A category-like structure of computational paths for parallel reduction*

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Abstract

We introduce a formal system of reduction paths as a category-like structure induced from a digraph. Our motivation behind this work comes from a quantitative analysis of reduction systems based on the perspective of computational cost and computational orbit. From the perspective, we define a formal system of reduction paths for parallel reduction, wherein reduction paths are generated from a quiver by means of three path-operators. Next, we introduce an equational theory and reduction rules for the reduction paths, and show that the rules on paths are terminating and confluent so that normal paths are obtained. Following the notion of normal paths, a graphical representation of reduction paths is provided. Then we prove that the reduction graph is a plane graph, and unique path and universal common-reduct properties are established. Based on this, a set of transformation rules from a conversion sequence to a reduction path leading to the universal common-reduct is given under a certain strategy. Finally, path matrices are defined as block matrices of adjacency matrices to count reduction orbits.

1 Introduction

Our motivation behind this work is to analyze quantitative properties of reduction systems, e.g., λ-calculi [3, 20, 13] in the context of the Church–Rosser theorem [5] from the perspective of (i) evaluation of computational cost (length), and (ii) evaluation of computational orbit (path).

Relating to the first perspective, the complexity of proofs and reduction length (steps) has been investigated in a wide range of fields such as proof theory and computer science, for instance, Statman [21] for deciding the $\beta\eta$-equality of typable λ-terms, Schwichtenberg [19] and Beckmann [4] for normalization in the simply typed λ-calculus, and so on. Concerning the complexity of confluence, there have been several investigations. Here, the confluence property states that

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if $M \twoheadrightarrow N_1$ and $M \twoheadrightarrow N_2$ then $N_1 \twoheadrightarrow P$ and $N_2 \twoheadrightarrow P$ for some P. Komori, Matsuda, and Yamakawa [16] investigated how to specify a common reduct of N_1 and N_2 in terms of M. They showed that a common reduct P can be given by an iteration of the so-called Takahashi translation [22], denoted by F, in terms of the number of reduction steps l from M to N_1 and r from M to N_2. That is, for $M \twoheadrightarrow^l N_1$ and $M \twoheadrightarrow^r N_2$, they obtained a common reduct $F^k(M)$ with $k = \max\{l, r\}$. Ketema and Simonsen [14] investigated the complexity of confluence by measuring reduction steps to a common reduct. They showed that the length leading to a common reduct can be bounded by a function in terms of M and the length from M to N_1 and from M to N_2. That is, for $M \twoheadrightarrow^l N_1$ and $M \twoheadrightarrow^r N_2$, they obtained a bound function $f(l, M, r)$ of $N_1 \twoheadrightarrow^{n_1} P$ and $N_2 \twoheadrightarrow^{n_2} P$ for some P such that $n_1, n_2 \le f(l, M, r)$ where f is in the fourth level of the Grzegorczyk hierarchy [12]. On the other hand, the Church–Rosser property [5] states that if $M =_\beta N$ then $M \twoheadrightarrow P$ and $N \twoheadrightarrow P$ for some P. Although confluence implies Church–Rosser, the two properties should be distinguished carefully on the complexity analysis [10]. For the complexity of Church–Rosser, we analyzed the equality (conversion) into *expansion* and *reduction*, and obtained an upper bound function still at the fourth level of the Grzegorczyk hierarchy [8, 9, 10].

In this paper, from the second perspective we analyze the complexity result obtained by the existing work in more detail. Compared with [16], the previous work [10] revealed a common reduct $F^k(P)$ with some P and $k \le \min\{l, r\}$ for $M \twoheadrightarrow^l N_1$ and $M \twoheadrightarrow^r N_2$. Here, we show that this common reduct can be considered as an optimal one for any common reduct generated by a so-called triangle property, by means of counting reduction paths via adjacency matrices. For counting computational paths, it is necessary to formalize paths so that we introduce a formal system of reduction paths together with an equational theory and reduction rules of paths. Then the existence of normal paths makes it possible to represent reduction paths as plane graphs via a context-free grammar, and structures of paths as adjacency matrices in an elegant way. The analysis also clarifies at most how many times F must be applied to obtain a common reduct. Moreover, the introduced formal system has an application not only to λ-calculi but also to abstract reduction systems with natural properties[1] with respect to parallel reductions. Although there have been many investigations on the *length* of reduction paths including [21, 19, 4, 14, 10], to the best of our knowledge, this paper makes a unique study on the *number* of reduction paths.

