数理解析研究所講究録 1372 巻 2004 年 144-150

Exponential and non-exponential convergence of solutions in some classes of nonlinear systems with application to neural networks

Jito Vanualailai (南太平洋大学 ジトー バニュアライライ)
Department of Mathematics and Computing Science, University of the South Pacific, FIJI.

Shin-ichi Nakagiri (神戸大学工学部 中桐 信一)
Department of Applied Mathematics, Faculty of Engineering, Kobe University, JAPAN.

1 Introduction

This paper is motivated to a large extent by modern applications of the Lyapunov method, especially those in artificial neural networks.

Historically, the rigorous application of the Lyapunov method to artificial neural networks can be traced back to the pioneering work of the mathematician Grossberg, who, in the 1970s, started a Lyapunov function- and Lyapunov functional-based methods for classifying the dynamical behaviours of a wide variety of competitive dynamical systems, and who, by 1988, had accumulated sufficient amount of important and fundamental results, which he then summarized in an excellent review on the then-current state of development [3]. Within the same period between the late 1970s and late 1980s, the physicist Hopfield concentrated on a particular form of dynamical systems that were being considered in general by Grossberg. Then, in 1984, Hopfield published a landmark paper [6] that, to this day, popularized the term *the Hopfield or Hopfield-type (artificial) neural network.* Hopfield designed his network using a Lyapunov function which is now recognized as a special form of the Lyapunov function that was proposed a year earlier by Cohen & Grossberg [1] for a more general system. In 1985 and 1986, Hopfield & Tank ([7], [8]) applied Hopfield's earlier findings to firmly establish the role of Hopfield-type neural networks as standard models that perform some computational task, such as recognition and association, on a given key pattern via interaction between a number of interconnected units having simple functions. As explained by Matsuoka [10], the key pattern presented to a Hopfield-type neural network is an initial state of the network. Then the network must be designed, using the Lyapunov method, for example, such that the network's state settles ultimately to an equilibrium which depends only on the key pattern. In this paper, we will also consider this model and discuss recent results.

We start by considering the autonomous system of the form

$$\mathbf{x}'(t) = \mathbf{g}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad t \geq t_0 \geq 0. \tag{1}$$

Guided by a well-known 1954 result of Krasovskii [9], we will strive to portray a simple method of generating the quadratic Lyapunov function for (1). The Lyapunov function guarantees global exponential stability. We then perturb (1) to include time-varying functions, and prove, by extending the quadratic Lyapunov function, that the same conditions for the perturbed system yield convergence of solutions to the equilibrium points of (1) when the perturbation is either L^2, or bounded and decays with time. We end by applying the stability and convergence criteria to Hopfield-type neural networks.

Throughout the article, we suppose that, in (1), the function $\mathbf{g} = (g_1, \dots, g_n)^T$ is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}_0)$, with $\mathbf{x} = (x_1, \dots, x_n)^T$. It is assumed that the readers of this article are familiar with the various standard definitions of Lyapunov stability. Thus, without loss of generality, we carry the assumption that $\mathbf{g}(\mathbf{0}) = \mathbf{0}$ so that $\mathbf{0}$ is the equilibrium point of (1).

2 Convergence criteria

In 1954, Krasovskii [9] established an asymptotic stability criterion that avoided the linearization principle, and in the process established a method of estimating the extent of asymptotic stability region for nonlinear systems. He assumed that $\mathbf{g} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$ and $\mathbf{g}(\mathbf{0}) = \mathbf{0}$. Then system (1) can be written as

$$\mathbf{x}'(t) = \int_0^1 \mathbf{J}(s\mathbf{x})\mathbf{x}ds\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0\,,$$

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J}(\mathbf{x}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}) = [J_{ij}(\mathbf{x})]_{n\times n}\,, \quad \text{where} \quad J_{ij}(\mathbf{x}) = \frac{\partial g_i}{\partial x_j}(\mathbf{x})\,.$$

The following result by Krasovskii is a fundamental one in control theory.

