Remarks on separation phenomena of radial solutions to Lane-Emden equation on the hyperbolic space

大阪大学大学院基礎工学研究科　長谷川 翔一

Shoichi Hasegawa
Graduate School of Engineering Science,
Osaka University

1 Introduction

This paper is devoted to an announcement about results in [27] and a survey on related topics to [27]. We shall introduce separation phenomena of radial solutions to the following Lane-Emden equation on the hyperbolic space $\mathbb{H}^N$:

$$-\Delta_g u = |u|^{p-1}u \quad \text{in} \quad \mathbb{H}^N, \tag{H}$$

where $N \geq 3$, and $p > 1$. Here, $\mathbb{H}^N$ is a manifold admitting a pole o and whose metric g is defined, in the polar coordinates around o, by

$$ds^2 = dr^2 + (\sinh r)^2 d\Theta^2, \quad r > 0, \quad \Theta \in \mathbb{S}^{N-1},$$

where $d\Theta^2$ denotes the canonical metric on the unit sphere $\mathbb{S}^{N-1}$, and r is the geodesic distance between o and a point (r, Θ). Moreover, Δ_g denotes the Laplace-Beltrami operator on $(\mathbb{H}^N, g)$ given by

$$\begin{aligned}\Delta_g f(r, \theta_1, ..., \theta_{N-1}) =& (\sinh r)^{-(N-1)} \partial_r \left\{ (\sinh r)^{N-1} \partial_r f(r, \theta_1, ..., \theta_{N-1}) \right\} \\ &+ (\sinh r)^{-2} \Delta_{\mathbb{S}^{N-1}} f(r, \theta_1, ..., \theta_{N-1}),\end{aligned}$$

where $f : \mathbb{H}^N \to \mathbb{R}$ is a scalar function and $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on the unit ball $\mathbb{S}^{N-1}$. Furthermore, we also define the exponents $p_s(N)$ and $p_{JL}(N)$, respectively, by

$$p_s(N) = \frac{N+2}{N-2}$$

and

$$p_{JL}(N) = \begin{cases} +\infty & \text{if} \quad N \leq 10, \\ \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if} \quad N > 10. \end{cases}$$

The exponents $p_s(N)$ and $p_{JL}(N)$ are called the Sobolev exponent and the Joseph-Lundgren exponent ([29]), respectively.

To begin with, we introduce known results on separation phenomena of radial solutions to the Lane-Emden equation in the Euclidean space

$$\text{(L)} \qquad -\Delta u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 3$, and $p > 1$. This equation was posed by J.H. Lane ([31]) in 1869 and was appeared in the astrophysical study of the structure of a singular star ([14, 17, 19]). There is also an extensive mathematical literature ([12, 13, 18, 20, 21, 23, 29, 36]). Concerning separation phenomena of radial solutions, X. Wang [44] and Y. Liu, Y. Li, Y. Deng [32] proved the existence of a critical exponent on separation and intersection properties of radial solutions to (L). Here, for each $\alpha > 0$, we denote by $u_\alpha^L = u_\alpha^L(r)$ the radial solution of (L) satisfying $u_\alpha^L(0) = \alpha$. Then, the following results were obtained:

Proposition 1.1 (Proposition 3.7 (iv) in [44], Theorem 1 (ii) in [32]). *Let $p > 1$. Then the following hold*:

(i) *If $p \in (p_s(N), p_{JL}(N))$, then for any $\alpha, \beta > 0$ with $\alpha \neq \beta$, u_α^L and u_β^L intersect infinitely many times in* $(0, \infty)$;

(ii) *If $p \geq p_{JL}(N)$, then for any $\alpha, \beta > 0$ with $\alpha \neq \beta$, u_α^L and u_β^L cannot intersect each other in $(0, \infty)$, i.e., $u_\alpha^L < u_\beta^L$ in $(0, \infty)$ if $\alpha < \beta$.*

Proposition 1.1 implies that $p_{JL}(N)$ is the critical exponent with respect to separation phenomena of radial solutions to (L). Thereafter, separation phenomena of radial solutions has been researched further in [1, 2, 3, 5, 16, 22, 34, 35] and was also studied for the equation (L) replacing u^p by e^u ([4, 6, 43]). Furthermore, making use of separation property of radial solutions, A. Farina [18] and E.N. Dancer, Y. Du, Z. Guo [15] showed the existence of stable solutions to (L). Separation property of radial solutions is also applicable to the research on asymptotic behavior of solutions to the corresponding semilinear parabolic equation to (L) ([23, 24, 39, 40]).

