example, Došlý and Marek [5] have presented the following nonoscillation theorem for (1.1).

Theorem A. *Suppose that* $\int_{t_0}^{\infty} \big(r(\rho(t))\big)^{1-q} \nabla t = \infty$, $\int_{t_0}^{\infty} c(t) \nabla t < \infty$ *and*

$$\lim_{t \to \infty} \frac{\nu(t)\big(r(\rho(t))\big)^{1-q}}{\int_{t_0}^{\rho(t)} \big(r(\rho(s))\big)^{1-q} \nabla s} = 0. \tag{1.5}$$

If

$$\liminf_{t \to \infty} A_p(\rho(t)) > -\frac{2p-1}{p} \left(\frac{p-1}{p}\right)^{p-1}$$

and

$$\limsup_{t \to \infty} A_p(\rho(t)) < \frac{1}{p} \left(\frac{p-1}{p}\right)^{p-1},$$

then all non-trivial solutions of (1.1) *are nonoscillatory, where*

$$A_p(\rho(t)) = \left(\int_{t_0}^{\rho(t)} \big(r(\rho(s))\big)^{1-q} \nabla s\right)^{p-1} \left(\int_{\rho(t)}^{\infty} c(s) \nabla s\right).$$

In Theorem A, Došlý and Marek [5] established a nonoscillation criterion by considering the lower boundary value

$$\liminf_{t \to \infty} A_p(\rho(t)) > -\frac{2p-1}{p} \left(\frac{p-1}{p}\right)^{p-1}$$

and other conditions. For the case $p = 2$, the lower boundary value is

$$\liminf_{t \to \infty} \left(\int_{t_0}^{\rho(t)} \frac{1}{r(\rho(s))} \nabla s\right) \left(\int_{\rho(t)}^{\infty} c(s) \nabla s\right) > -\frac{3}{4}.$$

The purpose of this talk is to report the extended result of Theorem A. We focus on finding the conditions that will extend the lower boundary value.

Our nonoscillation theorems are as follows.

Theorem 1.1. *Suppose that* $\int_{t_0}^{\infty}\big(r(\rho(t))\big)^{1-q}\nabla t=\infty$, $\int_{t_0}^{\infty}c(t)\nabla t<\infty$ *and* (1.5). *Let* $h(t)$ *be a* ∇*-differentiable, monotonically non-increasing and positive function. If there exists* $h(\rho(t))>0$ *such that*

$$\liminf_{t\to\infty} A_p(\rho(t)) > -(h(\rho(t)))^{\frac{1}{q}} - h(\rho(t)) \tag{1.6}$$

and

$$\limsup_{t\to\infty} A_p(\rho(t)) < (h(\rho(t)))^{\frac{1}{q}} - h(\rho(t)), \tag{1.7}$$

then all non-trivial solutions of (1.1) *are nonoscillatory, where*

$$A_p(\rho(t)) = \left(\int_{t_0}^{\rho(t)} \big(r(\rho(s))\big)^{1-q}\nabla s\right)^{p-1}\left(\int_{\rho(t)}^{\infty} c(s)\nabla s\right).$$

Theorem 1.2. *Suppose that* $\int_{t_0}^{\infty}\big(r(\rho(t))\big)^{1-q}\nabla t<\infty$ *and*

$$\lim_{t\to\infty}\frac{\nu(t)\big(r(\rho(t))\big)^{1-q}}{\int_{\rho(t)}^{\infty}\big(r(\rho(s))\big)^{1-q}\nabla s}=0. \tag{1.8}$$

Let $h(t)$ *be a* ∇*-differentiable, monotonically non-decreasing and positive function. If there exists* $h(\rho(t))>0$ *such that*

$$\liminf_{t\to\infty} B_p(\rho(t)) > -(h(\rho(t)))^{\frac{1}{q}} - h(\rho(t)) \tag{1.9}$$

and

$$\limsup_{t\to\infty} B_p(\rho(t)) < (h(\rho(t)))^{\frac{1}{q}} - h(\rho(t)), \tag{1.10}$$

then all non-trivial solutions of (1.1) *are nonoscillatory, where*

$$B_p(\rho(t)) = \left(\int_{\rho(t)}^{\infty} \big(r(\rho(s))\big)^{1-q}\nabla s\right)^{p-1}\left(\int_{t_0}^{\rho(t)} c(s)\nabla s\right).$$

