Spreading and vanishing for a free boundary problem of a reaction diffusion equation with a multi-stable type nonlinearity in high space dimensions

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1 Introduction and Main Results

This article is based on a joint work [10] with Dr.Yuki Kaneko(Japan Women's University) and Professor Yoshio Yamada(Waseda University). In this article we consider the following free boundary problem of reaction diffusion equation:

$$\begin{cases} u_t = \Delta u + f(u), & t>0,\ 0<r<h(t), \\ u_r(t,0) = u(t,h(t)) = 0, & t>0, \\ h'(t) = -\mu u_r(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,r)=u_0(r), & 0\le r\le h_0, \end{cases} \tag{1.1}$$

where $r=|x|$ for $x\in\mathbb{R}^N$, $r=h(t)$ denotes a free boundary and is to be determined together with $u(t,r)$, $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, μ and h_0 are given positive constants. Nonlinearity f is a C^1 function satisfying

(F) there exists $K>0$ such that $f(0)=0,\ f(K)=0$ and $f(u)<0$ for $u>K$.

For any given $h_0>0$, u_0 is assumed to belong to $\mathscr{K}(h_0)$, where

$$\mathscr{K}(h_0) := \{\phi\in C^2([0,h_0]) : \phi'(0)=\phi(h_0)=0,\ \phi(r)>0 \text{ in } [0,h_0)\}.$$

This problem was introduced first by Du and Lin [4], when $N=1$ and $f(u)=u(a-bu)$, as a population model which describes the spreading of a new or invasive species. They showed that, as $t\to\infty$, either spreading ($h(t)\to\infty$, $u(t,\cdot)\to a/b$ locally uniformly in $[0,\infty)$) or vanishing ($h(t)\to h_\infty<\infty$, $\|u(t,\cdot)\|_{C([0,h(t)])}\to 0$) occurs. This result is called the spreading-vanishing dichotomy. Since their work, this result have been extended by lots of researches (see for example [5, 7, 9, 11]). For $N\ge 2$, Du and Guo [3] considered logistic type nonlinearity, Du, Lou and Zhou [6] studied (1.1) with quite general nonlinearities, in particular, monostable and bistable nonlinearities. See [8] for more studies of (1.1) with $N\ge 2$.

In this talk, we consider f satisfying the following conditions:

$$
(f_{\rm PB}^{\rm H})\begin{cases}
\text{(i)} & f\in C^1([0,\infty)),\ f(0)=f(u_1^*)=f(u_2^*)=f(u_3^*)=0 \text{ with } 0<u_1^*<u_2^*<u_3^*,\\
& f'(0)>0,\ f'(u_1^*)<0 \text{ and } f'(u_3^*)<0;\\
\text{(ii)} & f(s)>0 \text{ for } s\in(0,u_1^*)\cup(u_2^*,u_3^*),\ f(s)<0 \text{ for } s\in(u_1^*,u_2^*)\cup(u_3^*,\infty)\\
& \text{and } \displaystyle\int_{u_1^*}^{u_3^*} f(s)ds>0;\\
\text{(iii)} & f(u)/(u-\bar{u}_2) \text{ is non-increasing in } u\in(\bar{u}_2,u_3^*) \text{ where } \bar{u}_2\in(u_2^*,u_3^*)\\
& \text{is uniquely determined by } \displaystyle\int_{u_1^*}^{\bar{u}_2} f(s)ds=0;\\
\text{(iv)} & \lim_{s\searrow u_2^*} f(s)/(s-u_2^*)^\kappa\in(0,\infty] \text{ for } \kappa=N/(N-2) \text{ when } N>2,\\
& \text{and for some } \kappa\in(0,\infty) \text{ when } N=2;
\end{cases}
$$

The typical example of f satisfying (i) and (ii) of $(f_{\rm PB}^{\rm H})$ is given by

$$f(u)=ku\left(1-\frac{u}{q}\right)-\frac{u^2}{1+u^2}$$

with positive parameters k and q being in certain parameter range. For $N\geq 2$ we impose additional condition (iii) and (iv). These conditions are used to guarantee the uniqueness of the "ground state solution" $V_{\rm dec}$ which will be mentioned below. We also note that $f'(u_2^*)>0$ implies (iv).