On the other hand, from 2000's, the study on elliptic equations on the hyperbolic space has attracted a great interest. In particular, the Lane-Emden equation (H) on the hyperbolic space has been well-investigated ([7, 8, 9, 10, 11, 25, 26, 28, 30, 33, 41, 42]). Now, we shall state known results on separation phenomena of radial solutions to (H). Here, for each $\alpha > 0$, we denote by $u_\alpha^H = u_\alpha^H(r)$ the radial solution of (H) satisfying $u_\alpha^H(0) = \alpha$, i.e., u_α^H is the solution of the following initial value problem:

$$\begin{cases} u''(r) + \dfrac{N-1}{\tanh r}u'(r) + |u(r)|^{p-1}u(r) = 0 \quad \text{in} \quad (0, +\infty), \\ u(0) = \alpha. \end{cases}$$

Regarding separation phenomena of radial solutions to (H), E. Berchio, A. Ferrero, G. Grillo [7] proved the following:

Proposition 1.2 (Theorem 2.14 in [7]). *Let $p > 1$ Then there exists $\alpha_0 = \alpha_0(N, p) > 0$ such that for any $\alpha, \beta \in (0, \alpha_0](\alpha \neq \beta)$, u_α^H and u_β^H cannot intersect each other in* $(0, \infty)$.

Differently from Proposition 1.1, Proposition 1.2 imply that even when $p \in (p_s(N), p_{JL}(N))$, there exist two regular radial solutions which cannot intersect each other in $(0, \infty)$. The difference is related to the positivity of the first eigenvalue of $-\Delta_g$. Indeed, in the proof of Proposition 1.2, letting the value at the origin less than the first eigenvalue sufficiently, they showed the separation phenomena of radial solutions to (H).

From Proposition 1.2 and the analogue of Proposition 1.1, we can expect that for $p \geq p_{JL}(N)$, any two regular radial solutions to (H) cannot intersect each other in $(0, \infty)$. Indeed, in [8], they state that by numerical analysis, for sufficiently large p and N, any two regular radial solutions do not intersect each other in $(0, \infty)$. Then, motivated by above, we are interested in the following problem:

Problem 1.1. *Is there a critical exponent with respect to separation phenomena of radial solutions to* (H)?

Following Problem 1.1, we shall investigate separation phenomena of radial solutions to the equation (H).

Our main results of [27] are the followings:

Theorem 1.1. *Let $p \geq p_{JL}(N)$. Then, for any $\alpha, \beta > 0$ with $\alpha \neq \beta$, u^H_α and u^H_β cannot intersect each other in* $(0, \infty)$.

Theorem 1.2. *Let $p \in (1, p_{JL}(N))$. Then, there exists $\alpha_1 = \alpha_1(N, p) > 0$ such that for any $\alpha, \beta > \alpha_1$ with $\alpha \neq \beta$, u^H_α and u^H_β intersect at least once in* $(0, \infty)$.

Theorems 1.1-1.2 imply that $p_{JL}(N)$ is the critical exponent with respect to separation phenomena of radial solutions to (H). Therefore, we obtain an affirmative answer to Problem 1.1.

As a consequence of Theorem 1.1, we shall also obtain the existence of a singular solution of (H). In [27], our result is the following:

Theorem 1.3. *Let $p \geq p_{JL}(N)$. Then, there exists a singular solution $U^H(r)$ of* (H) *such that*

$$\lim_{r \to +0} U^H(r)(\sinh r)^{\frac{2}{p-1}} = L,$$

and for any $\alpha > 0$,

$$u^H_\alpha(r) < U^H(r) < \frac{L}{(\tanh r)^{\frac{2}{p-1}}} \quad \text{in} \quad (0, \infty), \tag{1.1}$$

where

$$L = \left\{ \frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right\}^{\frac{1}{p-1}}.$$

Here, the inequality (1.1) in Theorem 1.3 implies that for $p \geq p_{JL}(N)$, the singular solution $U^H(r)$ and any regular radial solution to (H) also cannot intersect each other in $(0, \infty)$.

