Bifurcations of homoclinic orbits in reversible systems

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1 Introduction

Consider systems of the form

$$\dot{x} = f(x;\mu), \quad (x,\mu) \in \mathbb{R}^{2n} \times \mathbb{R}, \tag{1.1}$$

where $f : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^n$ is analytic, μ is a parameter and n is a positive integer. In [1] we studied bifurcations of homoclinic orbits to hyperbolic saddle equilibria in a class of systems of the form (1.1) including Hamiltonian systems. They also arise as bifurcations of solitons or pulses in partial differential equations (PDEs), and have attracted much attention even in the fields of PDEs and nonlinear waves. In this talk, we continue to discuss such bifurcations for reversible systems. The results are also used to study bifurcations of radially symmetric solutions in a coupled elliptic system [4]. See [3] for the details on our results including the proofs of the main theorems.

2 Assumptions

We first make the following assumptions.

(R1) The system (1.1) is *reversible*, i.e., there exists a linear involution R such that $R^2 = \mathrm{id}_{2n}$ and $f(Rx;\mu) + Rf(x;\mu) = 0$ for any $(x,\mu) \in \mathbb{R}^{2n} \times \mathbb{R}$, where id_{2n} is the $2n \times 2n$ identity matrix. Moreover, $\dim \mathrm{Fix}(R) = n$, where $\mathrm{Fix}(R) = \{x \in \mathbb{R}^{2n} \mid Rx = x\}$;

(R2) The origin O is an equilibrium in (1.1) for all $\mu \in \mathbb{R}$, i.e., $f(0;\mu) = 0$.

Note that $O \in \mathrm{Fix}(R)$ since $RO = O$. By assumption (R1) there exists a splitting $\mathbb{R}^{2n} = \mathrm{Fix}(R) \oplus \mathrm{Fix}(-R)$. Without loss of generality we can take the standard scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^{2n}$ such that $\mathrm{Fix}(-R) = \mathrm{Fix}(R)^\perp$. A fundamental characteristic of reversible systems is that if $x(t)$ is a solution, then so is $Rx(-t)$. We call a solution (and the corresponding orbit) *symmetric* if $x(t) = Rx(-t)$. It is a well-known fact that an orbit is symmetric if and only if it intersects the space $\mathrm{Fix}(R)$. Moreover, if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathrm{D}_x f(0;\mu)$, then so are $-\lambda$ and $\overline{\lambda}$.

(R3) The Jacobian matrix $\mathrm{D}_x f(0;0)$ has $2n$ eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ such that $0 < \mathrm{Re}\lambda_1 \le \cdots \le \mathrm{Re}\lambda_n$ (i.e., the origin is a hyperbolic saddle).

(R4) The equilibrium O has a symmetric homoclinic orbit $x^{\mathrm{h}}(t)$ with $x^{\mathrm{h}}(0) \in \mathrm{Fix}(R)$ at $\mu = 0$.

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The variational equation (VE) of (1.1) around $x^{\mathrm{h}}(t)$ at $\mu = 0$ is given by

$$\dot{\xi} = \mathrm{D}_x f(x^{\mathrm{h}}(t); 0)\xi, \tag{2.1}$$

to which $\xi = \dot{x}^{\mathrm{h}}(t)$ is a bounded solution tending to zero exponentially as $t \to \pm\infty$. Since $f(Rx; 0) + Rf(x; 0) = 0$, we have $\mathrm{D}_x f(x^{\mathrm{h}}(t); 0)R + R\mathrm{D}_x f(x^{\mathrm{h}}(t); 0) = 0$. Hence, if $\xi(t)$ is a solution to (2.1), then so are $\pm R\xi(-t)$ as well as $-\xi(t)$. For (2.1), we also say that a solution $\xi(t)$ is *symmetric* and *antisymmetric* if $\xi(t) = R\xi(-t)$ and $\xi(t) = -R\xi(-t)$, respectively, and show that it is symmetric and antisymmetric if and only if it intersects the spaces $\mathrm{Fix}(R)$ and $\mathrm{Fix}(-R) = \mathrm{Fix}(R)^{\perp}$, respectively, at $t = 0$. We easily see that $\xi = \dot{x}^{\mathrm{h}}(t)$ is antisymmetric since $\dot{x}^{\mathrm{h}}(t) = -R\dot{x}^{\mathrm{h}}(-t)$.