Theorem 1 (Krasovskii, 1954). *Let* $\mathbf{g} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$ *and* $\mathbf{g}(\mathbf{0}) = \mathbf{0}$. *If there exists a constant positive definite symmetric matrix* $\mathbf{P}$ *such that*

$$\mathbf{x}^T[\mathbf{P}\mathbf{J}(\mathbf{x}) + \mathbf{J}^T(\mathbf{x})\mathbf{P}]\mathbf{x}$$

is a negative definite function, then the zero solution of (1) is globally asymptotically stable.

For our purpose, we will need a criterion that explicitly uses each component of system (1). Thus, defining

$$\mathbf{D}(\mathbf{x}) = [d_{ij}(\mathbf{x})]_{n\times n}\,, \quad \text{where} \quad d_{ij}(\mathbf{x}) = \int_0^1 J_{ij}(s\mathbf{x})ds\,, \tag{2}$$

and given that $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, we can write system (1) as

$$\mathbf{x}'(t) = \mathbf{D}(\mathbf{x})\mathbf{x}\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0\,, \tag{3}$$

the i-th component of which is

$$x_i'(t) = d_{ii}(\mathbf{x})x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}(\mathbf{x})x_j \,.$$

Our first result gives a global exponential stability criterion for the autonomous system (1).

Theorem 2. *Let* $\mathbf{g} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$ *and* $\mathbf{g}(\mathbf{0}) = \mathbf{0}$*, and define*

$$\beta_i(\mathbf{x}) = d_{ii}(\mathbf{x}) + \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} |d_{ij}(\mathbf{x}) + d_{ji}(\mathbf{x})| \,.$$

Suppose there are constants $c_i > 0$ *such that* $\beta_i(\mathbf{x}) \leq -c_i < 0$ *for* $i = 1, \dots, n$ *and* $\mathbf{x} \in \mathbb{R}^n$*. Then the zero solution of (1) is globally exponentially stable.*

We can strengthen Theorem 2, but, as we shall see in the application to neural networks, this is necessary to produce practically useful results that are currently widely used. We have the following result, which is just another version of Krasovskii's theorem:

Theorem 3. *Let* $\mathbf{g} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$ *and* $\mathbf{g}(\mathbf{0}) = \mathbf{0}$*, and define*

$$\tau_i(\mathbf{x}) = J_{ii}(\mathbf{x}) + \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} |J_{ij}(\mathbf{x}) + J_{ji}(\mathbf{x})| \,.$$

Suppose there are constants $c_i > 0$ *such that* $\tau_i(\mathbf{x}) \leq -c_i < 0$ *for* $i = 1, \dots, n$ *and* $\mathbf{x} \in \mathbb{R}^n$*. Then the zero solution of (1) is globally exponentially stable.*

Let us next perturb (1) as follows:

$$\mathbf{x}'(t) = \mathbf{g}(\mathbf{x}) + \mathbf{h}(t)\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0\,, \tag{4}$$

where $\mathbf{h}(t) = (h_i(t), \dots, h_n(t))^T$, a vector function of t, need not be continuous on $[0, \infty)$. Then the following result maintains at least the convergence of solutions to the zero solution.

Theorem 4. *Let the conditions of either Theorem 2 or Theorem 3 hold. If*

$$\sum_{i=1}^{n} \int_t^{t+1} [h_i(s)]^2 ds \to 0 \ \text{as} \ t \to \infty,$$

then all solutions of (4) are uniformly bounded and tend to zero.

To prove Theorem 4, we need the following lemma which is a straighforward consequence of Theorem A in Hara [4]:

Lemma 1 (Hara, 1975). *Suppose that there exists a Liapunov function* $V(t, \mathbf{x})$ *of (4), continuous differentiable in* $[t_0, \infty) \times \mathbf{R}^n$*, satisfying the following conditions;*

(i) $a(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq b(\|\mathbf{x}\|)$, *where* $a(r) \in CIP$ *(the family of continuous and increasing positive definite functions),* $a(r) \to \infty$ *as* $r \to \infty$ *and* $b(r) \in CIP$,

(ii) $\frac{d}{dt}[V]_{(6)} \leq -cV+\lambda(t)(1+V)$, *where* $c > 0$ *is a constant and* $\lambda(t) \geq 0$ *satisfies* $\int_t^{t+1} \lambda(s)ds \to 0$ *as* $t \to \infty$.