Let us compare Theorem 1.1 with Theorem A. In the case that $h(\rho(t))\equiv(\frac{p-1}{p})^p$, by using $p/q=p-1$, we have the upper boundary value of

$$(h(\rho(t)))^{\frac{1}{q}} - h(\rho(t)) = \left(\frac{p-1}{p}\right)^{p-1}\left(1-\frac{p-1}{p}\right) = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$$

and the lower boundary value of

$$-(h(\rho(t)))^{\frac{1}{q}} - h(\rho(t)) = -\left(\frac{p-1}{p}\right)^{p-1}\left(1+\frac{p-1}{p}\right) = -\frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1}.$$

Hence, the condition of Theorem 1.1 becomes Theorem A. For the case $p = 2$, from Theorem A, we have

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\frac{3}{4} = -0.75 \quad \text{and} \quad \limsup_{t\to\infty} A_p(\rho(t)) < \frac{1}{4} = 0.25.$$

In the case that $p = 2$, from Theorem 1.1 ((1.6) and (1.7)), we assume that there exists $h(\rho(t)) \equiv k$ (positive constant) such that

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\sqrt{k} - k \quad \text{and} \quad \limsup_{t\to\infty} A_p(\rho(t)) < \sqrt{k} - k \le \frac{1}{4}.$$

Notice that there is parameter k remains, which gives us opportunity to get our desired value by setting it. If $k = 1/4$, then we have the same Došlý and Marek's result. As another example, we set $k = 1/2$, then we have

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\frac{\sqrt{2}+1}{2} \approx -1.207\cdots \quad \text{and} \quad \limsup_{t\to\infty} A_p(\rho(t)) < \frac{\sqrt{2}-1}{2} \approx 0.207\cdots.$$

We have the lower boundary value extended from -0.75 to $-1.207\cdots$. Therefore, we can conclude that by setting the parameter k, we can extend the lower boundary value. Moreover, in Theorem 1.2, we investigated the same boundary value with distinct conditions. Under those conditions, all non-trivial solutions of (1.1) are also nonoscillatory. However, since it is not the same conditions with Došlý and Marek's work, Theorem 1.1 and Theorem 1.2 can be considered as new results.

2 Proof of Theorems 1.1 and 1.2

We show some preliminary results that are used directly for proving the main results. The readers can find more preliminaries that support the proof in [5].

Lemma 2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function $g\colon \mathbb{T} \to \mathbb{R}$ be nabla differentiable. Then we have*

$$[f(g(t))]^\nabla = f'(\xi) g^\nabla(t),$$

where $g(\rho(t)) \le \xi(t) \le g(t)$.

Lemma 2.2. *Suppose that x is a solution of (1.1) such that $x(t) \neq 0$ in a time scale interval $\mathbb{I} = [a, b]$. Then $w = r\Phi_p(x^\nabla/x)$ is a solution of the Riccati-type equation*

$$w^\nabla(t) + c(t) = \begin{cases} -(p-1)\frac{|w(t)|^q}{\Phi_q(r(t))} & \text{if } \rho(t) = t, \\ -\frac{w(\rho(t))}{\nu(t)}\left(1 - \frac{r(\rho(t))}{\Phi_p\left(\Phi_q(r(\rho(t))) + \nu(t)\Phi_q(w(\rho(t)))\right)}\right) & \text{if } \rho(t) < t. \end{cases} \tag{2.1}$$

Moreover, if

$$x(\rho(t))x(t) > 0$$

for $t \in [a, b]_k$, holds, then

$$\Phi_q(r(\rho(t))) + \nu(t)\Phi_q(w(\rho(t))) > 0 \quad \text{for} \quad t \in [a, b]_k. \tag{2.2}$$