The main purpose of this article is to classify the asymptotic behavior of solutions. When $N=1$ and f satisfies (i), (ii) and $f'(u_2^*)>0$, Kawai and Yamada [12] have shown that asymptotic behaviors of solutions are classified into four cases, in particular, they discovered multiple spreading phenomenon corresponding to stable equilibrium points of f. The nonlinearity f satisfying (i), (ii) and $f'(u_2^*)>0$ is called *positive bistable* nonlinearity, which was originally defined in [12]. In this article we demonstrate that the results in [12] can be extended to the case where $N\geq 2$.

Theorem A. *Suppose that f satisfies $(f_{\rm PB}^{\rm H})$. Let $(u(t,r),h(t))$ be any solution to (1.1). Then exactly one of the following four cases occurs:*

(i) **Vanishing :** $\lim_{t\to\infty}h(t)\leq\sqrt{\lambda_1/f'(0)}$ *and* $\lim_{t\to\infty}\|u(t,\cdot)\|_{C([0,h(t)])}=0$*, where* λ_1 *is the first eigenvalue to the following eigenvalue problem with usual Laplacian* $\Delta=\sum_{i=1}^N\frac{\partial^2}{\partial x_i^2}$*:*

$$\begin{cases} -\Delta\varphi=\lambda\varphi & \text{in } B_1(0):=\{x\in\mathbb{R}^N:|x|<1\}\\ \varphi=0 & \text{on } \partial B_1(0)\end{cases}$$

(ii) **Small spreading :** $\lim_{t\to\infty}h(t)=\infty$ *and* $\lim_{t\to\infty}u(t,\cdot)=u_1^*$ *uniformly on* $[0,R]$ *for any* $R>0$*;*

(iii) **Big spreading:** $\lim_{t\to\infty}h(t)<\infty$ *and* $\lim_{t\to\infty}u(t,\cdot)=u_3^*$ *uniformly on* $[0,R]$ *for any* $R>0$*;*

(iv) **Transition :** $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,\cdot) = V_{\rm dec}$ *uniformly on* $[0, R]$ *for any* $R > 0$*, where* $V_{\rm dec} = V_{\rm dec}(r)$ *is a unique positive decreasing function satisfying*

$$V'' + \frac{N-1}{r}V + f(V) = 0 \ \ in \ \ (0,\infty), \quad V'(0) = 0 \ \ and \ \ \lim_{r\to\infty} V(r) = u_1^*.$$

To classify the asymptotic behavior of solutions to (1.1) it is important to investigate corresponding stationary problem. When $N = 1$ we can use a phase plane analysis to study the stationary problem. However when $N \geq 2$ we can not use this method. In this talk we will see that some results for some elliptic problems ([1, 2, 13, 14, 15]) can be used to investigate corresponding stationary problem.

For any fixed $h_0 > 0$ and $\phi \in \mathscr{K}(h_0)$ we consider a family of initial function $u_0 = \sigma\phi$ for $\sigma > 0$.

Theorem B. *Assume that* f *satisfies* $(f^{\rm H}_{\rm PB})$*. Let* $(u_\sigma(t, r), h_\sigma(t))$ *be the solution to* (1.1) *with* $u_0 = \sigma\phi$ *for* $h_0 > 0$*,* $\phi \in \mathscr{K}(h_0)$ *and* $\sigma > 0$*. Then there exist two numbers* $0 \leq \sigma_1^* < \sigma_2^*$ *such that*

