# On test function method for semilinear wave equations with scale-invariant damping

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## 1. Introduction

In this paper we consider the following semilinear wave equation with a space-dependent damping term

$$
(1.1)\qquad \begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \dfrac{a}{|x|}\partial_t u(x,t) = |u(x,t)|^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = \varepsilon f(x),\ \partial_t u(x,0) = \varepsilon g(x), & x \in \mathbb{R}^N, \end{cases}
$$

where $N \geq 3$ ($N \in \mathbb{N}$), $a \geq 0$ and $1 < p < \frac{N-2}{N-4}$ ($1 < p < \infty$ for $N = 3, 4$). The initial data $(f, g)$ is assumed to be smooth enough and compactly supported, that is, $f, g \in C_0^\infty(\mathbb{R}^N)$ with

$$
\operatorname{supp}(f,g) = \operatorname{supp} f \cup \operatorname{supp} g \subset \overline{B}(0,R_0) = \{x \in \mathbb{R}^N \ ;\ |x| \leq R_0\}.
$$

The parameter $\varepsilon > 0$ describes the smallness of initial data.

The semilinear wave equation ($a = 0$) has been studied from the pioneering work by John [**5**]. In [**5**], the problem (1.1) with $N = 3$ and $a = 0$ is discussed and the following assertion is shown

(i) If $1 < p < 1 + \sqrt{2}$, then there exists a pair $(f, g)$ such that the problem does not have global-in-time solutions of (1.1) for all $\varepsilon$.

(ii) If $p > 1 + \sqrt{2}$, then there exists a global-in-time solution of (1.1) with small $\varepsilon$.

After that, there are many subsequent papers dealing with the $N$-dimensional semilinar wave equation ($a = 0$) (see e.g., Kato [**6**], Yordanov–Zhang [**9**], and Zhou [**10**]). For the $N$-dimensional case, the following is proved in the literature.

(i) If $1 < p \leq p_S(N)$, then there exists a pair $(f, g)$ such that the problem does not have global-in-time solutions of (1.1) for all $\varepsilon$.

(ii) If $p > p_S(N)$, then there exists a global-in-time solution of (1.1) with small $\varepsilon$.

Here the exponent $p_S(n)$ is called the Strauss exponent defined as

$$
\begin{aligned}
\gamma(n,p) &:= 2 + (n+1)p - (n-1)p^2, \\
p_S(n) &:= \sup\{p > 1 \ ;\ \gamma(n,p) > 0\} \\
&= \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)} \qquad (n > 1).
\end{aligned}
$$

The study of maximal existence time (lifespan)

$$T_\varepsilon = T(\varepsilon f, \varepsilon g) = \sup\{T > 0\,; \text{there exists a solution of (1.1) in } (0,T)\}.$$

of blowup solutions to (1.1) has been also studied (see Lindblad [**7**], Takamura–Wakasa [**8**] and there references therein) as

$$T_\varepsilon \sim \begin{cases} C\varepsilon^{-\frac{p-1}{2}} & \text{if } N=1,\ 1<p<\infty, \\ C\varepsilon^{-\frac{p-1}{3-p}} & \text{if } N=2,\ 1<p<2, \\ Ca(\varepsilon^{-1}) & \text{if } N=2,\ p=2, \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } N=2,\ 2<p<p_S(2), \\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N,p)}} & \text{if } N\ge 3,\ 1<p<p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } N\ge 2,\ p=p_S(N), \end{cases} \tag{1.2}$$

where $a(s)$ denotes the inverse of the function $s(a) = a\sqrt{1+\log(1+a)}$. Therefore the blowup phenomena for solutions to (1.1) with small initial data and their lifespan estimate is already established.