We will denote $\mathcal{R}[w]$, the so-called Riccati operator (compare (2.1)), i.e.,

$$\mathcal{R}[w] := \begin{cases} w^{\nabla}(t) + c(t) + (p-1)(r(t))^{1-q}|w(t)|^q & \text{if } \rho(t) = t, \\ w^{\nabla}(t) + c(t) + \frac{w(\rho(t))}{\nu(t)}\left(1 - \frac{r(\rho(t))}{\Phi_p\left(\Phi_q(r(\rho(t))) + \nu(t)\Phi_q\left(w(\rho(t))\right)\right)}\right) & \text{if } \rho(t) < t. \end{cases}$$

Lemma 2.3. *Equation* (1.1) *is nonoscillatory if and only if there exists a* ∇*-differentiable function* w *satisfying* (2.2) *such that* $\mathcal{R}[w] \leq 0$ *for large* t*.*

In other words, we need only one function w and establish $\mathcal{R}[w] \leq 0$ for each case (left scattered case, and left dense case) to prove our main theorems.

Proof of Theorem 1.1. We denote

$$\tilde{r}(t) := r(\rho(t)), \quad \tilde{w}(t) := w(\rho(t)),$$

and

$$A_p(t) := \left(\int_0^t (\tilde{r}(s))^{1-q}\nabla s\right)^{p-1}\left(\int_t^{\infty} c(s)\nabla s\right).$$

Let

$$w(t) = h(t)\left(\int_0^t (\tilde{r}(s))^{1-q}\nabla s\right)^{p-1} + \int_t^{\infty} c(s)\nabla s.$$

By using Lemma 2.1, we can calculate

$$\left[\left(\int_0^t (\tilde{r}(s))^{1-q}\nabla s\right)^{1-p}\right]^{\nabla} = (1-p)(\tilde{r}(s))^{1-q}\,(\theta(t))^{-p},$$

where

$$\int_0^{\rho(t)} (\tilde{r}(s))^{1-q}\nabla s \leq \theta(t) \leq \int_0^t (\tilde{r}(s))^{1-q}\nabla s.$$

Also, by using Lagrange mean value, we have

$$\begin{aligned} &\frac{\tilde{w}(t)}{\nu(t)}\left(1 - \frac{\tilde{r}(t)}{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu(t)\Phi_q(\tilde{w}(t))\big)}\right) \\ &\quad = \frac{\tilde{w}(t)}{\nu(t)}\left(\frac{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu\Phi_q(\tilde{w}(t))\big) - \Phi_p(\Phi_q(\tilde{r}(t)))}{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu\Phi_q(\tilde{w}(t))\big)}\right) \\ &\quad = (p-1)\frac{|\eta(t)|^{p-2}|\tilde{w}(t)|^q}{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu(t)\Phi_q(\tilde{w}(t))\big)}, \end{aligned}$$

where

$$\Phi_q(\tilde{r}(t)) \leq \eta(t) \leq \Phi_q(\tilde{r}(t)) + \nu\Phi_q(\tilde{w}(t)).$$

From (1.6) and (1.7), there exists $\varepsilon > 0$ such that $|A_p(\rho(t)) + h(\rho(t))|^q(1+\varepsilon) < h(\rho(t))$. We also need to calculate

$$|\tilde{w}(t)|^q = \left(\int_0^{\rho(t)} (\tilde{r}(s))^{1-q}\nabla s\right)^{-p} |A_p(\rho(t)) + h(\rho(t))|^q.$$

We will divide the argument into two cases: (i) $t > \rho(t)$ and (ii) $t = \rho(t)$.

Case (i): Since $h(t)$ is a ∇-differentiable, monotonically non-increasing and positive function, we have