Then, all solution $\mathbf{x}(t)$ *of (6) is uniformly bounded and satisfies* $\mathbf{x}(t) \to 0$ *as* $t \to \infty$.

Remark 1. We note that the most recent result on the convergence of solutions for time-varying systems in the form of (4)was suggested by Vanualailai, Soma & Nakagiri [11], and which is to fit system (1) and system (4).

3 Application to neural networks

Artificial neural networks could be considered as dynamical systems for which the convergence of system trajectories to equilibrium states is a necessity. Moreover, it is best to guarantee exponential convergence since this implies that the rate of convergence to an equilibrium state can be measured, an important aspect in the stability analysis of neural networks.

In the first part of this section, we consider a continuous neural network that is described thoroughly in Hirsch [5], and provide a convergence criterion using Theorem 2. In the second part, we consider the Hopfield-type neural network, and provide convergence criteria using Theorems 3 and 4.

3.1 An artificial neural network of the type $\mathbf{x}'(t) = \mathbf{g}(\mathbf{x})$

The neural network in question has n units. To the ith unit, we associate its *activation state* at time t, a real number $x_i = x_i(t)$; an *output function* μ_i; a fixed *bias* θ_i; and an *output signal* $R_i = \mu_i(x_i + \theta_i)$. The *weight* or connection strength on the line from unit j to unit i is a fixed real number W_{ij}. When $W_{ij} = 0$, there is no transmission from unit j to unit i. The *incoming signal* from unit j to unit i is $S_{ij} = W_{ij}R_j$. In addition, there can be a vector $\mathbf{I}$ of any number of *external inputs* feeding into some or all units, so that we may write $\mathbf{I} = (I_1, \ldots, I_m)^T$.

A neural network with fixed weights is a dynamical system: given initial values of the activation of all units, the future activations can be computed. The future activation states are assumed to be determined by a system of n differential equations, the ith equation of which is

$$\begin{aligned} x_i'(t) &= G_i(x_i, S_{i1}, \ldots, S_{in}, \mathbf{I}) = G_i(x_i, W_{i1}R_1, \ldots, W_{in}R_n, \mathbf{I}) \\ &= G_i(x_i; W_{i1}\mu_1(x_1+\theta_1), \ldots, W_{in}\mu_n(x_n+\theta_n); I_1, \ldots, I_m) . \end{aligned} \tag{5}$$

With W_{ij}, θ_i, I_k and some initial value $x_i(t_0)$, $t_0 \geq 0$, assumed known, we can write (5) as

$$x_i'(t) = g_i(x_1, \ldots, x_n), \quad x_i(t_0) = x_{i0},$$

which is the ith component of the system

$$\mathbf{x}'(t) = \mathbf{g}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{6}$$

where $\mathbf{g}$ is a vector on Euclidean space $\mathbb{R}^n$. We assume that $\mathbf{g}$ is continuously differentiable and satisfies the usual theorems on existence, continuity and uniqueness of solutions. Thus, since

$\mathbf{g} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$, we can define $\mathrm{D}(\mathbf{x})$ as in (2) but using $\mathbf{g}$ in (6). Hence, if $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, then system (6) can be written as

$$\mathbf{x}'(t) = \mathbf{D}(\mathbf{x})\mathbf{x}\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0\,,$$

the ith component of which is

$$x_i'(t) = d_{ii}(\mathbf{x})x_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} d_{ij}(\mathbf{x})x_j\,.$$

If we apply Theorem 2, we obtain the following convergence criterion.

Corollary 1. *Let* $\mathbf{g} \in C^1[\mathbb{R}^n, \mathbb{R}^n]$ *and* $\mathbf{g}(\mathbf{0}) = \mathbf{0}$. *Define*

$$\beta_i(\mathbf{x}) = d_{ii}(\mathbf{x}) + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} |d_{ij}(\mathbf{x}) + d_{ji}(\mathbf{x})|\,.$$

Suppose there there are constants $c_i > 0$ *such that* $\beta_i(\mathbf{x}) \leq -c_i < 0$ *for* $i = 1, \ldots, n$ *and* $\mathbf{x} \in \mathbb{R}^n$. *Then the zero solution of (6) is globally exponentially stable.*