If $a > 0$, then the there are few works dealing with global existence and blowup of solutions to (1.1). If the damping term is milder, that is, we consider the problem

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + (1+|x|^2)^{-\frac{\alpha}{2}}\partial_t u(x,t) = |u(x,t)|^p, & (x,t)\in\mathbb{R}^N\times(0,T), \\ u(x,0)=\varepsilon f(x),\ \partial_t u(x,0)=\varepsilon g(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.3}$$

with $\alpha\in[0,1)$, then Ikehata–Todorova–Yordanov [**3**] consider the global existence and blowup of solutions to (1.3). In this case, they proved

(i) If $1 < p \le 1 + \frac{2}{N-\alpha}$, then there exists a pair $(f,g)$ such that the problem does not have global-in-time solutions of (1.1) for all $\varepsilon$.

(ii) If $p > 1 + \frac{2}{N-\alpha}$, then there exists a global-in-time solution of (1.1) with small $\varepsilon$.

This means the situation is close to the parabolic problem

$$\begin{cases} \partial_t v(x,t) - (1+|x|^2)^{\frac{\alpha}{2}}\Delta v(x,t) = 0, & (x,t)\in\mathbb{R}^N\times(0,T), \\ u(x,0)=\varepsilon f(x), & x\in\mathbb{R}^N \end{cases} \tag{1.4}$$

which has an unbounded diffusion. The case $\alpha = 1$ is more delicate. The linear problem

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + a(1+|x|^2)^{-\frac{1}{2}}\partial_t u(x,t) = 0, & (x,t)\in\mathbb{R}^N\times(0,\infty), \\ u(x,0)=u_0(x),\ \partial_t u(x,0)=u_1(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.5}$$

for $a > 0$. Ikehata–Todorova–Yordanov [**4**] discussed the decay property of energy function

$$\int_{\mathbb{R}^N}\Big(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\Big)\,dx \le \begin{cases} C(1+t)^{-a} & \text{if } 0<a<N, \\ C_\delta(1+t)^{-N+\delta} & \text{if } a\ge N. \end{cases}$$

Therefore the situation strongly depends on the size of the constant $a$ in front of the damping term $(1+|x|^2)^{-\frac{1}{2}}\partial_t u$.

Here we would like to consider the nonlinear problem (1.1) with $a>0$. It is remarkable that the equation in (1.1) has the scale-invariance, that is, if $u$ satisfies the equation on (1.1), then the scaled function $u_\lambda(x,t)=\lambda^{-\frac{2}{p-1}}u(\lambda x,\lambda t)$ also satisfies (1.1). This kind of structure helps us to analyse the dynamics of solutions.

Actually, in Ikeda–Sobajima [**1**] the finite time blowup of solutions is proved. More precisely, they showed

**Proposition 1.1** ([**1**]). *Let $N\geq 3$ and let $f$, $g$ be nonnegative, smooth and compactly supported with $g\not\equiv 0$. If $1<p<\infty$ for $N=3,4$, $1<p<\frac{N-2}{N-4}$ for $N\geq 5$, then there exists a unique solution*

$$u\in W^{2,\infty}([0,T_\varepsilon);L^2(\mathbb{R}^N))\cap W^{1,\infty}([0,T_\varepsilon);H^1(\mathbb{R}^N))\cap L^\infty([0,T_\varepsilon);H^2(\mathbb{R}^N)).$$

*Here $T_\varepsilon$ stands for the maximal existence time of solutions. Moreover, if $0<a<\frac{(N-1)^2}{N+1}$ and $\frac{N}{N-1}<p\leq p_S(N+a)$, then the maximal existence time $T_\varepsilon$ of solution $u$ is finite. In particular, the following estimates hold: there exists a positive constant $\varepsilon_0$ such that for every $\varepsilon\in(0,\varepsilon_0]$,*

$$T_\varepsilon\leq\begin{cases}C_\delta\varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}-\delta} & \text{if } p_S(N+a+2)<p<p_S(N+a),\\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p=p_S(N+a),\end{cases}$$

*where $C$ and $C_\delta$ are positive constants independent of $\varepsilon$ and $C_\delta\to\infty$ as $\delta\to 0$.*

We conjecture that $p_S(N+a)$ is the critical exponent for the problem (1.1) at least for small $a$, that is, it is expected that $p>p_S(N+a)$ implies the global existence for suitable initial data. From this viewpoint, it is natural that Proposition 1.1 gives the blowup result for the "critical" case $p=p_S(N+a)$ with an estimate for $T_\varepsilon$ of exponential type. However, in the subcritical case $\frac{N}{N-1}<p<p_S(N+a)$, the expected estimates should be $T_\varepsilon\leq C\varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}}$ (without $\delta$-loss) which could not prove in [**1**].

