A Commentary on the Work of Saks and Wigderson 1986: Correlated Distributions on an Unbalanced Binary AND-OR Tree

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Abstract

We discuss the game-theoretical equilibrium of an AND-OR tree. Here, correlated distributions on the truth assignments are taken into consideration. In the seminal paper of Saks and Wigderson (1986), they proposed a recurrence formula method to calculate the equilibrium value. They provided a proof only for the case of complete binary trees. As to unbalanced binary trees, without a proof, they wrote that the main result for the complete binary trees holds for nearly balanced trees. However, their definition of a nearly balanced tree is vague. This note is a commentary on the paper of Saks and Wigderson with a particular focus on unbalanced binary trees. We propose the concept of a weakly balanced tree as an alternative to the concept of a nearly balanced tree. We demonstrate that the recurrence formula method of Saks and Wigderson works well for weakly balanced trees.

1 Introduction and notation

An AND-OR tree is a rooted tree such that each leaf is a Boolean variable, each internal node is labeled with $\wedge$ (AND) or $\vee$ (OR), and AND layers and OR layers alternate. At the beginning of computation, the truth assignment is hidden. An algorithm is a Boolean decision tree. The goal of an algorithm is to find the Boolean value of the root. To this end, the algorithm makes queries to leaves. If it has enough information, it skips some leaves. For instance, if it knows the value of a child node of an AND node v is 0, then it knows that the value of v is 0, and it skips to probe the other child node and its descendants. We are interested in an AND-OR trees in the context of Boolean function complexity and in the context of game theory.

More on the background of AND-OR trees may be found in the survey paper of Suzuki [3]. In this research note, we get into the main subject immediately. All the undefined notions are found in [3] and in the paper of Saks and Wigderson [2].

Suppose that T is a binary AND-OR tree. We denote the Randomized complexity by $\mathcal{R}(f_T)$.

$$\mathcal{R}(f_T) = \min_{A_R} \max_{x} \mathrm{cost}(A_R, x)$$

Here, f_T is the Boolean function that T defines. A randomized algorithm denotes a probability distribution on the all deterministic algorithms that compute f_T. A_R runs over all randomized algorithms. x runs over all truth assignments to the leaves of T.

Yao's principle is a variant of von Neumann's minimax theorem. By Yao's principle, the following holds.

$$\mathcal{R}(f_T) = \max_{d} \min_{A_D} \mathrm{cost}(A_D, d)$$

Here, A_D runs over all deterministic algorithms that computes f_T. d runs over all probability distributions on the truth assignments to the leaves of T, including correlated distributions.

A significance the paper of Saks and Wigderson [2] is in that they took correlated distributions into account. Before [2], the researchers on AND-OR trees put focus on independent and identical distributions on the truth assignments. In [2], they gave a recurrence formula. We will review the recurrence formula immediately afterwords (Definition 1.1). They asserted that the recurrence formula gives the value of $\mathcal{R}(f_T)$, under reasonable assumptions on the shape of a tree. Although they provided a proof for the case of complete binary trees, they did not provide a proof for the case of unbalanced binary trees. They wrote, without a proof, that the above-mentioned result for the complete binary trees holds for nearly balanced trees. Nevertheless, their definition of a nearly balanced tree is vague. This note is a commentary on [2] with a particular focus on unbalanced binary trees. The goal of this note is to give an alternative to the concept of a nearly balanced tree.

In the case where we restrict ourselves to the truth assignments that make the value of the root 0, the resulting quantity is denoted by $\mathcal{R}_0(f_T)$. The quantity $\mathcal{R}_1(f_T)$ is defined in the same way.

To each node v, Saks and Wigderson assigned the quantities a_0 and a_1. We denote them by $a_i(v)$. In Defiition 1.1, for an internal node v, the symobols $L(v)$ and $R(v)$ denote the left and right child node of v, respectively. The symbols $\mathcal{R}(f_T)$ and $R(v)$ denote different things, although they would be confusing.

