Inverse scattering for an energy dependent, reflectionless potential

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1. Scattering theory

The primary subject of this note is a reflectionless inverse scattering theory for an energy dependent Schrödinger equation

$$f'' + [k^2 - (U(x) + 2k\,Q(x))]f = 0, \quad -\infty < x < \infty, \tag{1}$$

where $U(x)$, $Q(x)$ are real-valued, decreasing at $x \to \pm\infty$. In this equation k is a square root of energy and the potential $U(x) + 2k\,Q(x)$ is depending on energy; (1) is called an energy dependent Schrödinger equation.

The original Schrödinger equation is the case Q vanishes, which was introduced by Schrödinger in 1926. He published four papers concerning this topic in that year and this is the form in the so-called first paper: one-dimensional and time-independent. On the other hand, Heisenberg studied S-matrix in 1943 and 1944 in also four papers as an observable data in quantum scattering, because Heisenberg's philosophy is that physical theories should be built up based upon observable data.

Physical background can be explained by the following figure:

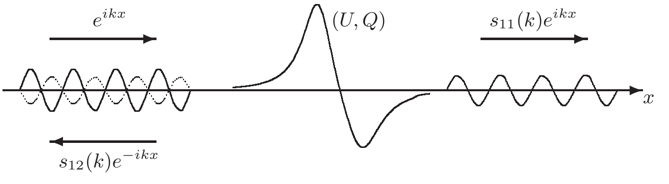

Figure 1: scattering matrix

We consider the situation that a free electron comes from $-\infty$ as a wave e^{ikx} in the direction to $+\infty$. In that case the potential works as a barrier to the electron, but this particle is a quantum, so some part of wave is transmitted in this direction with the scattering amplitude $s_{11}(k)$ without phase shift, which is a function of k, and some part of this wave is going back, namely, reflected in the opposite direction with the phase shift as a wave $s_{12}(k)e^{-ikx}$. Everything is converted by exchanging $-\infty$ to ∞; a wave e^{-ikx} coming from $+\infty$ is transmitted in the direction to $-\infty$ as a wave $s_{22}(k)e^{-ikx}$ and also is reflected as a wave $s_{21}(k)e^{ikx}$. So now we have the matrix, the so-called scattering matrix (S-matrix)

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \quad -\infty < k < \infty.$$

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The coefficients $s_{11}(k)$, $s_{22}(k)$ are called transmission coefficients and $s_{12}(k)$, $s_{21}(k)$ are called the reflection coefficients.

Now the inverse scattering problem is formulated as : to recover (U,Q) from the scattering matrix $S(k)$:

$$(U,Q) \longleftarrow S(k).$$

Important properties of this scattering matrix are:

- Firstly, $S(k)$ is unitary, for example, $|s_{11}(k)|^2+|s_{21}(k)|^2=1$. This is the conservation law in the scattering.
- Secondly, $s_{11}(k)=s_{22}(k)$. This is a symmetry.
- Thirdly, the transmission coefficient $s_{11}(k)(=s_{22}(k))$ is analytically extended to be an analytic function in the upper half plane $\boldsymbol{C}_+$ excepting at finitely many poles. If $s_{11}(k)$ has N poles $k_1,\cdots,k_N$ in $\boldsymbol{C}_+$ (see Figure 2) then the scattering is said to have N bound states.

In terms of the Jost solutions $f_\pm(x,k)\sim e^{\pm ikx}$ as $x\to\pm\infty$, the poles k_n are characterized as numbers for which there exist nonzero complex numbers d_n such that $f_-(x,k_n)=d_nf_+(x,k_n)$. These d_n are called coupling constants.

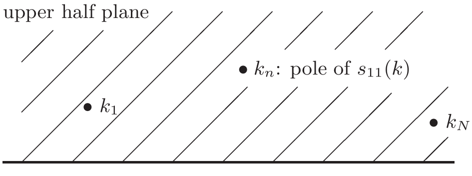

Figure 2: Bound states

Our subject, the inverse scattering, has a long history:

Table 1: Research history

Schrödinger equation $f''+[k^2-U(x)]f=0$	Energy dependent equation $f''+[k^2-(U(x)+2k\,Q(x))]f=0$	
Marchenko [8], 1955 Faddeev [3], 1964 Deift-Trubowitz [2], 1979	Jaulent-Jean [4], 1976 ($N=0$) Sattinger-Szmigielski [9], 1995 Kamimura [6], 2008 ($N=0$)	$N>0$ still open

Although in the case where $N=0$, namely in the absence of bound states, a complete generalization of the Marchenko-Faddeev-Deift and Trubowitz theory for the Schrödinger equation was given by [6], the inverse scattering problem for energy dependent equation in the presence of bound states ($N>0$) is still open.

We consider the reflectionless case: $s_{12}(k)\equiv s_{21}(k)\equiv 0$. In order to determine (U,Q) from $S(k)$ we employ complex constants c_n defined by

$$c_n := -i\,\mathrm{Res}_{\,k=k_n} s_{11}(k)\times d_n, \quad n=1,\cdots,N.$$

One can show that $c_n \neq 0$ by using the Poisson formula. The numbers c_n are corresponding to the norming constants in the standard Schrödinger case. In the reflectionless scattering, the triplet $\{0, k_n, c_n\}$, where 0 implies that $s_{21}(k) \equiv 0$, is used as *scattering data*.