This paper is organized as follows. Section 1 is devoted to background, related work, motivation, and contribution of this paper. Section 2 gives preliminaries including basic definitions and a guiding example. Section 3 introduces a formal system of reduction paths. Section 4 provides an equational theory and reduction rules for reduction paths, and then shows the normal path property. Based on this, section 5 provides a graphical representation of reduction paths, and here proves that the graph is a plane graph. Following this, section 6 shows the unique path and universal common-reduct properties. We also provide a set

[1] The properties will be defined soon after, called condition (A) in this paper.

of transformation rules from conversion sequences to reduction paths under a certain strategy. Section 6 introduces path matrices to count reduction paths, Finally, section 8 concludes with remarks and further work.

2 Preliminaries and a Guiding Example

The set of λ-terms denoted by Λ is defined referring to standard texts [3, 20, 13] as follows:

$$M, N, P \in \Lambda ::= x \mid (\lambda x.M) \mid (MN)$$

We write $M \equiv N$ for the syntactical identity under renaming of bound variables. The size $|M|$ of a term M is defined by $|x| = 1$, $|(\lambda x.M)| = 1 + |M|$, and $|(MN)| = 1 + |M| + |N|$. We use the notation $\rightarrow$ for one-step β-reduction, $\twoheadrightarrow$ for multiple-step β-reduction, and $=_\beta$ for β-equality (β-conversion).

We note that $M \twoheadrightarrow N$ iff there exists a finite sequence of terms $M_0, \ldots, M_n$ ($n \geq 0$) such that $M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \equiv N$. For this we also write $M \rightarrow^n N$ with the number of reduction steps displayed. Also note that $M =_\beta N$ iff there exists a finite sequence of terms $M_0, \ldots, M_n$ ($n \geq 0$) such that $M \equiv M_0 \leftrightarrow M_1 \leftrightarrow \cdots \leftrightarrow M_n \equiv N$ where $M_i \leftrightarrow M_{i+1}$ denotes either $M_i \rightarrow M_{i+1}$ or $M_{i+1} \rightarrow M_i$ ($i = 0, \ldots, n-1$). Here, the arrow $\rightarrow$ in the former case (*reduction*) is referred to as a right arrow, and that in the latter case (*expansion*) is referred to as a left arrow, denoted also by $M_i \leftarrow M_{i+1}$. The notation $\sharp(\leftarrow)[j,k]$ denotes the number of occurrences of left arrows between terms M_j and M_k ($0 \leq j \leq k \leq n$) in the sequence. For the sequence, we also write $M \overset{l\quad r}{\longleftrightarrow} N$ where $l = L(n)$ and $r = n - l$. Here, a shorthand notation $L(i)$ is often used for $\sharp(\leftarrow)[0,i]$.

Concerning computational cost (reduction length), we briefly review our previous result on the Church–Rosser theorem. For a reduction system with one-step reduction relation $\rightarrow$ and term size $|\ |$, suppose the following two conditions (A) and (B).

(A) We have a binary relation $\Rightarrow$ on terms and a translation F between terms as follows.

  (a) If $M \rightarrow N$ then $M \Rightarrow N$.
  (b) If $M \Rightarrow N$ then $M \twoheadrightarrow N$.
  (c) If $M \Rightarrow N$ then $N \Rightarrow F(M)$.

(B) We have two monotonic functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ as follows.

  If $M \Rightarrow N$ then $M \rightarrow^l N$ with $|N| \leq f(|M|)$ and $l \leq g(|M|)$, where f and g are respectively in the p-th and q-th levels of the Grzegorczyk hierarchy [12] with $p+1, q \geq 2$.