(R5) The VE (2.1) has two linearly independent bounded solution $\xi = \varphi_1(t)\ (= \dot{x}^{\mathrm{h}}(t)), \varphi_2(t)$, such that $\varphi_2(0) \in \mathrm{Fix}(R)$.

3 Main results

Under assumptions (R1)-(R5), we have the following.

Lemma 3.1. *There exist linearly independent solutions $\varphi_j(t)$, $j = 3, \ldots, 2n$, to (2.1) such that they are also linearly independent of $\varphi_j(t)$, $j = 1, 2$, and satisfy the following conditions:*

$$\begin{array}{lll}
\lim_{t\to+\infty} |\varphi_j(t)| = 0, & \lim_{t\to-\infty} |\varphi_j(t)| = \infty & \textit{for } j = 3, \ldots, n; \\
\lim_{t\to\pm\infty} |\varphi_j(t)| = \infty & & \textit{for } j = n+1, n+2; \\
\lim_{t\to+\infty} |\varphi_j(t)| = \infty, & \lim_{t\to-\infty} |\varphi_j(t)| = 0 & \textit{for } j = n+3, \ldots, 2n;
\end{array}$$

with $\varphi_{n+1}(0) \in \mathrm{Fix}(R)$ and $\varphi_{n+2}(0) \in \mathrm{Fix}(-R)$.

Let $\Phi(t) = (\varphi_1(t), \ldots, \varphi_{2n}(t))$. Then $\Phi(t)$ is a fundamental matrix to (2.1). Define $\psi_j(t)$, $j = 1, \ldots, 2n$, by

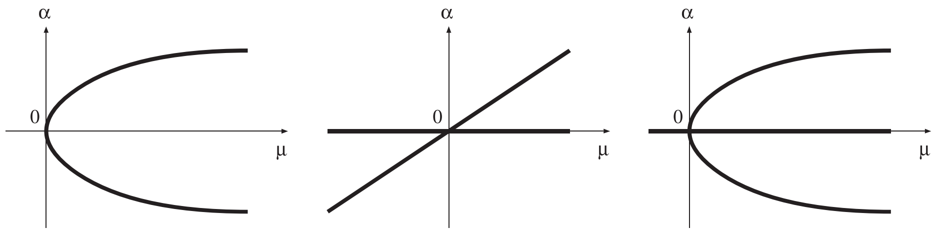
$$\langle \psi_j(t), \varphi_k(t) \rangle = \delta_{jk}, \quad j, k = 1, \ldots, 2n,$$

where δ_{jk} is Kronecker's delta. The functions $\psi_j(t)$, $j = 1, \ldots, 2n$, can be obtained by the formula $\Psi(t) = (\Phi^*(t))^{-1}$, where $\Psi(t) = (\psi_1(t), \ldots, \psi_{2n}(t))$ and the superscript $*$ represents the transpose operator. We can also have

$$\begin{array}{lll}
\lim_{t\to\pm\infty} |\psi_j(t)| = \infty & & \text{for } j = 1, 2; \\
\lim_{t\to+\infty} |\psi_j(t)| = \infty, & \lim_{t\to-\infty} |\psi_j(t)| = 0 & \text{for } j = 3, \ldots, n; \\
\lim_{t\to\pm\infty} |\psi_j(t)| = 0 & & \text{for } j = n+1, n+2; \\
\lim_{t\to\infty} |\psi_j(t)| = 0, & \lim_{t\to-\infty} |\psi_j(t)| = \infty & \text{for } j = n+3, \ldots, 2n.
\end{array}$$

with $\psi_1(0), \psi_{n+2}(0) \in \mathrm{Fix}(-R^*)$ and $\psi_2(0), \psi_{n+1}(0) \in \mathrm{Fix}(R^*)$. Moreover, $\Psi(t)$ is a fundamental matrix to the adjoint equation

$$\dot{\xi} = -\mathrm{D}_x f(x^{\mathrm{h}}(t); 0)^* \xi. \tag{3.1}$$



(a) Saddle-node bifurcation (b) Transcritical bifurcation (c) Pitchfork bifurcation

Figure 1. Bifurcation diagrams: Supercritical ones are plotted in Figs. (a) and (c).