For the proof of Theorems 1.1-1.3, see [27]. Here, in the proof of Theorem 1.1, applying Sturm-Liouville theory, we shall obtain separation property of radial solutions to (H). Then, the method of the proof of Theorem 1.1 is also applicable to analysis of separation phenomena of radial solutions to the following weighted Lane-Emden equation in $\mathbb{R}^N$:

$$\text{(M)} \qquad -\Delta u = \frac{1}{1+|x|^2}|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 3$, and $p > 1$. Here, the equation (M) is known as Matukuma's equation ([37, 38]). In the rest of this paper, we shall introduce the proof of separation property of radial solutions to (M). Remark that the result on separation property of radial solutions to (M) has been already obtained in [3, 32] and they employ phase plane method. In this paper, making use of the argument of Sturm-Liouville theory, we shall derive separation property of radial solutions to (M). In addition, we also remark that we use the modification of the proof of [5] and the result on separation property of radial solutions to (M) has also been derived in [5].

2 Matukuma's equation

2.1 Preliminaries

We shall consider the following Matukuma's equation in $\mathbb{R}^N$:

$$\text{(M)} \qquad -\Delta u = \frac{1}{1+|x|^2}|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 3$ and $p > 1$. Here, for each $\alpha > 0$, we denote by $u_\alpha = u_\alpha(r)$ the radial solution of (M) satisfying $u_\alpha(0) = \alpha$. Namely, u_α is the solution of the following initial value problem:

$$\begin{cases} u''(r) + \dfrac{N-1}{r}u'(r) + \dfrac{1}{1+r^2}|u(r)|^{p-1}u(r) = 0 \quad \text{in} \quad (0, +\infty), \\ u(0) = \alpha. \end{cases}$$

Concerning separation property of radial solutions to (M), the following result was obtained in [3, 32]:

Theorem 2.1 (Theorem 1.2 in [3], Theorem 1 in [32])**.** *Let $p \geq p_{JL}(N)$. Then, for any $\alpha, \beta > 0$ with $\alpha \neq \beta$, u_α and u_β cannot intersect each other in $(0, \infty)$.*

In this section, making use of Sturm-Liouville theory, we shall prove Theorem 2.1. Here, the following proof of Theorem 2.1 is the modification of that of [5]. To begin with, we define

$$t = \log r, \quad \text{and} \quad v_\alpha(t) = r^{\frac{2}{p-1}}u_\alpha(r).$$

Then $v = v_\alpha$ satisfies

$$v'' + av' - L^{p-1}v + \frac{1}{1+e^{2t}}v^p = 0,$$

where

$$a = N - 2 - \frac{4}{p-1}, \quad \text{and} \quad L = \left\{\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right\}^{\frac{1}{p-1}}.$$

Remark that $a > 0$ if and only if $p > p_s(N)$. Moreover, to the aim of the proof of Theorem 2.1, we prepare the following lemma:

Lemma 2.1 (Lemma 3.1 in [5])**.** *Let $p \geq p_{JL}(N)$ and $T \in \mathbb{R}$. Then there exists no function $z \in C^2(-\infty, T]$ satisfying the following* (i)-(iii):

(i) $z'' + az' + \left(\frac{a}{2}\right)^2 z > 0$ *for* $t \in (-\infty, T)$;

(ii) $z(t) > 0$ *for* $t \in (-\infty, T)$ *and* $z(T) = 0$;

(iii) $z(t)$ *and* $z'(t)$ *are bounded on* $(-\infty, T)$.

Lemma 2.1 has been already proved in [5].