$$\begin{aligned}
\mathcal{R}[w] &= w^{\nabla}(t) + c(t) + \frac{\tilde{w}(t)}{\nu(t)}\left(1 - \frac{\tilde{r}(t)}{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu(t)\Phi_q(\tilde{w}(t))\big)}\right)\\
&= -(p-1)h(\rho(t))(\theta(t))^{-p}(\tilde{r}(t))^{1-q} + h^{\nabla}(t)\left(\int_{t_0}^{t}(\tilde{r}(s))^{1-q}\nabla s\right)^{1-p} - c(t)\\
&\quad + c(t) + (p-1)\frac{|\eta(t)|^{p-2}|\tilde{w}(t)|^q}{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu(t)\Phi_q(\tilde{w}(t))\big)}\\
&\le (p-1)(\tilde{r}(t))^{1-q}\Bigg[-h(\rho(t))\left(\int_{t_0}^{t}(\tilde{r}(s))^{1-q}\nabla s\right)^{-p}\\
&\quad + \left(\int_{t_0}^{\rho(t)}(\tilde{r}(s))^{1-q}\nabla s\right)^{-p}\frac{|\eta(t)|^{p-2}(\tilde{r}(t))^{q-1}}{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu(t)\Phi_q(\tilde{w}(t))\big)}|A_p(\rho(t)) + h(\rho(t))|^q\Bigg]\\
&= \frac{(p-1)(\tilde{r}(t))^{1-q}}{\left(\int_0^t(\tilde{r}(s))^{1-q}\nabla s\right)^p}[-h(\rho(t)) + S(t)|A_p^{\rho}(t) + h(\rho(t))|^q],
\end{aligned}$$

where

$$S(t) := \left(\frac{\int_0^t(\tilde{r}(s))^{1-q}\nabla s}{\int_0^{\rho(t)}(\tilde{r}(s))^{1-q}\nabla s}\right)^p \frac{|\eta(t)|^{p-2}(\tilde{r}(t))^{1-q}}{\Phi_p\big(\Phi_q(\tilde{r}(t)) + \nu(t)\Phi_q(\tilde{w}(t))\big)}.$$

We can see that

$$\begin{aligned}
\nu(t)\left|\frac{\tilde{w}(t)}{\tilde{r}(t)}\right|^{q-1} &= \nu(t)\frac{\left|h(\rho(t))\left(\int_0^{\rho(t)}(\tilde{r}(s))^{1-q}\nabla s\right)^{1-p} + \int_{\rho(t)}^{\infty}c(s)\nabla s\right|^{q-1}}{(\tilde{r}(t))^{1-q}}\\
&= \frac{\nu(t)(\tilde{r}(t))^{1-q}}{\int_{t_0}^{\rho(t)}(\tilde{r}(s))^{1-q}\nabla s}\left|h(\rho(t)) + \left(\int_{t_0}^{\rho(t)}(\tilde{r}(s))^{1-q}\nabla s\right)^{p-1}\left(\int_{\rho(t)}^{\infty}c(s)\nabla s\right)\right|^{q-1}\\
&\to 0
\end{aligned}$$

as $t \to \infty$ because of (1.5). Therefore, we can estimate

$$\begin{aligned}
|S(t)| &= \left(\frac{\int_0^{\rho(t)} (\tilde{r}(s))^{1-q} \nabla s + \nu(t)(\tilde{r}(t))^{1-q}}{\int_0^{\rho(t)} (\tilde{r}(s))^{1-q} \nabla s} \right)^p \frac{|\Phi_q(\tilde{r}(t)) + \nu \Phi_q(\tilde{w}(t))|^{p-2} (\tilde{r}(t))^{1-q}}{\Phi_p(\Phi_q(\tilde{r}(t)) + \nu(t)\Phi_q(\tilde{w}(t)))} \\
&= \left(1 + \frac{\nu(t)(\tilde{r}(t))^{1-q}}{\int_0^{\rho(t)} (\tilde{r}(s))^{1-q} \nabla s} \right)^p \frac{(\tilde{r}(t))^{(q-1)(p-1)} |1 + \nu(t)\Phi_q(\tilde{w}(t)/\tilde{r}(t))|^{p-2}}{\tilde{r}(t)\Phi_p(1 + \nu(t)\Phi_q(\tilde{w}(t)/\tilde{r}(t)))} \\
&= \left(1 + \frac{\nu(t)(\tilde{r}(t))^{1-q}}{\int_0^{\rho(t)} (\tilde{r}(s))^{1-q} \nabla s} \right)^p \frac{1}{1 + \nu(t)\Phi_q(\tilde{w}(t)/\tilde{r}(t))} \\
&\to 1
\end{aligned}$$

as $t \to \infty$. Summarizing all estimates, we have

$$\mathcal{R}[w] \le \frac{(p-1)(\tilde{r}(t))^{1-q}}{\left(\int_0^t (\tilde{r}(s))^{1-q} \nabla s \right)^p} [-h(\rho(t)) + S(t)|A_p(\rho(t)) + h(\rho(t))|^q (1+\varepsilon)] < 0$$

for large t.