Definition 1.1. [2]

1. For positive real numbers x, y, z and w, we define $\Psi(x, y, z, w)$ as follows.

$$\Psi(x, y, z, w) = \frac{xz + yw + zw}{z + w} \tag{1}$$

2. To each node v, we assign quantities $a_0(v)$ and $a_1(v)$ as follows. If v is a

leaf, $a_0(v) = a_1(v) = 1$. If v is an internal node labeled with $\wedge$,

$$a_1(v) = a_1(L(v)) + a_1(R(v)), \tag{2}$$
$$a_0(v) = \Psi(a_0(L(v)), a_0(R(v)), a_1(L(v)), a_1(R(v))). \tag{3}$$

If v is an internal node labeled with $\vee$,

$$a_0(v) = a_0(L(v)) + a_0(R(v)), \tag{4}$$
$$a_1(v) = \Psi(a_1(L(v)), a_1(R(v)), a_0(L(v)), a_0(R(v))). \tag{5}$$

In addition, we define $a(v)$ by

$$a(v) = \max\{a_0(v),\ a_1(v)\}. \tag{6}$$

3. Throughout the rest of the paper, we let P denote the conjunction of the following two assertions. If v is an internal node labeled with $\wedge$ then we have (7) and (8). If v is an internal node labeled with $\vee$ then we have (9) and (10).

$$a_0(L(v)) + a_1(R(v)) \geq a_0(R(v)), \tag{7}$$
$$a_0(R(v)) + a_1(L(v)) \geq a_0(L(v)) \tag{8}$$

$$a_1(L(v)) + a_0(R(v)) \geq a_1(R(v)), \tag{9}$$
$$a_1(R(v)) + a_0(L(v)) \geq a_1(L(v)) \tag{10}$$

Saks and Wigderson [2] provided a sufficient condition for $a_i = \mathcal{R}$, in other words, the randomized complexity is given by the recurrence.

Theorem 1.2. *[2] Suppose that the property P (of Definition 1.1) holds at every internal node v of T. Then we have $a_i(T) = \mathcal{R}_i(f_T)$. Here, the left-hand side denotes the value of a_i at the root of T.*

Proof. In the original paper, the last formula is $a_i = \ell_i = d_i = \mathcal{R}_i$. Here, ℓ_i and d_i are quantities defined by means of similar recurrence formulas as a_i. In this note, we omit the definitions of them. The proof of this theorem is omitted in the original paper. We include a proof.

We prove it by induction on the height k of the tree T. The base case is trivial. Assume that if the height of a tree is k or less, then the theorem holds. Suppose that the height of the tree T is $k+1$ and the root of T is labeled $\wedge$. When the root is labeled $\vee$, it can be shown similarly. Assume that each internal node of T satisfies the property P. By the induction hypothesis, $a_i(T_L) = \ell_i(T_L) = d_i(T_L)$, $a_i(T_R) = \ell_i(T_R) = d_i(T_R)$ holds. Note that the values of $\Psi(x, y, z, w)$ are the same for $(x, y, z, w) = (a_0(T_L), a_0(T_R), a_1(T_L), a_1(T_R))$,

$(\ell_0(T_L), \ell_0(T_R), \ell_1(T_L), \ell_1(T_R))$, and $(d_0(T_L), d_0(T_R), d_1(T_L), d_1(T_R))$. Throughout the rest of the proof, we denote the common value by Ψ.

Proof of $a_1(T) = \ell_1(T) = d_1(T)$: Since the root is labeled with $\wedge$, the recurrence formulas defining $a_1(T)$, $\ell_1(T)$ and $d_1(T)$ are the same. Thus they are equal.

Proof of $a_0(T) = \ell_0(T) = d_0(T)$: The following holds by the definition of d_i.

$$d_0(T) = \max\{d_0(T_L),\ d_0(T_R),\ \Psi\} \tag{11}$$

Now, we are going to show the following.

$$d_0(T_L) \leq \Psi \tag{12}$$
$$d_0(T_R) \leq \Psi \tag{13}$$

Then by (3) and (11), we will get $d_0(T) = \Psi = a_0(T)$.