2.2 Proof of Theorem 2.1

To the aim of the proof of Theorem 2.1, we shall show the following proposition:

Proposition 2.1. *Let $p \geq p_{JL}(N)$. Then, for any $\alpha > 0$, v_α satisfies*

$$v_\alpha(t) < L(1+e^{2t})^{\frac{1}{p-1}} \quad \textit{for} \quad t \in (-\infty, \infty).$$

Proof. We prove the assertion by contradiction. Assume that there exists $T \in \mathbb{R}$ such that

$$v_\alpha(t) < L(1+e^{2t})^{\frac{1}{p-1}} \quad \text{for} \quad t \in (-\infty, T), \quad \text{and} \quad v_\alpha(T) = L(1+e^{2T})^{\frac{1}{p-1}}.$$

Now, we take

$$V(t) = L(1+e^{2t})^{\frac{1}{p-1}},$$

and $V(t)$ satisfies, for $t \in (-\infty, \infty)$,

$$\begin{aligned} &V'' + aV' - L^{p-1}V + \frac{1}{1+e^{2t}}V^p \\ =&\frac{2}{p-1}Le^{2t}(1+e^{2t})^{\frac{1}{p-1}-2}\left\{2 + \frac{2}{p-1}e^{2t} + a(1+e^{2t})\right\} > 0. \end{aligned}$$

Here, the last inequality is followed from $a > 0$. Moreover, setting

$$w_\alpha(t) = V(t) - v_\alpha(t),$$

we have

$$w_\alpha'' + a w_\alpha' - L^{p-1} w_\alpha + \frac{1}{1+e^{2t}}\Theta > 0, \tag{2.1}$$

where

$$\Theta(t) = V^p(t) - v_\alpha^p(t). \tag{2.2}$$

Then, applying Lemma 2.1, we shall show the non-existence of w_α. Indeed, we can verify that w_α satisfies (ii)-(iii) of Lemma 2.1 directly. Furthermore, using the mean-value theorem, we observe from (2.2) that

$$\Theta(t) < pV^{p-1}(t) w_\alpha(t) \quad \text{for} \quad t \in (-\infty, T). \tag{2.3}$$

Hence, combining (2.1) with (2.3), we derive

$$w_\alpha'' + a w_\alpha' + (p-1)L^{p-1} w_\alpha > 0 \quad \text{for} \quad t \in (-\infty, T).$$

Since $p \geq p_{JL}(N)$ is equivalent to

$$(p-1)L^{p-1} \leq \left(\frac{a}{2}\right)^2,$$

we see that w_α satisfies (i) of Lemma 2.1. Thus, we observe from Lemma 2.1 that there exists no function w_α. This is a contradiction and we complete the proof. □

Proof of Theorem 2.1. To begin with, we shall show that for any $\alpha, \beta > 0$ with $\alpha < \beta$,

$$v_\alpha(t) < v_\beta(t) \quad \text{for} \quad t \in (-\infty, \infty). \tag{2.4}$$

We prove this assertion by contradiction. Assume that there exists $T \in \mathbb{R}$ such that

$$v_\alpha(t) < v_\beta(t) \quad \text{for} \quad t \in (-\infty, T), \quad \text{and} \quad v_\alpha(T) = v_\beta(T).$$

Then, setting

$$w_{\alpha,\beta}(t) = v_\beta(t) - v_\alpha(t),$$

we have

$$w_{\alpha,\beta}'' + a w_{\alpha,\beta}' - L^{p-1} w_{\alpha,\beta} + \frac{1}{1+e^{2t}}\Theta_{\alpha,\beta} > 0, \tag{2.5}$$

where

$$\Theta_{\alpha,\beta}(t) = v_\beta^p(t) - v_\alpha^p(t). \tag{2.6}$$

Then, applying Lemma 2.1, we shall show the non-existence of $w_{\alpha,\beta}$. Indeed, we can check that $w_{\alpha,\beta}$ satisfies (ii)-(iii) of Lemma 2.1 directly. Furthermore, using the mean-value theorem and Proposition 2.1, we observe from (2.6) that

$$\Theta_{\alpha,\beta}(t) < p v_\beta^{p-1}(t) w_{\alpha,\beta}(t) < pL^{p-1}(1+e^{2t}) w_{\alpha,\beta}(t) \quad \text{for} \quad t \in (-\infty, T). \tag{2.7}$$

Then, combining (2.5) with (2.7), we obtain

$$w''_{\alpha,\beta} + aw'_{\alpha,\beta} + (p-1)L^{p-1}w_{\alpha,\beta} > 0 \quad \text{for} \quad t \in (-\infty, T).$$