- *the vanishing occurs for* $\sigma \in [0, \sigma_1^*]$*;*
- *the small spreading occurs for* $\sigma \in (\sigma_1^*, \sigma_2^*)$*;*
- *the transition occurs for* $\sigma = \sigma_2^*$*;*
- *the big spreading occurs for* $\sigma \in (\sigma_2^*, \infty)$*.*

Moreover $\sigma_1^* > 0$ *if* $h_0 < \sqrt{\lambda_1/f'(0)}$*, and* $\sigma_1^* = 0$ *if* $h_0 \geq \sqrt{\lambda_1/f'(0)}$*, where* λ_1 *is defined as in Theorem A.*

Finally, I will give a result about the spreading speed of the free boundary, when the small spreading, big spreading or the transition occurs. To investigate the spreading speed of the free boundary, we have to consider the following *semi-wave problem*:

$$({\rm SWP})_{u^*} \begin{cases} q_{zz} - cq_z + f(q) = 0, \ q(z) > 0 \ \text{ for } \ z > 0. \\ q(0) = 0, \ \mu q_z(0) = c, \ \ \lim\limits_{z\to\infty} q(z) = u^*, \end{cases}$$

where $u^* \in \mathbb{R}$ is a positive and stable equilibrium of f. Since $f|_{[0,u_1^*]}$ is a monostable nonlinearity, it follows from [5, Proposition 1.9] that $({\rm SWP})_{u_1^*}$ admits a unique solution pair $(c_{\rm S}, q_{\rm S})$ for any $\mu > 0$. Moreover, $c_{\rm S} = c_{\rm S}(\mu)$ is monotone increasing in μ, $\lim_{\mu\to 0} c_{\rm S}(\mu) = 0$ and $\lim_{\mu\to\infty} c_{\rm S}(\mu) = c_0^{\rm S}$ holds, where $c_0^{\rm S}$ is the minimal speed of traveling waves determined by $f|_{[0,u_1^*]}$.

The solvability of $({\rm SWP})_{u_3^*}$ is delicate and it was shown in [12] that one of the following two cases holds true:

(Case A) For every $\mu > 0$, there exists a unique solution pair $(c, q) = (c_{\rm B}, q_{\rm B})$ to $({\rm SWP})_{u_3^*}$.

(Case B) There exists a positive number μ^* such that $(\mathrm{SWP})_{u_3^*}$ admits a unique solution $(c,q) = (c_{\mathrm{B}}, q_{\mathrm{B}})$ for $\mu \in (0,\mu^*)$ and no solution for every $\mu \in [\mu^*, \infty)$.

It was also shown in [12] that if $(c_{\mathrm{B}}, q_{\mathrm{B}})$ exists, then $c_{\mathrm{S}} < c_{\mathrm{B}}$ holds. Moreover, by the phase plane analysis given in [12], it can be shown that Case B occurs if and only if $c_1^{\mathrm{B}} < c_0^{\mathrm{S}}$ where c_1^{B} is the unique speed of traveling waves determined by the bistable nonlinearity $f|_{[u_1^*, u_3^*]}$.

Theorem C. *Suppose that f satisfies $(f^{\mathrm{H}}_{\mathrm{PB}})$ and let (u,h) be the solution.*

(i) *If a small spreading occurs for (u,h), then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_{\mathrm{S}},$$

where c_{S} is the number determined by unique solution pair to $(\mathrm{SWP})_{u_1^*}$.

(ii) *If a transition occurs for (u,h),*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_{\mathrm{S}}.$$

(iii) *If a big spreading occurs for (u,h), then the following properties holds true :*

In Case A,

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_{\mathrm{B}},$$

and in Case B,

$$\lim_{t\to\infty} \frac{h(t)}{t} = \begin{cases} c_{\mathrm{B}} & \text{when } \mu < \mu^*, \\ c_{\mathrm{S}} & \text{when } \mu > \mu^*, \end{cases}$$

where c_{B} is the number determined by unique solution pair to $(\mathrm{SWP})_{u_3^*}$ and μ^* is the number given in Case B.