The purpose of this paper is to deal with the estimate for $T_\varepsilon$ of solutions to (1.1) in the subcritical case $\frac{N}{N-1}<p<p_S(N+a)$. The result is the following.

**Theorem 1.1.** *Let $f$, $g$ be nonnegative, smooth and compactly supported with $g\not\equiv 0$ and let $u$ be the solution of (1.1) in Proposition 1.1. Then there exists a positive constant $\varepsilon_0$ such that for every $\varepsilon\in(0,\varepsilon_0]$,*

$$T_\varepsilon\leq\begin{cases}C\varepsilon^{-(\frac{2}{p-1}-N+1)^{-1}} & \text{if } \frac{N}{N-1}<p<\frac{N+1}{N-1},\\ C\varepsilon^{-\frac{2p(p-1)}{\gamma(N+a,p)}} & \text{if } \frac{N+1}{N-1}<p<p_S(N+a),\end{cases}$$

*where $C$ is a positive constant independent of $\varepsilon$.*

*Remark* 1.1. We can directly check the following identity:

$$p_S(N+a_*)=\frac{N+1}{N-1},\quad a_*=\frac{(N-1)^2}{N+1}.$$

(see also Ikeda–Sobajima [**1**]). Therefore we have $0 \le a < a_*$ implies $\frac{N+1}{N-1} < p_S(N+a)$. At this moment, we may regard Theorem 1.1 as an extension of the result for (upper) lifespan estimates for the usual semilinear wave equations ($a = 0$) with small initial data.

The proof is based on a test function method for wave equations developed in Ikeda–Sobajima–Wakasa [**2**]. In particular, for the problem (1.1) we use positive solutions to the corresponding linear conjugate equation

$$\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|}\partial_t \Phi = 0.$$

In Section 2, we prove Theorem 1.1 by using positive solutions to the corresponding conjugate equation $\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|}\partial_t \Phi = 0$.

# 2 Proof of Theorem 1.1

To prove Theorem 1.1, we use the following structure. We will only give an idea for the proof.

**Lemma 1.** *Let $u$ be a solution of* (1.1). *Assume that for every $t \ge 0$, $u(t)$ is compactly supported. Then for every $T \in (0, T_\varepsilon)$ and $\Phi \in C^\infty(\mathbb{R}^N \times [0, T_\varepsilon))$ satisfying $\partial_t \Phi(\cdot, T) = \Phi(\cdot, T) = 0$,*

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^N} \Big(g + \frac{a}{|x|} f\Big)\Phi(x,0) - f(x)\partial_t \Phi(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} |u|^p \Phi\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^N} u\Big(\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|}\partial_t \Phi\Big)\,dx\,dt.
\end{aligned}$$

*Sketch of the proof.* Multiplying the equation in (1.1) and $\Phi$ and integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^p \Phi\,dx &= \int_{\mathbb{R}^N} \Big(\partial_t^2 u - \Delta u + \frac{a}{|x|}\partial_t u\Big)\Phi\,dx. \\
&= \frac{d}{dt}\int_{\mathbb{R}^N} \Big(\partial_t u + \frac{a}{|x|} u\Big)\Phi - u\partial_t \Phi\,dx + \int_{\mathbb{R}^N} u\Big(\partial_t^2 \Phi - \frac{a}{|x|}\partial_t \Phi\Big) - \Delta u \Phi\,dx.
\end{aligned}$$

Employing integration by parts and integrating it over $[0, T]$, we obtain the desired equality. □

Next, we fix $\eta \in C^\infty([0,\infty); [0,1])$ as follows:

$$\eta(s) = \begin{cases} 1 & \text{if } s \le 1/2, \\ \text{decreasing} & \text{if } 1/2 < s < 1, \\ 0 & \text{if } s \ge 1, \end{cases} \qquad \eta_T(t) = \eta(t/T).$$

Since $\varphi(x,t) = 1$ satisfies $\partial_t^2 \varphi - \Delta \varphi - \frac{a}{|x|}\partial_t \varphi = 0$, we first choose $\Phi = \varphi \eta_T^{2p'} = \eta_T^{2p'}$, where $p' = p/(p-1)$ is the Hölder conjugate of $p$. Then we have the following.