Since $p \geq p_{JL}(N)$ is equivalent to

$$(p-1)L^{p-1} \leq \left(\frac{a}{2}\right)^2,$$

we see that $w_{\alpha,\beta}$ satisfies (i) of Lemma 2.1. Therefore, we observe from Lemma 2.1 that there exists no function $w_{\alpha,\beta}$. This is a contradiction and we obtain (2.4). Then, the inequality (2.4) is equivalent to

$$u_\alpha(t) < u_\beta(t) \quad \text{for} \quad t \in (-\infty, \infty).$$

Thus, we complete the proof. □

References

[1] S. Bae, *Separation structure of positive radial solutions of a semilinear elliptic equation in* $\mathbb{R}^n$, J. Differential Equations, **194** (2003), no. 2, 460–499.

[2] S. Bae, *Infinite multiplicity and separation structure of positive solutions for a semilinear elliptic equation in* $\mathbb{R}^n$, J. Differential Equations, **200** (2004), no. 2, 274–311.

[3] S. Bae, T.-K. Chang, *On a class of semilinear elliptic equations in* $\mathbb{R}^n$, J. Differential Equations, **185** (2002), no. 1, 225–250.

[4] S. Bae, *Entire solutions with asymptotic self-similarity for elliptic equations with exponential nonlinearity*, J. Math. Anal. Appl., **428** (2015), no. 2, 1085–1116.

[5] S. Bae, Y. Naito, *Existence and separation of positive radial solutions for semilinear elliptic equations*, J. Differential Equations, **257** (2014), no. 7, 2430–2463.

[6] S. Bae, Y. Naito, *Separation structure of radial solutions for semilinear elliptic equations with exponential nonlinearity*, Discrete Contin. Dyn. Syst., **38** (2018), no. 9, 4537–4554.

[7] E. Berchio, A. Ferrero, G. Grillo, *Stability and qualitative properties of radial solutions of the Lane-Emden-Fowler equation on Riemannian models*, J. Math. Pures Appl. (9) **102** (2014), no. 1, 1–35.

[8] M. Bonforte, F. Gazzola, G. Grillo, J. L. Vázquez, *Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space*, Calc. Var. Partial Differential Equations **46** (2013), no. 1-2, 375–401.

[9] C. Bandle, Y. Kabeya, *On the positive, "radial" solutions of a semilinear elliptic equation in* $\mathbb{H}^N$, Adv. Nonlinear Anal. **1** (2012), no. 1, 1–25.

[10] C. Bandle, M.A. Pozio, A. Tesei, *The Fujita exponent for the Cauchy problem in the hyperbolic space*, J. Differential Equations **251** (2011), 2143–2163.

[11] M. Bhakta, K. Sandeep, *Poincaré-Sobolev equations in the hyperbolic space*, Calc. Var. Partial Differential Equations **44** (2012), no. 1-2, 247–269.

[12] H. Brézis, *Elliptic equations with limiting Sobolev exponents-the impact of topology*, Comm. Pure Appl. Math. **39** (1986), no. S, suppl., S17–S39.

[13] H. Brézis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[14] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Dover, New York (1967).

[15] E.N. Dancer, Y. Du, Z. Guo, *Finite Morse index solutions of an elliptic equation with supercritical exponent*, J. Differential Equations **250** (2011), no. 8, 3281–3310.

[16] Y. Deng, Y. Li, F. Yang, *On the positive radial solutions of a class of singular semilinear elliptic equations*, J. Differential Equations, **253** (2012), no. 2, 481–501.

[17] V.R. Emden, *Gaskugeln, Anwendungen der mechanischen Warmentheorie auf Kosmologie und meteorologische Probleme*, Teubner, Leipzig (1907), Chap. XII.

[18] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl. (9) **87** (2007), no. 5, 537–561.

[19] R. H. Fowler, *Further studies of Emden's and similar differential equations*, Q. J. Math. (Oxford Series) **2** (1931), 259–288.

[20] B. Gidas, J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), no. 8, 883–901.

[21] B. Gidas, J. Spruck, *Global and local behavior of positive solution of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525–598.