Case (ii): If $\rho(t) = t$, then $\tilde{r} = r$ and $\tilde{w} = w$. Hence, the Riccati-type equation is

$$\begin{aligned}
\mathcal{R}[w] &= w^\nabla(t) + c(t) + (p-1)\frac{|w(t)|^q}{\Phi_q(r(t))} \\
&= -(p-1)h(\rho(t)) \left(\int_0^t (r(s))^{1-q} \nabla s \right)^{-p} (r(t))^{1-q} + h^\nabla(t) \left(\int_0^t (r(s))^{1-q} \nabla s \right)^{1-p} \\
&\quad - c(t) + c(t) + (p-1) \frac{\left(\int_0^t (r(s))^{1-q} \nabla s \right)^{-p} |A_p(\rho(t)) + h(\rho(t))|^q}{\Phi_q(r(t))} \\
&= (p-1) \left(\int_0^t (r(s))^{1-q} \nabla s \right)^{-p} (r(t))^{1-q} [-h(\rho(t)) + |A_p(\rho(t)) + h(\rho(t))|^q] \\
&< 0
\end{aligned}$$

for large t. From Lemma 2.3, this completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. One can show in the same way as in the proof of Theorem 1.2 that the function

$$w(t) = -h(t) \left(\int_t^\infty (\tilde{r}(s))^{1-q} \nabla s \right)^{p-1} - \int_0^t c(s) \nabla s$$

satisfies $\mathcal{R}[w] \le 0$. □

3 Linear difference equation

In this section, let $\mathbb{T} = \mathbb{N}$ and $p = 2$. Then, we consider the linear difference equation

$$\Delta(r(t-1)\Delta x(t-1)) + c(t)x(t) = 0. \tag{3.1}$$

Needless to say, from $\mathbb{T} = \mathbb{N}$, $p = 2$ and $\nabla(r(t)\Delta x(t)) = \Delta(r(t-1)\Delta x(t-1))$, we see that equation (1.1) becomes (3.1). We present an example of which all non-trivial solutions of (3.1) are nonoscillatory even if $\liminf_{t\to\infty} B_p(\rho(t))$ is less than the lower limit value $-3/4$.

In Theorem 1.2, we assume that $\mathbb{T} = \mathbb{N}$, $p = 2$ and $h(\rho(t)) \equiv k$ (positive constant). Then, we have the following corollary.

Corollary 3.1. *Suppose that*

$$\sum_{t=1}^{\infty} \frac{1}{r(t-1)} < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\frac{1}{r(t-1)}}{\sum_{j=t}^{\infty} \frac{1}{r(j-1)}} = 0. \tag{3.2}$$

If there exists a constant $k > 0$ such that

$$\liminf_{t\to\infty} B_2(t-1) > -\sqrt{k} - k \tag{3.3}$$

and

$$\limsup_{t\to\infty} B_2(t-1) < \sqrt{k} - k \le \frac{1}{4}, \tag{3.4}$$

then all non-trivial solutions of (3.1) *are nonoscillatory, where*

$$B_2(t-1) = \sum_{j=t}^{\infty} \frac{1}{r(j-1)} \sum_{j=1}^{t-1} c(j).$$

Example 3.1. we consider the

$$\Delta\left(t(t+1)\Delta x(t-1)\right) + \left(-\frac{1}{2} + \frac{\sqrt{2}}{2} \sin\left(\log t + \frac{\pi}{4}\right)\right) x(t) = 0 \tag{3.5}$$

for $t \in \mathbb{N}$. Then all non-trivial solutions of (3.5) are nonoscillatory.