Note that if $\xi(t)$ is a solution to (3.1), then so are $\pm R^*\xi(-t)$ as well as $-\xi(t)$.

We look for a symmetric homoclinic orbit of the form

$$x = x^{\mathrm{h}}(t) + \alpha\varphi_2(t) + \mathrm{O}(\sqrt{|\alpha|^4 + |\mu|^2}) \tag{3.2}$$

satisfying $x(0) \in \mathrm{Fix}(R)$ in (1.1) when $\mu \neq 0$, where $\alpha \in \mathbb{R}$. Let κ be a positive real number such that $\kappa < \frac{1}{4}\mathrm{Re}\lambda_1$, and define the Banach space

$$\mathscr{Z}^0 = \{z \in C^0(\mathbb{R}, \mathbb{R}^n) \mid \sup_{t\in\mathbb{R}} |z(t)|e^{\kappa|t|} < \infty, z(t) = -Rz(-t), t \in \mathbb{R}\}$$

where the supremum is taken as the norm. Let $\Pi : \mathscr{Z}^0 \to \mathscr{Z}^0$ be a projection given by

$$\Pi z(t) = q(t)\varphi_{n+2}(t) \int_{-\infty}^{\infty} \langle \psi_{n+2}(\tau), z(\tau)\rangle \,\mathrm{d}\tau,$$

where $q : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\sup_t |q(t)|e^{\kappa|t|} < \infty, \quad q(t) = q(-t) \quad \text{and} \quad \int_{-\infty}^{\infty} q(t)\mathrm{d}t = 1.$$

Define two constants a_2, b_2 as

$$a_2 = \int_{-\infty}^{\infty} \langle \psi_{n+2}(t), \mathrm{D}_\mu f(x^{\mathrm{h}}(t); 0)\rangle \,\mathrm{d}t,$$
$$b_2 = \frac{1}{2}\int_{-\infty}^{\infty} \langle \psi_{n+2}(t), \mathrm{D}_x^2 f(x^{\mathrm{h}}(t); 0)(\varphi_2(t), \varphi_2(t))\rangle \,\mathrm{d}t$$

We obtain the following result as in Theorem 2.7 of [1].

Theorem 3.1. *Under assumptions* (R1)-(R5), *suppose that* $a_2, b_2 \neq 0$. *Then a saddle-node bifurcation of symmetric homoclinic orbits occurs at* $\mu = 0$. *Moreover, it is super- and subcritical if* $a_2b_2 < 0$ *and* > 0, *respectively. See Fig.* 1(a).

We next assume the following instead of (R4).

(R4') The equilibrium $x = 0$ has a symmetric homoclinic orbit $x^{\mathrm{h}}(t;\mu)$ in an open interval $I \ni \mu = 0$. Moreover, $\langle \psi_{n+2}(t), \dot{x}^{\mathrm{h}}(t;\mu)\rangle = 0$ for any $t \in \mathbb{R}$ and $\mu \in I$.

Under assumption (R4') we have $a_2 = 0$, so that we cannot apply Theorem 3.1. Let $\xi = \xi^\mu(t)$ be a unique solution to

$$\dot{\xi} = \mathrm{D}_x f(x^{\mathrm{h}}(t); 0)\xi + (\mathrm{id} - \Pi)\mathrm{D}_\mu f(x^{\mathrm{h}}(t); 0)$$

with $\langle \varphi_{n+2}(0), \xi(0)\rangle = 0$ and $\xi(0) \in \mathrm{Fix}(R)$, and define

$$\bar{a}_2 = \int_{-\infty}^{\infty} \langle \psi_{n+2}(t), \mathrm{D}_\mu \mathrm{D}_x f(x^{\mathrm{h}}(t);0)\varphi_2(t) + \mathrm{D}_x^2 f(x^{\mathrm{h}}(t);0)(\xi^\mu(t), \varphi_2(t))\rangle \, \mathrm{d}t,$$

where $x^{\mathrm{h}}(t) = x^{\mathrm{h}}(t;0)$.