[22] C. Gui, *Positive entire solutions of the equation* $\Delta u + f(x,u) = 0$, J. Differential Equations, **99** (1992), no. 2, 245–280.

[23] C. Gui, W.-M. Ni, X. Wang, *On the stability and instability of positive steady states of a semilinear heat equation in* $\mathbb{R}^n$, Comm. Pure Appl. Math. **45** (1992), no. 9, 1153–1181.

[24] C. Gui, W.-M. Ni, X. Wang, *Further study on a nonlinear heat equation*, J. Differential Equations, **169** (2001), 588–613.

[25] S. Hasegawa, *A critical exponent for Hénon type equation on the hyperbolic space*, Nonlinear Anal. **129** (2015), 343–370.

[26] S. Hasegawa, *A critical exponent of Joseph-Lundgren type for an Hénon equation on the hyperbolic space*, Communications on Pure and Applied Analysis, **16** (2017), no. 4, 1189–1198.

[27] S. Hasegawa, *Separation phenomena of radial solutions to Lane-Emden equation on the hyperbolic space*, in preparation.

[28] H. He, *The existence of solutions for Hénon equation in hyperbolic space*, Proc. Japan Acad. Ser. A Math. Sci. **89** (2013), no. 2, 24–28.

[29] D. D. Joseph, T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal., **49** (1972/73), 241–269.

[30] Y. Kabeya, *A unified approach to Matukuma type equations on the hyperbolic space or on a sphere*, Discrete Contin. Dyn. Syst. 2013, Dynamical systems, differential equations and applications. 9th AIMS Conference. Suppl., 385–391.

[31] J.H. Lane, *In the theoretical temperature of the Sun under the hypothesis of a gaseous mass maintaining ots volume by its internal heat and depending on the laws of gases known to terrestrial experiment*, Am. J. Sci. Ser. II **50** (1869), 57–74.

[32] Y. Liu, Y. Li, Y. Deng, *Separation property of solutions for a semilinear elliptic equation*, J. Differential Equations, **163** (2000), no. 2, 381–406.

[33] G. Mancini, K. Sandeep, *On a semilinear elliptic equation in* $\mathbb{H}^n$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 4, 635–671.

[34] Y. Miyamoto, *Intersection properties of radial solutions and global bifurcation diagrams for supercritical quasilinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl., **23** (2016), 1–24.

[35] Y. Miyamoto, K. Takahashi, *Generalized Joseph-Lundgren exponent and intersection properties for supercritical quasilinear elliptic equations*, Arch. Math. (Basel), **108** (2017), no. 1, 71–83.

[36] W.-M. Ni, *On the elliptic equation* $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$*, its generalizations, and applications in geometry*, Indiana Univ. Math. J. **31** (1982), no. 4, 493–529.

[37] W.-M. Ni, S. Yotsutani, *On Matukuma's equation and related topics*, Proc. Japan Acad. Ser. A Math. Sci., **62** (1986), no. 7, 260–263.

[38] W.-M. Ni, S. Yotsutani, *Semilinear elliptic equations of Matukuma-type and related topics*, Japan J. Appl. Math. **5** (1988), no. 1, 1–32.

[39] P. Poláčik, E. Yanagida, *On bounded and unbounded global solutions of a supercritical semilinear heat equation*, Math. Ann., **327** (2003), 745–771.

[40] P. Poláčik, E. Yanagida, *A Liouville property and quasiconvergence for a semilinear heat equation*, J. Differential Equations, **208** (2005), 194–214.

[41] F. Punzo, *On well-posedness of semilinear parabolic and elliptic problems in the hyperbolic space*, J. Differential Equations **251** (2011), no. 7, 1972–1989.

[42] S. Stapelkamp, *The Brézis-Nirenberg problem on* $\mathbb{H}^n$. *Existence and uniqueness of solutions*, Elliptic and parabolic problems (Rolduc/Gaeta, 2001), World Sci. Publ., River Edge, NJ, (2002), 283–290.

[43] J.I. Tello, *Stability of steady states of the Cauchy problem for the exponential reaction-diffusion equation*, J. Math. Anal. Appl., **324** (2006), no. 1, 381–396.

[44] X. Wang, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc., **337** (1993), no. 2, 549–590.