By the property P, $d_0(T_L) \leq d_0(T_R) + d_1(T_L)$ holds. Therefore, we have the following.

$$\begin{aligned}
d_0(T_L) &\leq d_0(T_R) + d_1(T_L) \\
d_0(T_L)d_1(T_R) &\leq d_0(T_R)d_1(T_R) + d_1(T_L)d_1(T_R) \\
d_0(T_L)d_1(T_R) + d_0(T_L)d_1(T_L) &\leq d_0(T_R)d_1(T_R) + d_1(T_L)d_1(T_R) + d_0(T_L)d_1(T_L) \\
d_0(T_L) &\leq \frac{d_0(T_R)d_1(T_R) + d_1(T_L)d_1(T_R) + d_0(T_L)d_1(T_L)}{d_1(T_L) + d_1(T_R)} \\
&= \Psi
\end{aligned}$$

Therefore the inequality (12) is shown. In the same way, we get the inequality (13) by means of $d_0(T_R) \leq d_0(T_L) + d_1(T_R)$. Thus, by we have shown the following.

$$d_0(T) = \Psi = a_0(T) \tag{14}$$

Now, recall the following.

$$\ell_0(T) = \min\{\ell_0(T_L) + \ell_1(T_R),\ \ell_0(T_R) + \ell_1(T_L),\ \Psi\} \tag{15}$$

Again, by the property P, we have the following.

$$\ell_0(T_R) \leq \ell_0(T_L) + \ell_1(T_R) \tag{16}$$
$$\ell_0(T_L) \leq \ell_0(T_R) + \ell_1(T_L) \tag{17}$$

By multiplying the both sides of (16) by positive real number $\ell_1(T_R)$, we get the following.

$$\ell_0(T_R)\ell_1(T_R) \leq \ell_0(T_L)\ell_1(T_R) + \ell_1(T_R)^2$$

Then it is not difficult to see the following.

$$\ell_0(T_L)\ell_1(T_L)+\ell_0(T_R)\ell_1(T_R)+\ell_1(T_L)\ell_1(T_R) \leq (\ell_0(T_L)+\ell_1(T_R))(\ell_1(T_L)+\ell_1(T_R))$$

By dividing the both sides of the above by positive real number $\ell_1(T_L)+\ell_1(T_R)$, we have the following.

$$\ell_0(T_L) + \ell_1(T_R) \geq \Psi \tag{18}$$

In the same way, by means of (17), we have the following.

$$\ell_0(T_R) + \ell_1(T_L) \geq \Psi \tag{19}$$

By (15), (18) and (19), it holds that $\ell_0(T) = \Psi$. Hence, by (14), it holds that $a_0(T) = \ell_0(T) = d_0(T)$.

Proof of $a_i(T) = \mathcal{R}_i(f_T)$: We have shown that $\ell_i(T) = a_i(T) = d_i(T)$ for $i = 0, 1$. By the definition of d_i and [2, Theorem 4.3], it holds that $\ell_i(T) \leq \mathcal{R}_i(f_T) \leq d_i(T)$. Hence, $a_i(T) = \mathcal{R}_i(f_T)$. This completes the induction step. $\square$

2 Weakly balanced trees

We are going to propose the concept of a weakly balanced tree as an alternative to the concept of a nearly balanced tree. We will show that weakly balanced trees have the property P, and thus their equilibrium are given by the recurrence formula of Saks and Wigderson.

Definition 2.1. A binaly tree T is *weakly balanced* if each internal node v satisfies the following.

If v is labeled with $\wedge$ then $a_0(R(v)) \leq 2a_0(L(v))$ *and* $a_0(L(v)) \leq 2a_0(R(v))$.
If v is labeled with $\vee$ then $a_1(R(v)) \leq 2a_1(L(v))$ *and* $a_1(L(v)) \leq 2a_1(R(v))$.

Lemma 2.2. *If tree T is weakly balanced then each node v satisfies the following condition.*

If v is labeled $\wedge$ then $\frac{3}{4}a_1(v) \leq a_0(v) \leq a_1(v)$.
If v is labeled $\vee$ then $\frac{3}{4}a_0(v) \leq a_1(v) \leq a_0(v)$.

We abbriviate $a_0(L(v))$, $a_1(R(v))$ and $a_1(L(R(v)))$ to L_0, R_1, LR_1 and so on.

Proof. We prove the lemma by induction on the height k of T. The base case is trivial. Assume that if the height of a tree is k or less, then the theorem holds. Suppose that the height of T is $k+1$ and the root of T is labeled with $\wedge$. The case where the root is labeled with $\vee$ is shown in the same way.