Proof. Comparing equation (3.5) with equation (3.1), we see that $r(t-1) = t(t+1)$ and

$$c(t) = -\frac{1}{2} + \frac{\sqrt{2}}{2} \sin\left(\log t + \frac{\pi}{4}\right) = -\frac{1}{2} + \frac{1}{2}\Big(\sin(\log(t)) + \cos(\log(t))\Big).$$

From $r(t-1)$, it is easy to check that

$$\sum_{t=1}^{\infty} \frac{1}{r(t-1)} = \sum_{t=1}^{\infty} \left(\frac{1}{t(t+1)}\right) = \sum_{t=1}^{\infty} \left(\frac{1}{t} - \frac{1}{t+1}\right) = 1 < \infty,$$

$$\sum_{j=t}^{\infty} \frac{1}{r(j-1)} = \sum_{j=t}^{\infty} \left(\frac{1}{j(j+1)}\right) = \sum_{j=t}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1}\right) = \frac{1}{t},$$

$$\lim_{t\to\infty}\frac{\frac{1}{r(t-1)}}{\sum_{j=t}^{\infty}\frac{1}{r(j-1)}}=\lim_{t\to\infty}\frac{\frac{1}{t(t+1)}}{\frac{1}{t}}=\lim_{t\to\infty}\frac{1}{t+1}=0.$$

Hence, conditions (3.2) are sutisfied. By a straightforward calculation, it follows that

$$\begin{aligned}\sum_{j=1}^{t-1}c(j)&=-\sum_{j=1}^{t-1}\frac{1}{2}+\sum_{j=1}^{t-1}\frac{1}{2}\Big(\sin(\log(j))+\cos(\log(j))\Big)\\&=-\frac{t-1}{2}+\frac{t}{2}\sum_{j=1}^{t}\frac{1}{t}\left[\sin\left(\log\left(\frac{j}{t}t\right)\right)+\cos\left(\log\left(\frac{j}{t}t\right)\right)\right]\\&\quad-\frac{1}{2}\left(\sin(\log(t))+\cos(\log(t))\right)\\&=-\frac{t-1}{2}+\frac{t}{2}\sum_{j=1}^{t}\frac{1}{t}\left[\sin\left(\log\left(\frac{j}{t}\right)+\log(t)\right)+\cos\left(\log\left(\frac{j}{t}\right)+\log(t)\right)\right]\\&\quad-\frac{1}{2}\left(\sin(\log(t))+\cos(\log(t))\right).\end{aligned}$$

By using addition theorem of trigonometric functions, we have

$$\begin{aligned}\sum_{j=1}^{t-1}c(j)=-\frac{t-1}{2}&+\frac{t}{2}\sum_{j=1}^{t}\frac{1}{t}\left[\cos(\log(t))\left\{\sin\left(\log\left(\frac{j}{t}\right)\right)+\cos\left(\log\left(\frac{j}{t}\right)\right)\right\}\right]\\&+\frac{t}{2}\sum_{j=1}^{t}\frac{1}{t}\left[\sin(\log(t))\left\{\cos\left(\log\left(\frac{j}{t}\right)\right)-\sin\left(\log\left(\frac{j}{t}\right)\right)\right\}\right]\\&-\frac{1}{2}\left(\sin(\log(t))+\cos(\log(t))\right).\end{aligned}$$

Hence, we see that

$$\begin{aligned}\lim_{t\to\infty}B_2(t-1)&=\lim_{t\to\infty}\sum_{j=t}^{\infty}\frac{1}{r(j-1)}\sum_{j=1}^{t-1}c(j)\\&=\lim_{t\to\infty}\frac{t-1}{2t}\\&\quad+\lim_{t\to\infty}\frac{1}{2}\cos(\log(t))\sum_{j=1}^{t}\frac{1}{t}\left[\sin\left(\log\left(\frac{j}{t}\right)\right)+\cos\left(\log\left(\frac{j}{t}\right)\right)\right]\\&\quad+\lim_{t\to\infty}\frac{1}{2}\sin(\log(t))\sum_{j=1}^{t}\frac{1}{t}\left[\cos\left(\log\left(\frac{j}{t}\right)\right)-\sin\left(\log\left(\frac{j}{t}\right)\right)\right]\\&\quad-\lim_{t\to\infty}\frac{1}{2t}\left(\sin(\log(t))+\cos(\log(t))\right).\end{aligned}$$