We should remark that the literature expounds the condition (A) by the properties (1), (2), and (5) in [22], and that the condition (c) of (A) is also called the triangle property.

The n-fold iteration of f is written as usual: $f^0(x) = x$, $f^n(x) = f(f^{n-1}(x))$.

Then, as demonstrated in [9, 10], the enriched form of the Church–Rosser theorem holds.

Theorem 1 (Quantitative Church–Rosser [9]) *If $M \overset{l\quad r}{\longleftrightarrow} N$ then there exists a term P such that $M \to^m F^k(P)$ and $N \to^n F^k(P)$ where*

1. $k = \sharp(\leftarrow)[0, r] \leq \min\{l, r\}$,
2. $m \leq \displaystyle\sum_{i=0}^{r-1} g(f^i(|M|))$, $\quad n \leq \displaystyle\sum_{i=0}^{l-1} g(f^i(|N|))$, *and*
3. *m, n are bounded by functions in the level of* $\max\{p + 1, q\}$ *of the Grzegorczyk hierarchy.*

As an instance of the theorem, we can take not only type-free λ-calculus with $\beta\eta$-reduction, but also typed calculi such as Gödel's system **T**, Girard's system **F**, and so on, by setting $f(x) = 2^x$ and $g(x) = x$, so that the length m, n are bounded by an iteration of the elementary function, i.e., functions in the fourth level of the hierarchy.

The essence of the proof relies upon the harmonized property of the condition (c) of (A) with respect to $\Rightarrow$ (parallel reduction) and F (so-called Takahashi translation[2] [22]). We also write $N \Leftarrow M$ for $M \Rightarrow N$. We write $M \Rightarrow^n N$, if $M \equiv M_0 \Rightarrow M_1 \Rightarrow \cdots \Rightarrow M_n \equiv N$ for some $n \geq 0$ and M_i $(i = 0, 1, \ldots, n)$. We write $M \overset{l\quad r}{\Longleftrightarrow} N$, if $M \equiv M_0 \Leftrightarrow M_1 \Leftrightarrow \cdots \Leftrightarrow M_n \equiv N$ for some $n \geq 0$ and M_i $(i = 0, 1, \ldots, n)$ where $\Leftrightarrow$ denotes either $\Rightarrow$ or $\Leftarrow$ together with $l = \sharp(\Leftarrow)[0, n]$ and $r = n - l$. Here, by $\sharp(\Leftarrow)[j, k]$ we mean the number of occurrences of $\Leftarrow$ between M_j and M_k $(0 \leq j \leq k \leq n)$ in the sequence.

Now the essence of Theorem 1 is extracted by the following proposition in terms of $\Rightarrow$.

Proposition 1 ([8, 9, 10]) *If $M \overset{l\quad r}{\Longleftrightarrow} N$, then there exists a term P such that $M \Rightarrow^r F^k(P)$ and $N \Rightarrow^l F^k(P)$ where $k = \sharp(\Leftarrow)[0, r]$.*

This paper will show that there exist unique paths of parallel reductions from M to $F^k(P)$ and from N to $F^k(P)$ for some P, respectively, within a certain reduction graph. Before this, we demonstrate the key idea of the formal system by way of example. From a given conversion sequence, reduction paths can be generated by means of an iterated application of the condition (c) of (A). We show a simple example of $b_0 \overset{2\quad 3}{\Longleftrightarrow} b_5$ with a common reduct $F^2(b_3)$ of b_0 and

[2] This translation is a complete development.

b_5, where b_i and $\to$ are used instead of M_i and $\Rightarrow$, respectively.