In terms of the scattering data, we define an $N \times N$ matrix B and a column vector $\boldsymbol{v}$ by

$$B := \left(\overline{c_m} \, \frac{e^{(\overline{ik_m} + ik_n)x}}{\overline{ik_m} + ik_n} \right), \quad \boldsymbol{v} := \left(\frac{c_m}{ik_m} e^{ik_m x} \right)$$

and introduce a function $\Delta(x)$ by

$$\Delta(x) := \det(I - B\,\overline{B}\,) + (e^{\overline{ik_1 x}} \cdots e^{\overline{ik_N x}})(I - B\overline{B}\,)\tilde{}\,(B\boldsymbol{v} - \overline{\boldsymbol{v}}), \tag{2}$$

where $(I - B\overline{B}\,)\tilde{}$ denotes the cofactor matrix of $I - B\overline{B}$. By definition, $\Delta(x)$ is written by only exponential functions.

In the reflectionless case, the scattering for the energy dependent equation is completely controlled by this function Δ:

Theorem 1 ([7]) *A triplet $\{0, k_n, c_n\}$ with $k_n \in \boldsymbol{C}_+$, $c_n \in \boldsymbol{C} \setminus \{0\}$, $n = 1, \cdots, N$, is the scattering data for some pair $(U, Q) \in \mathcal{S} \times \mathcal{S}$ if and only if $\Delta(x)$ has no zeros on $\boldsymbol{R}$, and, under this condition, (U, Q) is (uniquely) determined by*

$$\begin{cases} Q(x) = -\frac{d}{dx} \arg \Delta(x), \\ U(x) + Q(x)^2 = -\frac{d^2}{dx^2} \log |\Delta(x)|. \end{cases} \tag{3}$$

Here $\mathcal{S}$ denotes the Schwartz class on $\boldsymbol{R}$.

A data $\{0, k_n, c_n\}$ is said to be *regular* if $\Delta(x)$ has no zeros on $\boldsymbol{R}$.

In the standard Schrödinger case where $c_n > 0$, $ik_n < 0$, it turns out that the matrix B is real and $\Delta(x) = (\det I - B))^2 > 0$. Hence, as a special case of Theorem 1, we draw the following:

Corollary *If $c_n > 0$, $ik_n < 0$ then $Q \equiv 0$, $U(x) = -2\frac{d^2}{dx^2} \log \det(I - B)$.*

This is a well-known representation (see, e.g., [1], [10]) of reflectionless potentials in the Schrödinger equation via Hirota's transformation. In other words, Theorem 1 gives a (complex) generalization of reflectionless scattering theory for the Schrödinger equation:

Table 2: Summary in Section 1

	Schrödinger equation	Energy dependent equation
Poles k_n	on imaginary axis	in $\boldsymbol{C}_+$
Constants c_n	$c_n > 0$, any	$c_n \in \boldsymbol{C} \setminus \{0\}$, regular
Potential	$(U, 0)$, single	(U, Q), system
Inversion formula	$U(x) = -2\frac{d^2}{dx^2} \log \det(I - B)$	Eq. (3)

2. A nonlinear evolution system

In general, a nonlinear evolution equation along which locations of bound states k_n of a spectral problem are invariant in the time is said to be an isospectral flow of the spectral problem. Isospectral flows of the energy dependent Schrödinger equation (1) was formally found by Jaulent and Miodek [5]. Under the transformation

$$u = -4Q, \quad w = 4(U + Q^2),$$

the flow with the lowest degree is written as the nonlinear evolution system

$$\begin{cases} u_t + w_x + uu_x = 0, \\ w_t - u_{xxx} + (uw)_x = 0. \end{cases} \tag{4}$$

This is corresponding to the KdV equation which is an isospectral flow of the Schrödinger equation.

Based upon Theorem 1, we establish the following inverse scattering method to find N-soliton solutions of the system (4):

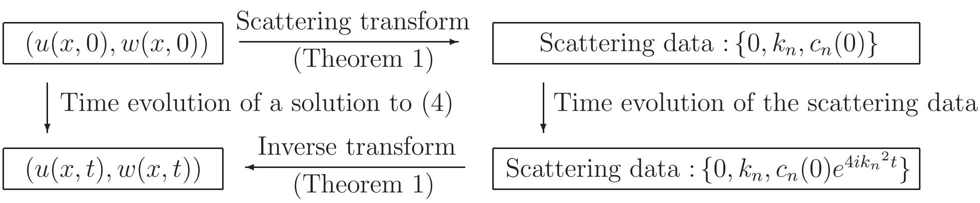

Figure 3: Inverse scattering method

The pair $(u(x,t), w(x,t))$ constructed by the above recipe is indeed a solution of (4):