**Lemma 2.** *Let $f$, $g$ be nonnegative and smooth with* $\operatorname{supp}(f,g) \subset \overline{B}(0,R_0)$ *and* $g \not\equiv 0$. *If* $T_\varepsilon > 2R_0$, *then for every* $T \in (2R_0, T_\varepsilon)$,

$$C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\, dx\, dt \le C T^{(N-1-\frac{2}{p-1})\frac{1}{p'}} \left( \int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\, dx\, dt \right)^{\frac{1}{p}},$$

*where* $C_{f,g} = \int_{\mathbb{R}^N} g + a|x|^{-1} f\, dx > 0$. *In particular, we have*

$$C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\, dx\, dt \le C^p T^{N-1-\frac{2}{p-1}}.$$

*Sketch of the proof.* Applying Lemma 1 with $\Phi = \eta^{2p'}$, we have

$$\begin{aligned} & C_{f,g}\varepsilon + \int_0^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\, dx\, dt \\ & = \int_{T/2}^T \int_{\mathbb{R}^N} u \Big( \partial_t^2 \eta_T^{2p'} - \Delta \eta_T^{2p'} - \frac{a}{|x|} \partial_t \eta_T^{2p'} \Big)\, dx\, dt. \\ & \le C_1 \int_{T/2}^T \int_{\operatorname{supp} u(t)} |u| \eta_T^{2p'-2} \Big( \frac{1}{T^2} + \frac{1}{T|x|} \Big)\, dx\, dt. \\ & \le C_1 \left( \int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\, dx\, dt \right)^{\frac{1}{p}} \left( \int_{T/2}^T \int_{B(0,R_0+t)} \Big( \frac{1}{T^2} + \frac{1}{T|x|} \Big)^{p'} dx\, dt \right)^{\frac{1}{p'}}. \end{aligned}$$

It should be mentioned that the restriction $p > \frac{N}{N-1}$ comes from the integrability of $|x|^{-p'}$ in $B(R_0+t)$. The remaining part is just a straight forward computation. □

Next, to find a good test function, we introduce

$$\widetilde{\varphi}(x,t) = (2R_0 + t + |x|)^{-\frac{N-1-a}{2}} (2R_0 + t - |x|)^{-\frac{N-1+a}{2}}, \quad x \in B(0, 2R_0 + t),$$

which is a self-similar solution of the equation $\partial_t^2 \Phi - \Delta \Phi - \frac{a}{|x|} \partial_t \Phi = 0$ given by

$$\Phi_\beta(x,t) = (2R_0 + t + |x|)^{-\beta} F\left( \beta, \frac{N-1+a}{2}, N-1; \frac{2|x|}{2R_0 + t - |x|^2} \right)$$

with a particular choice $\beta = N-1$. The function $F(\cdot,\cdot,\cdot,z)$ stands for the Gauss hypergeometric function ($\Phi_\beta$ for general $\beta$ is introduced in [**1**]). But because of the simple structure of $\widetilde{\varphi}$, by direct computation we can verify that $\widetilde{\varphi}$ satisfies the linear conjugate equation $\partial_t^2 \widetilde{\varphi} - \Delta \widetilde{\varphi} - \frac{a}{|x|} \partial_t \widetilde{\varphi} = 0$ on $\operatorname{supp} u$. The following lemma is a consequence of the choice of $\Phi = \widetilde{\varphi} \eta_T^{2p'}$. This lemma can be understood as the concentration phenomena to the wave front $\{|x| \sim t\}$ for the wave equation (with scale-invariant damping term).