$$\begin{array}{ccccccccccc}
b_0 & \leftarrow & b_1 & \rightarrow & b_2 & \leftarrow & b_3 & \rightarrow & b_4 & \rightarrow & b_5 \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & & \swarrow & \\
F(b_0) & \leftarrow & F(b_1) & \rightarrow & F(b_2) & \leftarrow & F(b_3) & \rightarrow & F(b_4) & \rightarrow & F(b_5) \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & & \swarrow & \\
F^2(b_0) & \leftarrow & F^2(b_1) & \rightarrow & F^2(b_2) & \leftarrow & F^2(b_3) & \rightarrow & F^2(b_4) & \rightarrow & F^2(b_5) \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & & \swarrow & \\
F^3(b_0) & \leftarrow & F^3(b_1) & \rightarrow & F^3(b_2) & \leftarrow & F^3(b_3) & \rightarrow & F^3(b_4) & \rightarrow & F^3(b_5)
\end{array}$$

Observe that there exists a unique path from b_0 to $F^2(b_5)$, i.e., $b_0 \Rightarrow^5 F^2(b_5)$, and also a unique path from b_5 to $F^3(b_0)$, i.e., $b_5 \Rightarrow^5 F^3(b_0)$. Here, all the common reducts of b_0, b_5 are points which occur in the area below the two paths including the boundary, i.e., $F^2(b_2)$, $F^2(b_3)$, $F^2(b_4)$, $F^2(b_5)$, $F^3(b_0)$, $F^3(b_1)$, $F^3(b_2)$, *etc.* The common reduct $F^2(b_3)$ is the unique crossing point of the two paths $b_0 \Rightarrow^5 F^2(b_5)$ and $b_5 \Rightarrow^5 F^3(b_0)$. In particular, the point $F^2(b_3)$ can lead to all the common reducts of b_0, b_5, which are generated by the condition (c) of (A). The example is not a special case, but shows a general property on reduction paths generated by the condition. We study the following fundamental properties of reduction paths generated by the triangle property.

1. *Unique path property* of common reduct (UPP): If $a \overset{l\quad r}{\Longleftrightarrow} b$, then there exist m, n and a common reduct c such that we have unique reduction paths of $a \Rightarrow^n c$ and $b \Rightarrow^m c$.

2. *Universal common-reduct property* (UCR): If $a \overset{l\quad r}{\Longleftrightarrow} b$, then there exists a common reduct c of a and b such that for any common reduct d of a and b we have $c \Rightarrow^n d$ for some n.

As a result of UPP and UCR, the common reduct $F^k(P)$ in Proposition 1 can be considered as the optimum one for any common reduct within the reduction graph generated by the triangle property.

3 Formal System of Reduction Paths

Based on a monoid-like structure such as [15, 17], we introduce a formal system of reduction paths for parallel reduction. First, we define a quadruple $\vec{\Delta} = \langle \Delta_0, \Delta_1, s, t \rangle$, called a quiver[3] [1], consisting of two sets Δ_0, Δ_1 and two maps s, t. Here, Δ_0 is the set of points or vertices denoted by a, b, c, and Δ_1 is the set of arrows or atomic paths denoted by α, β, γ. The maps $s, t : \Delta_1 \to \Delta_0$ are provided such that $s(\alpha)$ is called a source of $\alpha \in \Delta_1$ and $t(\alpha)$ is called a target of $\alpha \in \Delta_1$, respectively, which is denoted by $\alpha : s(\alpha) \to t(\alpha)$ or $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. For this case, we also write $\vec{\Delta} \vdash \alpha : s(\alpha) \to t(\alpha)$.

[3]This quadruple is nothing but a directed multigraph, and the notion of quiver has been used for the graphical representation of finite dimensional algebras [1].

Let $\vec{\Delta}$ be a quiver $\langle \Delta_0, \Delta_1, s, t\rangle$, Base be a countable set with $\Delta_0 \subseteq \mathsf{Base}$, and $F : \mathsf{Base} \to \mathsf{Base}$ be a point constructor that is an injective mapping such that $\Delta_0 \cap F(\mathsf{Base}) = \emptyset$. Well-formed paths, denoted by $p, q, r \in \mathsf{RP}$, are constructed by the following formation rules together with a mapping, called a measure function $\mathsf{k} : \mathsf{RP} \to \mathbb{N}$, where $a, b, c \in \mathsf{Base}$, $b_1, b_2 \in \Delta_0$, $\alpha, \beta, \gamma \in \Delta_1$, and $m, n, k \in \mathbb{N}$.