Taking into account that

$$\begin{aligned}&\lim_{t\to\infty}\sum_{j=1}^{t}\frac{1}{t}\left[\sin\left(\log\left(\frac{j}{t}\right)\right)+\cos\left(\log\left(\frac{j}{t}\right)\right)\right]\\&=\int_0^1(\sin(\log x)+\cos(\log x))dx\\&=\lim_{\varepsilon\to0^+}x\sin(\log x)\Big|_{\varepsilon}^{1}=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{t\to\infty}\sum_{j=1}^{t}\frac{1}{t}\left[\cos\left(\log\left(\frac{j}{t}\right)\right)-\sin\left(\log\left(\frac{j}{t}\right)\right)\right]\\&=\int_0^1(\cos(\log x)-\sin(\log x))dx\\&=\lim_{\varepsilon\to0^+}x\cos(\log x)\Big|_{\varepsilon}^{1}=1,\end{aligned}$$

we can check that

$$\liminf_{t\to\infty}B_2(t-1)=-1<-\frac{3}{4}\quad\text{and}\quad\limsup_{t\to\infty}B_2(t-1)=0.$$

Form Corollary 3.1, if we set $k=81/100$, then

$$\liminf_{t\to\infty}B_2(t-1)>-\frac{171}{100}$$

and

$$\limsup_{t\to\infty}B_2(t-1)=0<\frac{9}{100}<\frac{1}{4}.$$

Thus, conditions (3.3) and (3.4) hold. Then all non-trivial solutions of (3.5) are nonoscillatory. □

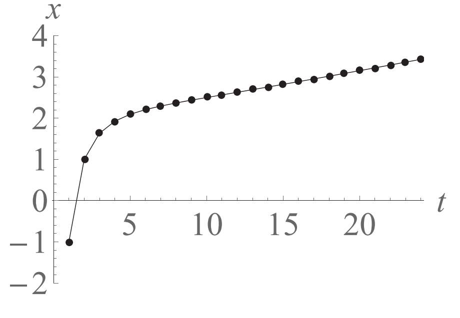

Figure 1: The initial condition of the solution of Eq.(3.5) is $(x(1),x(2))=(-1,1)$.

4 Appendix

Let $\mathbb{T} = \mathbb{R}$ and $p = 2$. Then, equation (1.1) becomes linear differential equation

$$(r(t)x'(t))' + c(t)x(t) = 0. \tag{4.1}$$

As a condition to guarantee that all non-trivial solutions of (4.1) are nonoscillatory, it is known by Moore [9], Wray [10] and Wu and Sugie [11] that the existence of the lower limit value $-3/4$ is not important. For example, Moore [9] gave the following nonoscillation theorems for (4.1).

Theorem B. *Suppose that* $\int_{t_0}^{\infty} r^{-1}(t)dt = \infty$ *and* $\int_{t_0}^{\infty} c(t)dt$ *converges. If there exists a constant* $k > 0$ *such that*

$$\left(1 + \int_{t_0}^{t} \frac{1}{r(s)} ds\right)\left(\int_{t}^{\infty} c(s)ds\right) \geq -\sqrt{k} - k$$

and

$$\left(1 + \int_{t_0}^{t} \frac{1}{r(s)} ds\right)\left(\int_{t}^{\infty} c(s)ds\right) \leq \sqrt{k} - k \leq \frac{1}{4},$$

then all nontrivial solutions of (4.1) *are nonoscillatory.*

Theorem C. *Suppose that* $\int_{t_0}^{\infty} r^{-1}(t)dt$ *converges. If there exists a constant* $k > 0$ *such that*

$$\left(1 + \int_{t}^{\infty} \frac{1}{r(s)} ds\right)\left(\int_{t_0}^{t} c(s)ds\right) \geq -\sqrt{k} - k$$

and

$$\left(1 + \int_{t}^{\infty} \frac{1}{r(s)} ds\right)\left(\int_{t_0}^{t} c(s)ds\right) \leq \sqrt{k} - k \leq \frac{1}{4},$$

then all nontrivial solutions of (4.1) *are nonoscillatory.*

Theorems 1.1 and 1.2 are generalization to Theorems B and D. Indeed, we assume that $\mathbb{T} = \mathbb{R}$, $p = 2$ and $h(\rho(t)) \equiv k$ (positive constant) for Theorems 1.1 and 1.2. Then, we have the upper boundary value of $\sqrt{k} - k$ and the lower boundary value of $-\sqrt{k} - k$.

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