**Lemma 3.** *Let $f$, $g$ be nonnegative and smooth with* $\operatorname{supp}(f,g) \subset \overline{B}(0,R_0)$ *and* $g \not\equiv 0$. *If* $T_\varepsilon > 2R_0$, *then for every* $T \in (2R_0, T_\varepsilon)$,

$$\delta \varepsilon^p T^{N - \frac{N-1+a}{2} p} \le \int_{T/2}^T \int_{\mathbb{R}^N} |u|^p \eta_T^{2p'}\, dx\, dt.$$

*where $\delta$ is a positive constant independent of $\varepsilon$.*

*Sketch of the proof.* Applying Lemma 1 with $\Phi = \widetilde{\varphi}\eta^{2p'}$, we have

$$\begin{aligned}
\widetilde{C}_{f,g}\varepsilon &\le \varepsilon \int_{\mathbb{R}^N} \Big(g + \frac{a}{|x|}f\Big)\widetilde{\varphi}(x,0) - f(x)\partial_t\widetilde{\varphi}(x,0)\,dx + \int_0^T \int_{\mathbb{R}^N} |u|^p \widetilde{\varphi}\eta_T^{2p'}\,dx\,dt \\
&= \int_{T/2}^T \int_{\mathbb{R}^N} u\Big(\partial_t^2\eta_T^{2p'}\widetilde{\varphi} + 2\partial_t\eta_T^{2p'}\partial_t\widetilde{\varphi} - \frac{a}{|x|}\partial_t\eta_T^{2p'}\widetilde{\varphi}\Big)\,dx\,dt. \\
&\le C_2 \int_{T/2}^T \int_{\operatorname{supp} u(t)} |u|\eta_T^{2p'-2}\Big(\frac{\widetilde{\varphi}}{T^2} + \frac{\widetilde{\varphi}}{T|x|} + \frac{\partial_t\widetilde{\varphi}}{T}\Big)\,dx\,dt. \\
&\le C_1 \Big(\int_{T/2}^T \int_{\mathbb{R}^N} |u|^p\eta_T^{2p'}\,dx\,dt\Big)^{\frac{1}{p}} \Big(\int_{T/2}^T \int_{B(0,R_0+t)} \Big(\frac{\widetilde{\varphi}}{T^2} + \frac{\widetilde{\varphi}}{T|x|} + \frac{\partial_t\widetilde{\varphi}}{T}\Big)^{p'}\,dx\,dt\Big)^{\frac{1}{p'}},
\end{aligned}$$

where we have used $\partial_t\widetilde{\varphi} \le 0$ and the conjugate equation for $\widetilde{\varphi}$. The remaining part is just a straight forward computation. □

Finally, we give a proof of Theorem 1.1.

*Proof of Theorem 1.1.* Assume that $T_\varepsilon > 2R_0$. Then combining Lemmas 2 and 3, we already have the following inequality:for every $T \in (2R_0, T_\varepsilon)$,

$$C_{f,g}\varepsilon + \delta\varepsilon^p T^{N-\frac{N-1+a}{2}p} \le CT^{N-1-\frac{2}{p-1}}.$$

Then we see that if $p < \frac{N+1}{N-1}$, then $\kappa = -(N-1-\frac{2}{p-1}) > 0$ and therefore

$$T \le \left(\frac{C}{C_{f,g}\varepsilon}\right)^{\frac{1}{\kappa}}.$$

On the other hand, if $p < p_S(N+a)$, then $\frac{N-1+a}{2} - 1 - \frac{2}{p-1} = -\frac{\gamma(N+a,p)}{2(p-1)} < 0$ and therefore

$$T \le \left(\frac{C}{\delta\varepsilon^p}\right)^{\frac{2(p-1)}{\gamma(N+a,p)}}.$$

Since $T_\varepsilon$ is the maximal existence time, we can choose $T$ arbitrary close to $T_\varepsilon$. This means that $T_\varepsilon$ satisfies the same estimate as $T$ as above. The proof is complete. □

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