Theorem 3.2. *Under assumptions* (R1)-(R3), (R4') *and* (R5). *suppose that* $\bar{a}_2, b_2 \neq 0$. *Then a transcritical bifurcation of symmetric homoclinic orbits occurs at* $\mu = 0$. *See Fig.* 1(b).

Finally we consider the $\mathbb{Z}_2$-equivalent or equivariant case, and assume the following.

(R6) Eq. (1.1) is $\mathbb{Z}_2$-*equivalent* or *equivariant*, i.e., there exists a $2n \times 2n$ matrix S such that $S^2 = \mathrm{id}_{2n}$ and $Sf(x;\mu) = f(Sx;\mu)$.

Especially, if $x = \bar{x}(t)$ is a solution to (1.1), then so is $x = S\bar{x}(t)$. We say that the pair $\bar{x}(t)$ and $S\bar{x}(t)$ are S-conjugate if $\bar{x}(t) \neq S\bar{x}(t)$. We have the decomposition $\mathbb{R}^{2n} = X^+ \oplus X^-$, where $Sx = x$ for $x \in X^+$ and $Sx = -x$ for $x \in X^-$. We also need the following assumption.

(R7) We have $X^- = (X^+)^\perp$. For every $t \in \mathbb{R}$, $x^{\mathrm{h}}(t), \psi_{n+1}(t) \in X^+$ and $\varphi_2(t), \psi_{n+2}(t) \in X^-$.

Assumption (R7) also means that $\varphi_1(t) \in X^+$. Moreover, a symmetric homoclinic orbit of the form (3.2) has an S-conjugate counterpart for $\alpha \neq 0$ since it is not included in X^+. In this situation, we have $a_2, b_2 = 0$ and cannot apply Theorems 3.1 and 3.2. Let $\xi = \xi^\alpha(t)$ be a unique solution to

$$\dot{\xi} = \mathrm{D}_x f(x^{\mathrm{h}}(t);0)\xi + \tfrac{1}{2}(\mathrm{id} - \Pi)\mathrm{D}_x^2 f(x^{\mathrm{h}}(t);0)(\varphi_2(t), \varphi_2(t))$$

with $\langle \varphi_{n+2}(0), \xi(0)\rangle = 0$ and $\xi(0) \in \mathrm{Fix}(R)$, and define

$$\bar{b}_2 = \int_{-\infty}^{\infty} \Big\langle \psi_{n+2}(t), \frac{1}{6}\mathrm{D}_x^3 f(x^{\mathrm{h}}(t);0)(\varphi_2(t), \varphi_2(t), \varphi_2(t)) + \mathrm{D}_x^2 f(x^{\mathrm{h}}(t);0)(\xi^\alpha(t), \varphi_2(t))\Big\rangle \, \mathrm{d}t.$$

We obtain the following result as in Theorem 2.9 of [1].

Theorem 3.3. *Under assumptions* (R1)-(R7), *suppose that* $\bar{a}_2, \bar{b}_2 \neq 0$. *Then a pitchfork bifurcation of homoclinic orbits occurs at* $\mu = 0$. *Moreover, it is super- and subcritical if* $\bar{a}_2\bar{b}_2 < 0$ *and* > 0, *respectively. See Fig.* 1(c).

4 Example

We now illustrate our theory for the four-dimensional system

$$\begin{aligned} &\dot{x}_1 = x_3, \quad \dot{x}_3 = x_1 - (x_1^2 + 8x_2^2)x_1 - \beta_2 x_2, \\ &\dot{x}_2 = x_4, \quad \dot{x}_4 = s x_2 - \beta_1(x_1^2 + 2x_2^2)x_2 - \beta_2 x_1 - \beta_3 x_2^2, \end{aligned} \tag{4.1}$$

where $s > 0$ and β_j, $j = 1, 2, 3$, are constants. Eq. (4.1) is reversible with the involution

$$R : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4),$$

for which $\mathrm{Fix}(R) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3, x_4 = 0\}$, and has an equilibrium at the origin $x = 0$. Thus, assumptions (R1) and (R2) hold. Moreover, the Jacobian matrix of the right hand side of (4.1) at $x = 0$ has two pairs of positive and negative eigenvalues with the same absolute values so that the origin $x = 0$ is a hyperbolic saddle. Thus, assumption (R3) holds.