In this article I will give a sketch of proof of Theorem A. For the proof of Theorems B and C, please see [10].

2 Sketch of proof of Theorem A

We first recall a general result obtained by Du, Lou and Zhou [6].

Proposition 2.1 ([6]). *Assume* (F) *and let $(u(t,r), h(t))$ be the solution to* (1.1). *Then, exactly one of the following two cases occurs.*

(1) $\lim_{t\to\infty} h(t) < \infty$ *and* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0$.

(2) $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,r) = v(r)$ *in* $C_{\rm loc}([0,\infty))$*, where* $v(r)$ *satisfies*

$$\text{(S)}\quad v'' + \frac{N-1}{r}v' + f(v) = 0 \quad \textit{for} \quad r > 0, \quad v'(0) = 0$$

and either $v \equiv$ *const. or* $v'(r) < 0$ *for* $r \geq h_0$*; in the former case, the constant is necessarily a nonnegative zero of* f*.*

In what follows we assume that f satisfies $(f^{\rm H}_{\rm PB})$.

Comparing $u(t,r)$ with a solution to the corresponding ordinary differential equation, we can obtain the following lemma.

Lemma 2.2. *For any solution* $(u(t,r), h(t))$ *and* $\delta \in (0, -f'(u_3^*))$ *there exist* $M > 0$ *and* $T > 0$ *such that*

$$u(t,r) \leq u_3^* + Me^{-\delta t} \quad t \geq T \quad \textit{and} \quad r \in [0, h(t)].$$

Therefore, to prove Theorem A, it is important to investigate problem (S), in particular, constant solution to (S) and solution $v = v(r)$ to (S) with $0 \leq v(r) \leq u_3^*$ for $r \geq 0$, $v'(r) < 0$ for $r \geq h_0$.

Lemma 2.3 ([13, 14])**.** *Problem* (S) *has a unique ground state solution, namely a positive solution* $V_{\boldsymbol{dec}} = V_{\boldsymbol{dec}}(r)$ *satisfying* $V'_{\rm dec}(r) < 0$ *for* $r > 0$ *and* $\lim_{r\to\infty} V_{\rm dec}(r) = u_1^*$*. Moreover* $V_{\rm dec}(0) \in (u_2^*, u_3^*)$ *holds.*

To investigate the solution set of problem (S) we consider the following initial value problem:

$$\text{(IVP)}\begin{cases} v'' + \dfrac{N-1}{r}v_r + f(v) = 0 \quad \text{for} \quad r > 0, \\ v(0) = \zeta, \quad v'(0) = 0. \end{cases}$$

For given $\zeta \in [0, u_3^*]$, let $v = v(r) = v(r;\zeta)$ be the unique solution to (IVP). The following proposition is a key to prove Theorem A.

Proposition 2.4. (i) *If* $\zeta \in (V_{\rm dec}(0), u_3^*)$*, then there exists* $R_1(\zeta) > 0$ *such that* $v(R_1(\zeta);\zeta) = 0$ *and* $v(r;\zeta) > 0$ *for* $r \in [0, R_1(\zeta))$ *and* $v'(r;\zeta) < 0$ *for* $r \in (0, R_1(\zeta)]$.

(ii) *If* $\zeta = V_{\rm dec}(0)$*, then* $v(r;\zeta) = V_{\rm dec}(r)$ *for* $r \geq 0$*.*

(iii) *If* $\zeta \in (u_1^*, V_{\rm dec}(0)) \backslash \{u_2^*\}$*, then the one of the following two cases occurs;*

(iii-a) v *is monotone for all* $r >> 1$ *and* $\lim\limits_{r\to\infty} v(r;\zeta) = u_2^*$*, or*

(iii-b) $v(r;\zeta) \in (u_1^*, u_3^*)$ *takes local maximum and minimum infinitely many times and oscillates.*