Theorem 2 ([7]) *Let $\Delta(x,t)$ be a function defined by the inversion formula* (2) *with $c_n = c_n(0)e^{4ik_n{}^2 t}$. Then $(u(x,t), w(x,t))$ defined by*

$$\begin{cases} u(x,t) = 4\frac{\partial}{\partial x} \arg \Delta(x,t), \\ w(x,t) = -4\frac{\partial^2}{\partial x^2} \log |\Delta(x,t)|, \end{cases}$$

satisfies the system (4) *as long as $\{0, k_n, c_n(t)\}$ is regular.*

Example $(N = 1)$ Set

$$k_1 = a + bi, \quad x_0 = \frac{1}{2b} \log \frac{|c_1(0)|}{2b}, \quad \Theta = \arg c_1(0) + 2\tan^{-1}\frac{a}{b} + \frac{a}{b}\log\frac{|c_1(0)|}{2b},$$

assume $\Theta \in (-\pi, \pi)$ without loss of generality, and put

$$\xi := 2b(x - x_0 + 4at), \quad \eta := 2a(x - x_0 + 4at) - (4(a^2 + b^2)t - \Theta).$$

Then 1-soliton solution found by Theorem 2 is written as:

$$u(x,t) = 8b\frac{(1 - (\frac{a}{b})^2)\sinh\xi\sin\eta - 2\frac{a}{b}(\cosh\xi\cos\eta + 1)}{(1 + (\frac{a}{b})^2)\cosh^2\xi + 2\left(\cosh\xi\cos\eta + \frac{a}{b}\sinh\xi\sin\eta\right) + 1 - (\frac{a}{b})^2},$$

$$\begin{aligned} w(x,t) = -16(a^2+b^2)&\Big\{(1 - 3(\tfrac{a}{b})^2)\cosh^3\xi\cos\eta + \tfrac{a}{b}(3 - (\tfrac{a}{b})^2)\sinh^3\xi\sin\eta \\ &+3(1 - (\tfrac{a}{b})^2)\cosh^2\xi + 3(1 + (\tfrac{a}{b})^2)\cosh\xi\cos\eta + \cos 2\eta + 3(\tfrac{a}{b})^2\Big\} / \\ &\left((1 + (\tfrac{a}{b})^2)\cosh^2\xi + 2\left(\cosh\xi\cos\eta + \tfrac{a}{b}\sinh\xi\sin\eta\right) + 1 - (\tfrac{a}{b})^2\right)^2. \end{aligned}$$

The life span of the solution $(u(x,t), w(x,t))$ is finite; it exists only in the interval

$$t_{\min} := \frac{\Theta - \pi}{4(a^2+b^2)} < t < \frac{\Theta + \pi}{4(a^2+b^2)} =: t_{\max},$$

since it has a singularity $x = x_0 - 4at$ at $t = t_{\max}, t_{\min}$ ($\Leftrightarrow \cos(\Theta - 4(a^2+b^2)) = -1$).

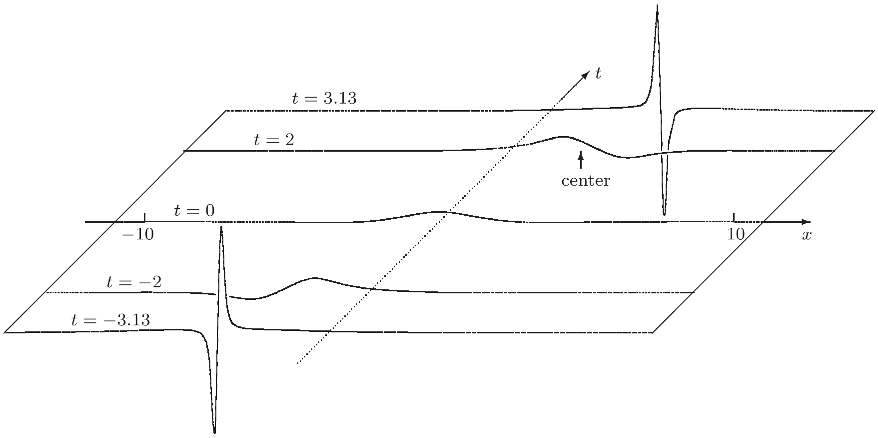

Figure 4: Profile of $u(x,t)$ for $-\pi < t < \pi$ in the case $a = -\frac{3}{10}$, $b = \frac{4}{10}$, $x_0 = 0$, $\Theta = 0$.

Table 3: Summary of Section 2

	Schrödinger equation	Energy dependent equation
Isospectral flow	KdV eq : $u_t - 6uu_x + u_{xxx} = 0$ single	(4) system
Constants $c_n(t)$	$c_n(0)e^{8ik_n{}^3t}$: positive	$c_n(0)e^{4ik_n{}^2t}$: complex
Life span	infinite	no longer infinite

There are many open issues left to be settled in connection with the subject of this note. We wish to pick out some of them:

- To obtain inverse scattering methods for isospectral flows of (1) with higher degrees.
- To show that the life span of N-Soliton solutions of (4) is necessarily finite even for $N \geq 2$.
- To develop inverse scattering theory for energy dependent equation (1), involving the case with reflection.
- To develop a reflectionless inverse scattering theory for other energy dependent Schrödinger equations.

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