Suppose that $\beta_2 = 0$. Then there exist a pair of symmetric homoclinic orbits

$$x^{\mathrm{h}}_{\pm}(t) = (\pm\sqrt{2}\ \mathrm{sech}\, t, 0, \mp\sqrt{2}\ \mathrm{sech}\ t \tanh t, 0)$$

to $x = 0$. Thus, assumption (R4) holds. Henceforth we only treat the homoclinic orbit $x^{\mathrm{h}}_{+}(t)$ for simplification and denote it by $x^{\mathrm{h}}(t)$. The VE (2.1) around $x = x^{\mathrm{h}}(t)$ for (4.1) is given by

$$\dot{\xi}_1 = \xi_3, \quad \dot{\xi}_3 = (1 - 6\ \mathrm{sech}^2\, t)\xi_1, \tag{2a}$$

$$\dot{\xi}_2 = \xi_4, \quad \dot{\xi}_4 = (s - 2\beta_1\ \mathrm{sech}^2\, t)\xi_2. \tag{2b}$$

Eq. (2b) has a bounded symmetric solution, so that assumption (R5) holds, if and only if

$$\beta_1 = \frac{(2\sqrt{s} + 4\ell + 1)^2 - 1}{8}, \quad \ell \in \mathbb{N} \cup \{0\}, \tag{4.3}$$

while Eq. (2a) always has a bounded solution corresponding to $\xi = \dot{x}^{\mathrm{h}}(t)$. Moreover, if condition (4.3) holds, then the differential Galois group of the VE given by (2a) and (2b) is triangularizable.

Fix the values of β_1 and $\beta_3 \neq 0$ such that Eq. (4.3) holds. Take $\mu = \beta_2$ as a control parameter. Applying Theorem 3.1, we show that a saddle-node bifurcation of symmetric homoclinic orbits occurs at $\beta_2 = 0$ for almost all values of s at least. We next assume that $\beta_2 = 0$. Then assumption (R4') holds. Take $\mu = \beta_1$ as a control parameter. Applying Theorem 3.2, we show that a transcritical bifurcation of symmetric homoclinic orbits occurs at the values of β_1 given by (4.3) for almost all values of s at least. Finally, we assume that $\beta_2, \beta_3 = 0$. Then Eq. (4.1) is $\mathbb{Z}_2$-equivariant with the involution

$$S : (x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, x_3, -x_4)$$

and assumption (R6) and (R7) hold. In particular, $X^+ = \{x_2, x_4 = 0\}$ and $X^- = \{x_1, x_3 = 0\}$. Applying Theorem 3.3, we see that a pitchfork bifurcation of symmetric homoclinic orbits occurs at the values of β_1 given by (4.3) if $\bar{b}_2 \neq 0$ for almost all values of s at least.

Finally we give numerical computations for pitchfork bifurcations of homoclinic orbits in (4.1). We take $s = 2$ so that Eq. (4.3) gives $\beta_1 = 1.70710678\ldots$ for $\ell = 0$, $\beta_1 = 7.5355339\ldots$ for $\ell = 1$ and $\beta_1 = 17.36396103\ldots$ for $\ell = 2$ as the value of β_1 for which assumption (R5) holds. To numerically compute symmetric homoclinic orbits, we used the computer tool AUTO [2] to solve the bondary value problem for (4.1) with the boundary conditions