Definition 1 (Reduction paths) *1. Syntax of reduction paths* (RP):

$$
\begin{array}{rcll}
p, q \in \mathsf{RP} & ::= & \mathsf{id}_a & (*\ \textit{identity path}\ *) \\
& | & \alpha & (*\ \textit{atomic path}\ *) \\
& | & (p; q) & (*\ \textit{composition, concatenation}\ *) \\
& | & \mathsf{mon}_{a,b}(p) & (*\ \textit{monotonic path}\ *) \\
& | & \mathsf{flip}_{a,b}(p) & (*\ \textit{flipped path}\ *)
\end{array}
$$

2. Measure function $\mathsf{k} : \mathsf{RP} \to \mathbb{N}$:

- $\mathsf{k}(\mathsf{id}) = 0$ *for the identity path, and* $\mathsf{k}(\alpha) = 1$ *for atomic* $\alpha \in \Delta_1$.
- $\mathsf{k}(p; q) = \mathsf{k}(p) + \mathsf{k}(q)$, *and* $\mathsf{k}(\mathsf{mon}(p)) = \mathsf{k}(\mathsf{flip}(p)) = \mathsf{k}(p)$.

3. Formation rules for well-typed reduction paths:

- *Identity paths for* $a \in \mathsf{Base}$:

$$\vec{\Delta} \vdash \mathsf{id}_a : a \rightsquigarrow a$$

- *Atomic paths for* $\alpha \in \Delta_1$ *with* $s(\alpha) = b_1 \in \Delta_0$ *and* $t(\alpha) = b_2 \in \Delta_0$:

$$\vec{\Delta} \vdash \alpha : b_1 \rightsquigarrow b_2$$

- *Concatenation of paths*:

$$\frac{\vec{\Delta} \vdash p : a \rightsquigarrow b \quad \vec{\Delta} \vdash q : b \rightsquigarrow c}{\vec{\Delta} \vdash (p; q) : a \rightsquigarrow c}$$

- *Monotonic paths*:

$$\frac{\vec{\Delta} \vdash p : a \rightsquigarrow b}{\vec{\Delta} \vdash \mathsf{mon}_{a,b}(p) : F(a) \rightsquigarrow F(b)}$$

- *Flipped paths*:

$$\frac{\vec{\Delta} \vdash p : a \rightsquigarrow b}{\vec{\Delta} \vdash \mathsf{flip}_{a,b}(p) : b \rightsquigarrow F^{\mathsf{k}(p)}(a)}$$

Our intuitive idea for the formal system is based on the following observations. The mapping F would represent a reduction strategy as that of [3], wherein a cofinal strategy F is defined such that if $M \twoheadrightarrow N$ then $N \twoheadrightarrow F^n(M)$ for some natural number n. In our case, the condition (c) of (A) implies that if $M \Rightarrow^n N$ then $N \Rightarrow^n F^n(M)$, like the Z-property [7, 18]. Here, the mapping $\mathsf{k} : \mathsf{RP} \to \mathbb{N}$ should provide the number of steps in terms of $\Rightarrow$, so that the value $\mathsf{k}(p)$ should be associated to the length of the path p. From the definition of k, we indeed have the commutative and associative properties with respect to concatenations:

1. $\mathsf{k}(p; q) = \mathsf{k}(q; p)$ and $\mathsf{k}((p; q); r) = \mathsf{k}(p; (q; r))$.
2. If $\mathsf{k}(p) = \mathsf{k}(q)$ then $\mathsf{k}(p; s) = \mathsf{k}(q; s)$ and $\mathsf{k}(s; p) = \mathsf{k}(s; q)$.

4 Equational Theory and Reduction Rules of Paths

Here, a term is handled as a point, and then we do not care about the structure of terms, but just consider reduction paths generated by path constructors. Next, we consider what paths should be equivalent to each other so that equality rules on reduction paths are introduced, which are compatible with the path constructors ;, mon, and flip.