3.2 The Hopfield-type neural network

Next, we look at the Hopfield-type neural network, which is a specific case of (5). It is modeled by the nonlinear differential equation

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{n} W_{ij}\, \mu_j(x_j(t) + \theta_j) + I_i(t) \\ &= -a_i x_i(t) + \sum_{j=1}^{n} W_{ij}\, \nu_j(x_j(t)) + I_i(t)\,, \end{aligned} \tag{7}$$

where $a_i > 0$ is the constant *decay rate*, $I_i(t)$ is the time-varying external input (to the ith neuron) defined almost everywhere on $[0, \infty)$ and ν_i is the suppressed notation for the fixed θ_i by having θ_i incorporated into ψ_i. The function ν_i is called the *neuron activation function*.

Now, define $\mathbf{A} = \mathrm{diag}(-a_1, \ldots, -a_n)$, $\mathbf{x} = (x_1, \ldots, x_n)^T$,

$$\mathbf{v}(\mathbf{x}) = (\nu_1(x_1), \ldots, \nu_n(x_n))^T\,,$$

$\mathbf{W} = [W_{ij}]_{n\times n}$ and $\mathbf{h}(t) = (I_i(t), \ldots, I_n(t))^T$. Then (7) is the ith component of the system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x} + \mathbf{W}\mathbf{v}(\mathbf{x}) + \mathbf{h}(t)\,, \quad \mathbf{x}(t_0) = \mathbf{x}_0\,. \tag{8}$$

3.2.1 Hopfield-type neural networks with constant external inputs

Let us first look at the case of constant external input vector, $\mathbf{h}(t) \stackrel{def}{=} \mathbf{k} = (I_i, \ldots, I_n)^T$. Assume that $\mathbf{x} = \mathbf{x}^*$ is the corresponding equilibrium point of (8) when the input vector is constant, so that $\mathbf{A}\mathbf{x}^* + \mathbf{W}\mathbf{v}(\mathbf{x}^*) + \mathbf{k} = \mathbf{0}$. Introduce the vector $\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{x}^*$. Then

$$\mathbf{u}'(t) \;=\; \mathbf{A}[\mathbf{u} + \mathbf{x}^*] + \mathbf{W}\mathbf{v}(\mathbf{u} + \mathbf{x}^*) + \mathbf{k}$$

$$
\begin{aligned}
&= \mathbf{A}[\mathbf{u}+\mathbf{x}^*] + \mathbf{W}\mathbf{v}(\mathbf{u}+\mathbf{x}^*) + \mathbf{k} - [\mathbf{A}\mathbf{x}^* + \mathbf{W}\mathbf{v}(\mathbf{x}^*) + \mathbf{k}] \\
&= \mathbf{A}[\mathbf{u}+\mathbf{x}^* - \mathbf{x}^*] + \mathbf{W}\,[\mathbf{v}(\mathbf{u}+\mathbf{x}^*) - \mathbf{v}(\mathbf{x}^*)] \\
&\overset{def}{=} \mathbf{A}\mathbf{u} + \mathbf{W}\mathbf{r}(\mathbf{u}) \\
&\overset{def}{=} \tilde{\mathbf{g}}(\mathbf{u})\,, \quad \mathbf{u}(t_0) = \mathbf{u}_0\,,
\end{aligned} \tag{9}
$$

the ith component of which is

$$
u_i'(t) = -a_i u_i + \sum_{j=1}^{n} W_{ij}[\nu_j(u_j + x_j^*) - \nu_j(x_j^*)] = -a_i u_i + \sum_{j=1}^{n} W_{ij} r_j(u_j)\,.
$$

It is clear that $\tilde{\mathbf{g}}(\mathbf{0}) = \mathbf{0}$, so that Theorem 2 is applicable to the zero solution of (9), and therefore to the equilibrium point $\mathbf{x} = \mathbf{x}^*$ of (8) with constant external inputs. Now, the Jacobian matrix yields

$$
J_{ii}(\mathbf{u}) = \frac{\partial \tilde{g}_i}{\partial u_i}(\mathbf{u}) = -a_i + W_{ii}\nu_i'(u_i + x_i^*) = -a_i + W_{ii} r_i'(u_i)\,,
$$

and

$$
J_{ij}(\mathbf{u}) = \frac{\partial \tilde{g}_i}{\partial u_j}(\mathbf{u}) = W_{ij}\nu_j'(u_j + x_j^*) = W_{ij} r_j'(u_j)\,, \quad i \neq j\,.
$$