$$L_{\mathrm{s}} x(-T) = 0, \quad x(0) \in \mathrm{Fix}(R),$$

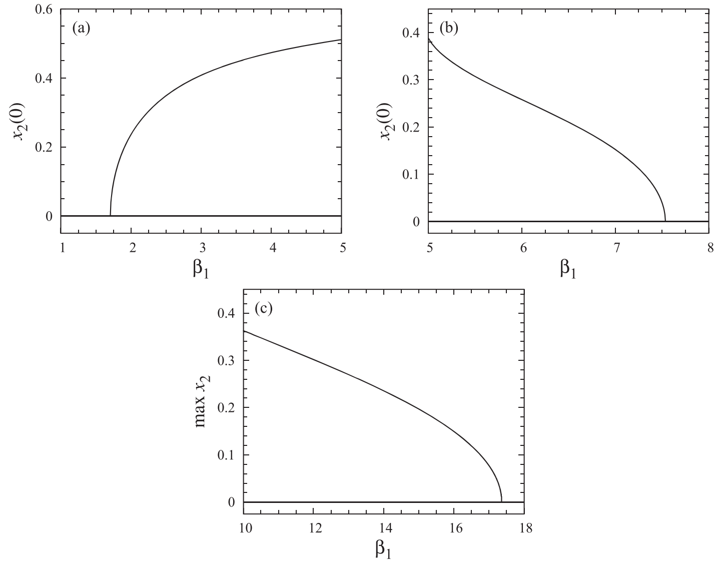


Figure 2. Bifurcation diagrams for $s = 2$ and $\beta_2, \beta_3 = 0$: (a) $\ell = 0$; (b) $\ell = 1$; (c) $\ell = 2$. Here β_1 is taken as a control parameter.

where $T = 20$ and L_s is the 2×4 matrix consisting of two row eigenvectors with negative eigenvalues for the Jacobian matrix of (4.1) at the origin,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -\beta_2 & 0 & 0 \\ -\beta_2 & s & 0 & 0 \end{pmatrix}.$$

Figure 2 shows bifurcation diagrams for $\beta_2, \beta_3 = 0$ when β_1 is taken as a control parameter. Note that there exist a branch of $x_2(= x_4) = 0$ for all values of β_1, and a pair of branches of solutions which are symmetric about $x_2 = 0$. We observe that a pitchfork bifurcation occurs at values of β_1 satisfying (4.3) for $\ell = 0, 1, 2$. The x_2-components of symmetric homoclinic orbits born at the bifurcation in Fig. 2 are also plotted in Fig. 3.

References

[1] D. Blázquez-Sanz and K. Yagasaki, Analytic and algebraic conditions for bifurcations of homoclinic orbits I: Saddle equilibria, J. Differential Equations, 253 (2012), 2916–2950.

[2] E. Doedel and B.E. Oldeman, AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations, 2012, available online from `http://cmvl.cs.concordia.ca/auto`.

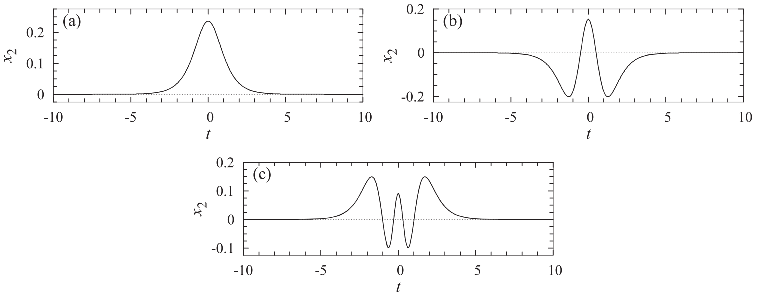


Figure 3. Profiles of symmetric homoclinic orbits on the branches for $s = 2$ and $\beta_2, \beta_3 = 0$: (a) $\beta_1 = 2$; (b) $\beta_1 = 7$; (c) $\beta_1 = 16$.

[3] K. Yagasaki, Analytic and algebraic conditions for bifurcations of homoclinic orbits II: Reversible systems, submitted for publication.

[4] K. Yagasaki and T. Stachowiak, Bifurcations of radially symmetric solutions to a coupled elliptic system with critical growth in $\mathbb{R}^d$ for $d = 3, 4$, J. Math. Anal. App., 484 (2020), 123726.

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