Our goal is the following two inequalities.

$$\frac{3}{4}a_1(v) \leq a_0(v) \tag{20}$$

$$a_0(v) \leq a_1(v) \tag{21}$$

In order to show (20), it is enough to show the following.

$$3(L_1 + R_1)^2 \leq 4(L_0 L_1 + R_0 R_1 + L_1 R_1)$$

This is equivalent to the following.

$$0 \leq 4(L_0L_1 + R_0R_1 + L_1R_1) - 3(L_1 + R_1)^2 \tag{22}$$

Since the tree is alternating, $L_0,\ L_1,\ R_0,\ R_1$ are given by the recurrence formula for $\vee$ nodes. Therefore, by the induction hypothesis, we have $L_1 \leq L_0,\ R_1 \leq R_0$. The following inequalities show that the right side of (22) is non-negative.

$$\begin{aligned}
4(L_0L_1 + R_0R_1 + L_1R_1) - 3(L_1 + R_1)^2 &\geq 4(L_1^2 + R_1^2 + L_1R_1) - 3(L_1 + R_1)^2 \\
&= L_1^2 - 2L_1R_1 + R_1^2 \\
&= (L_1 - R_1)^2
\end{aligned}$$

Thus we get (20). In order to show (21), it is enough to show the following.

$$L_0L_1 + R_0R_1 + L_1R_1 \leq L_1^2 + R_1^2 + 2L_1R_1$$

This is equivalent to the following.

$$L_1^2 + R_1^2 + L_1R_1 - L_0L_1 - R_0R_1 \geq 0$$

We are going to show the above under the assumption of $L_0 \leq R_0$.

$$\begin{aligned}
& L_1^2 + R_1^2 + L_1R_1 - L_0L_1 - R_0R_1 \\
=\ & L_1(L_1 - L_0) \quad + R_1^2 + L_1R_1 - R_0R_1 \\
\geq\ & L_1(\frac{3}{4}L_0 - L_0) + R_1^2 + L_1R_1 - R_0R_1 \quad \text{[Induction hypothesis]} \\
=\ & -\frac{1}{4}L_0L_1 \quad + R_1^2 + L_1R_1 - R_0R_1 \\
\geq\ & -\frac{1}{4}R_0L_1 \quad + R_1^2 + L_1R_1 - R_0R_1 \quad \text{[Assumption]} \\
\geq\ & -\frac{1}{4}\cdot\frac{4}{3}R_1L_1 \quad + R_1^2 + L_1R_1 - R_0R_1 \quad \text{[Induction hypothesis]} \\
=\ & -\frac{1}{3}R_1L_1 \quad + R_1^2 + L_1R_1 - R_0R_1 \\
=\ & \frac{2}{3}L_1R_1 \quad + R_1^2 - R_0R_1 \\
=\ & R_1(R_1 + \frac{2}{3}L_1 - R_0) \\
\geq\ & R_1(R_1 + \frac{2}{3}\cdot\frac{3}{4}L_0 - R_0) \quad \text{[Induction hypothesis]} \\
=\ & R_1(R_1 + \frac{1}{2}L_0 - R_0) \\
\geq\ & R_1(\frac{3}{4}R_0 + \frac{1}{2}L_0 - R_0) \quad \text{[Induction hypothesis]} \\
=\ & R_1(\frac{1}{2}L_0 - \frac{1}{4}R_0) \\
=\ & \frac{1}{4}R_1(2L_0 - R_0) \geq 0 \quad \text{[Weakly balanced]}
\end{aligned}$$

The case of $R_0 < L_0$ is shown in the similar way. □

By using the above lemma, we are going to show the following theorem.

Theorem 2.3. *If T is weakly balanced, then T satisfies P at every internal node v.*

Theorem 2.3 and Theorem 1.2 immediately imply the following.

Theorem 2.4. *(The main theorem) If v is a root of a weakly balanced tree T, then $\mathcal{R}(f_T) = a(v)$. Recall that $a(v) = \max\{a_0(v),\ a_1(v)\}$.*

In the reminder of this section, we prove Theorem 2.3.

Proof. (of Theorem 2.3) Suppose that T is a weakly balanced tree. We prove the theorem by induction on the height k of T. The base case is trivial. For the induction step, let T be a weakly balanced tree with height $k+1$ where the root v is labeled with $\wedge$. The case of $\vee$ is similar. By the induction hypothesis, the property P holds at each internal node other than the root v. Our goal is to show that the property P holds at the root v.