(iv) *If* $\zeta \in (0, u_1^*)$*, then there exists* $R_2(\zeta) > 0$ *such that* $v(R_2(\zeta);\zeta) = 0$ *and* $v(r;\zeta) > 0$ *for* $r \in [0, R_2(\zeta))$ *and* $v'(r;\zeta) < 0$ *for* $r \in (0, R_2(\zeta)]$*.*

(v) *If* $\zeta = 0, u_1^*, u_2^*$ *or* u_3^*, *then* $v(r;\zeta) \equiv 0, u_1^*, u_2^*$ *or* u_3^*, *respectively.*

Remark . I will give a brief remark about how the results [1, 2, 15] are used to get the above lemma. Please see [10] for details.

- Since $f|_{[0,u_1^*]}$ is monostable nonlinearity, we can get (iv) by using a Liouville type theorem for monostable nonlinearity in [2].
- Since $f|_{[u_1^*,u_3^*]}$ is bistable nonlinearity, for $\zeta \in (u_1^*, u_3^*)$ we can use the result for classification of solution to (IVP) with bistable nonlinearity in [1].
- To get (i) we use a nonexistence of positive upper solution to $-\Delta u = \varepsilon u$ on exterior domains in [15].

By Proposition 2.4 we get the following corollary.

Corollary 2.5. *Let* v *be any solution to* (S) *which satisfies* $0 \leq v(r) \leq u_3^*$ *for* $r \geq 0$ *and*

$$v \equiv const. \quad or \quad v'(r) < 0 \quad for \quad r > h_0.$$

Then $v \in \{0,\ u_1^*,\ u_2^*,\ u_3^*,\ V_{\mathrm{dec}}\}$, *or* $v = v(r)$ *satisfies*

$$v(0) \in (u_1^*, V_{\mathrm{dec}}(0)) \backslash \{u_2^*\}, \quad v'(r) < 0 \quad for \quad r \geq h_0 \quad and \quad \lim_{r\to\infty} v(r) = u_2^*.$$

Now I will give the proof of Theorem A.

Proof of Theorem A. **Step 1.** We show that if $\lim_{t\to\infty} h(t) < \infty$, then $\lim_{t\to\infty} h(t) \leq \sqrt{\lambda_1/f'(0)}$ and $\lim_{t\to\infty} \|u(t,\,\cdot\,)\|_{C([0,h(t)])} = 0$.

By Proposition 2.1 we have that if $h_\infty = \lim_{t\to\infty} h(t) < \infty$ then $\lim_{t\to\infty} \|u(t,\,\cdot\,)\|_{C([0,h(t)])} = 0$. Hence it remains to show that $h_\infty \leq \sqrt{\lambda_1/f'(0)}$. Suppose that $h_\infty > \sqrt{\lambda_1/f'(0)}$. There exists $T > 0$ such that $h(T) > \sqrt{\lambda_1/f'(0)}$. Take $\ell \in (\sqrt{\lambda_1/f'(0)}, h(T))$ and consider the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \ B_\ell(0), \\ \varphi = 0 & \text{on } \ \partial B_\ell(0). \end{cases}$$

Let λ_1^ℓ and $\varphi_\ell(x)$ (normalize $\max_{x\in B_\ell(0)} \varphi_\ell(x) = 1$) are the first eigenvalue and the corresponding eigenfunction. Then we have $\lambda_1^\ell = \lambda_1/\ell^2$ and $\lambda_1^\ell < f'(0)$. Then it is easy to show that for sufficiently small $\varepsilon > 0$, $\varepsilon\varphi_\ell$ becomes a lower solution and then $\liminf_{t\to\infty} \|u(t,\,\cdot\,)\|_{C([0,h(t)])} > 0$ which is contradiction. Now Step 1 has been completed.