Using Theorem 3 we can show the following result:

Corollary 2. *Let the neuron activation functions $r_i(u_i)$ be of C^1-class and $r_i(0) = 0$. Assume there exist constants $\rho_i > 0$ such that $0 \leq r_i'(u_i) \leq \rho_i$ for $i = 1, \ldots, n$ and $\mathbf{u} \in \mathbf{R}^n$. Define $y^+ = \max\{y, 0\}$ for all real numbers y and*

$$
\psi_i = -a_i + W_{ii}^+ \rho_i + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} (|W_{ij}|\rho_j + |W_{ji}|\rho_i)\ .
$$

If $\psi_i < 0$ for $i = 1, \ldots, n$, then the zero solution of (9) is globally exponentially stable.

Remark 2. Corollary 2 corresponds to a well-known 1996 result by Fang & Kincaid [2], Theorem 3.8 ii-d), page 1001.

3.2.2 Hopfield-type neural networks with time-varying external inputs

Let us next consider the perturbed systems:

$$
\mathbf{u}'(t) = \tilde{\mathbf{g}}(\mathbf{u}) + \mathbf{h}(t)\,, \quad \mathbf{u}(t_0) = \mathbf{u}_0\,. \tag{10}
$$

We can thus apply Theorem 4, giving the convergence of all solutions of (10) to the zero solution, and hence convergence of all solutions of (8) to the solution $\mathbf{x}^*$.

Corollary 3. *Let the conditions of Corollary 2 hold. If*

$$
\sum_{i=1}^{n} \int_t^{t+1} [h_i(s)]^2 ds \to 0 \ \textit{as}\ t \to \infty,
$$

then all solutions of (10) tend to zero.

Acknowledgements

The authors would like to thank Professor Tadayuki Hara, of Osaka Prefecture University, for pointing out his 1975 result which was used to improve Theorem 4. The first author would like to thank the French Government and the French Embassy, Fiji, for providing funds and assistance to allow the author to spend his leave (December, 2002 – March, 2003) at the Laboratoire des Signaux et Systèmes, Supélec, France, where the first version of this paper was written.

References

[1] M. A. Cohen and S. Grossberg, Absolute stability of global pattern formation and parallel memory storage by competitive neural networks, *IEEE Transactions on Systems, Man and Cybernetics*, SMC-13, (1983), 815–826.

[2] Y. Fang and T. G. Kincaid, Stability analysis of dynamical neural networks, *IEEE Transactions on Neural Networks*, 7, (1996), 996–1005.

[3] S. Grossberg, Nonlinear neural networks: principles, mechanisms and architectures. *Neural Networks*, 1, (1988), 17–61.

[4] T. Hara, On the asymptotic behaviour of solutions of certain non-autonomous differential equations, *Osaka J. Math.*, 12, (1975), 267–282.

[5] M. W. Hirsch, Convergent activation dynamics in continuous time networks, *Neural Networks*, 2, (1989), 331–349.

[6] J. J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, *Proceedings of the National Academy of Sciences*, 81, (1984), 3058–3092.

[7] J. J. Hopfield and D. W. Tank, "Neural" computation of decisions in optimization problems, *Biological Cybernetics*, 52, (1985), 141–152.

[8] J. J. Hopfield and D. W. Tank, Computing with neural networks: a model. *Science*, 233, (1986), 625–633.

[9] K. K. Krasovskii, On the stability in the large of a system of nonlinear differential equations. *Prikladnaya Matematika i Mekhanika*, 18, (1954), 735–737.

[10] K. Matsuoka, Stability conditions for nonlinear continuous neural networks with asymmetric connection weights, *Neural Networks*, 5, (1992), 495–500.

[11] J. Vanualailai, T. Soma and S. Nakagiri, Convergence of solutions and practical stability of Hopfield-type neural networks with time-varying external inputs, *Nonlinear Studies*, 9, (2002), 109–122.