Suppose that $R_0 \geq L_0$. The other case can be shown similarly. In the case where the right child of the root v is a leaf, then $R_0 = 1$ holds. By the assumption of $R_0 \geq L_0$, it holds that $L_0 = 1$. Since $R_0 = L_0 = 1$, the property P holds at the root. In the rest of the proof, we assume that the right child of the root is not a leaf.

Now, the property P is equivalent to the conjunction of $L_0 + R_1 \geq R_0 \cdots (\Diamond)$ and $R_0 + L_1 \geq L_0$. However, the latter is obvious because of the assumption of $R_0 \geq L_0$. Thus, it is enough to show the former $(\Diamond)$. From the definition of a_i, we get the following.

$$R_0 = RL_0 + RR_0$$
$$R_1 = \frac{RL_0 RL_1 + RR_0 RR_1 + RL_0 RR_0}{RL_0 + RR_0}$$

By substituting the above two right-hand sides for R_0 and R_1 in $(\Diamond)$, we know that the following is a sufficient condition for $(\Diamond)$.

$$L_0 + \frac{RL_0 RL_1 + RR_0 RR_1 + RL_0 RR_0}{RL_0 + RR_0} \geq RL_0 + RR_0$$

We multiply both sides of the above inequality by positive real number $RL_0 + RR_0$. Then the following is a sufficient condition for $(\Diamond)$.

$$L_0 RL_0 + L_0 RR_0 + RL_0 RL_1 + RR_0 RR_1 + RL_0 RR_0 \geq (RL_0 + RR_0)^2$$

This is equivalent to the following.

$$L_0 RL_0 + L_0 RR_0 + RL_0 RL_1 + RR_0 RR_1 - RL_0^2 - RR_0^2 - RR_0 RL_0 \geq 0$$

By Lemma 2.2, $RL_1 \geq RL_0$ and $RR_1 \geq RR_0$ hold. Therefore, the following inequality is a sufficient condition for $(\Diamond)$.

$$L_0 RL_0 + L_0 RR_0 - RR_0 RL_0 \geq 0$$

Each of the following is equivalent to the above inequality.

$$RR_0(L_0 - RL_0) + L_0 RL_0 \geq 0$$
$$RL_0(L_0 - RR_0) + L_0 RR_0 \geq 0$$

Thus, our goal is achieved if we can show that at least one of the following two inequalities hold.

$$L_0 - RL_0 \geq 0, \quad L_0 - RR_0 \geq 0$$

Case 1, $RR_0 \geq RL_0$. Since the tree is weakly balanced, we have $L_0 \geq \frac{1}{2}R_0$. Thus we get the following.

$$L_0 - RL_0 \geq \frac{1}{2}R_0 - RL_0 = \frac{1}{2}(RR_0 + RL_0) - RL_0 = \frac{1}{2}(RR_0 - RL_0) \geq 0$$

Case 2, otherwise. Then we have $RR_0 < RL_0$. In this case, we can show $L_0 - RR_0 \geq 0$. This completes the proof of ($\Diamond$). Thus, we have shown Theorem 2.3. $\square$

3 Examples of Weakly Balanced Tree

Saks and Wigderson [2] gave examples of nearly balanced trees. Among them, we look at two types of trees. One is a tree such that there is a positive integer h and for every leaf, the height is either h or $h + 1$. It is possible that some leaves have height h and the others have height $h+1$. The other is a Fibonacci tree. In [2], they did not give a definition of a Fibonacci tree. We will give a definition. In this section, we observe that the above-mentioned two types of trees are weakly balanced.

By means of induction on the height and Lemma 2.2, we can show $R_0 \leq 2L_0$ in a tree such that every leaf has height h or $h + 1$. Therefore, such a tree is weakly balanced.

Definition 3.1. We define a *Fibonacci tree* as follows.

A Fibonacci tree of height 1 is the tree consists of just one leaf.

A Fibonacci tree of height 2 is the tree consists of just one leaf.

A Fibonacci tree of height $n + 2$ is a binary AND-OR tree T such that the left subtree of the root is a Fibonacci tree of height n, and the right subtree of the root is a Fibonacci tree of height $n + 1$. Here, the left subtree is the subtree which consists of $L(v)$, the left child of the root of the original tree, and all descendants of $L(v)$ in the original tree. The right subtree is defined in the same way.

By means of induction on the height and Lemma 2.2, we can show $R_0 \leq 2L_0$ in a Fibonacci tree. Therefore, a Fibonacci tree is weakly balanced.

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