Suppose that $h_\infty = \infty$. By Proposition 2.1 we have

$$\lim_{t\to\infty} u(t,r) = v(r) \ \text{ locally uniformly in } \ [0,\infty)$$

where $v(r)$ is a solution to (S) with $0 \leq v(r) \leq u_3^*$ and either $v \equiv$ *const.* or $v'(r) \leq 0$ in (h_0, ∞). By Corollary 2.5, to complete the proof of Theorem A, it is enough to exclude the possibility that $v \equiv 0$ and $\lim_{r\to\infty} v(r) = u_2^*$.

Step 2. $v \not\equiv 0$.

Since $h_\infty = \infty$, there exists $T > 0$ such that $h(T) > \sqrt{\lambda_1/f'(0)}$. Then we can show $\liminf_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} > 0$ and then $v \not\equiv 0$ by the same argument as in Step 1.

Step 3. v does not satisfy $\lim_{r\to\infty} v(r) = u_2^*$.

Let $v_{\rm dec}(r)$ be $V_{\rm dec}(r)$ with $N = 1$, that is $v_{\rm dec}$ is the unique solution to

$$v'' + f(v) = 0,\ v'(0) = 0,\ \lim_{r\to\infty} v(r) = u_1^*,\quad v'(r) < 0 \ \text{ for } \ r > 0.$$

For any $\xi > 0$, $\overline{u}(t,r) := v_{\rm dec}(r-\xi)$ satisfies

$$\begin{aligned} &\overline{u}_t - \overline{u}_{rr} - \frac{N-1}{r}\overline{u}_r - f(\overline{u}) \\ =& - v''_{\rm dec}(r-\xi) - \frac{N-1}{r}v'_{\rm dec}(r-\xi) - f(v_{\rm dec}(r-\xi)) \\ =& - \frac{N-1}{r}v'_{\rm dec}(r-\xi) > 0 \ \text{ for } \ t > 0, \quad \xi < r < \infty. \end{aligned}$$

We will see that if $\lim_{r\to\infty} v(r) = u_2^*$ then for sufficiently large $\xi > 0$, $\overline{u}$ becomes an upper solution.

Take $\varepsilon > 0$ any small such that $u_2^* + \varepsilon < v_{\rm dec}(0)$ and suppose that $\lim_{r\to\infty} v(r) = u_2^*$. Then there exists $K > h_0$ such that $v(K) \le u_2^* + \varepsilon/2$. Since $\lim_{t\to\infty} u(t,r) = v(r)$, there exists $T > 0$ such that

$$u(t,K) \le v(K) + \varepsilon/2 \le u_2^* + \varepsilon < v_{\rm dec}(0) \ \text{ for } \ t \ge T.$$

On the other hand, since $u(T,h(T)) = 0$ and $v_{\rm dec} > u_1^*$, there exists $\xi > K$ such that $\overline{u}(t,r)$ satisfies

$$\begin{aligned} &u(T,r) \le u_1^* < v_{\rm dec}(r-\xi) = \overline{u}(T,r) \ \text{ for } \ \xi \le r \le h(T). \\ &u(t,h(t)) = 0 < u_1^* < v_{\rm dec}(h(t)-\xi) = \overline{u}(t,h(t)) \ \text{ for } \ t \ge T. \end{aligned}$$

Moreover, since $u_r(t,r) < 0$ for $r > h_0$, we have

$$u(t,\xi) \le u(t,K) \le v_{\rm dec}(0) = \overline{u}(t,\xi) \ \text{ for } \ t \ge T.$$

Therefore simple comparison principle applied over region $\{(t,r) | t \ge T \text{ and } \xi \le r \le h(t)\}$ yields that

$$u(t,r) \le v_{\rm dec}(r-\xi) \ \text{ for } \ t \ge T \ \text{ and } \ \xi \le r \le h(t).$$

Letting $t \to \infty$ we obtain $v(r) \le v_{\rm dec}(r-\xi)$ for $r \ge \xi$. Since $\lim_{r\to\infty} v_{\rm dec}(r) = u_1^*$, for sufficiently large $r > 0$ the above inequality leads to a contradiction.

Now the proof of Step 3 and Theorem A has been completed. □

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