Definition 2 (Equational theory $\mathcal{E}$ of reduction paths) *Let $p, q, s \in \mathsf{RP}$, and $a, b \in \mathsf{Base}$.*

(E_0) $p \approx p$. *If* $q \approx p$ *then* $p \approx q$. *If* $p \approx q$ *and* $q \approx s$ *then* $p \approx s$.

(E_1) *If* $p \approx q$ *then* $\mathsf{flip}(p) \approx \mathsf{flip}(q)$, $\mathsf{mon}(p) \approx \mathsf{mon}(q)$, $(p; s) \approx (q; s)$, *and* $(s; p) \approx (s; q)$.

(E_2) $(\mathsf{id}_a; p) \approx p \approx (p; \mathsf{id}_b)$ *for a path* $p : a \rightsquigarrow b$.

(E_3) $((p; q); s) \approx (p; (q; s))$ *for paths* $p : a \rightsquigarrow b$, $q : b \rightsquigarrow c$, *and* $s : c \rightsquigarrow d$.

(E_4) $\mathsf{mon}_{a,a}(\mathsf{id}_a) \approx \mathsf{id}_{F(a)}$.

(E_5) $\mathsf{mon}(p; q) \approx (\mathsf{mon}(p); \mathsf{mon}(q))$ *for paths* $p : a \rightsquigarrow b$ *and* $q : b \rightsquigarrow c$.

(E_6) $\mathsf{flip}(\mathsf{id}_a) \approx \mathsf{id}_a$.

(E_7) $\mathsf{flip}^2(\alpha) \approx \mathsf{mon}_{a,b}(\alpha)$ *for an atomic path* $\alpha : a \rightsquigarrow b$.

(E_8) $\mathsf{flip}(p; q) \approx (\mathsf{flip}(q); \mathsf{mon}^{\mathsf{k}(q)}(\mathsf{flip}(p)))$ *for paths* $p : a \rightsquigarrow b$ *and* $q : b \rightsquigarrow c$.

(E_9) $\mathsf{flip}(\mathsf{mon}(p)) \approx \mathsf{mon}(\mathsf{flip}(p))$.

The intuitive meaning of the rules can be explained by the following diagrams.

1. (E_4) $\mathsf{mon}(\mathsf{id}_a) \approx \mathsf{id}_{F(a)}$, (E_6) $\mathsf{flip}(\mathsf{id}_a) \approx \mathsf{id}_a$, (E_7) $\mathsf{flip}^2(\alpha) \approx \mathsf{mon}(\alpha)$:

$$\begin{tikzcd} a \arrow[r, "\mathsf{id}_a", "\mathsf{flip}(\mathsf{id}_a)"'] & a \\ F(a) \arrow[r, "\mathsf{mon}(\mathsf{id}_a)", "\mathsf{id}_{F(a)}"'] & F(a) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"'] & b \arrow[d, "\mathsf{flip}(\alpha)"'] & \\ & F(a) \arrow[r, "\mathsf{flip}^2(\alpha)", "\mathsf{mon}(\alpha)"'] & F(b) \end{tikzcd}$$

2. (E_5) $(\mathsf{mon}(p); \mathsf{mon}(q)) \approx \mathsf{mon}(p; q)$ with $p : a \rightsquigarrow b$ and $q : b \rightsquigarrow c$:

$$\begin{tikzcd} a \arrow[r, "p"'] & b \arrow[r, "q"'] & c \\ F(a) \arrow[r, "\mathsf{mon}(p)"] & F(b) \arrow[r, "\mathsf{mon}(q)"] & F(c) \end{tikzcd} \approx \begin{tikzcd} a \arrow[r, "(p;q)"'] & c \\ F(a) \arrow[r, "\mathsf{mon}(p;q)"] & F(c) \end{tikzcd}$$

3. (E_8) $(\mathsf{flip}(q); \mathsf{mon}^{\mathsf{k}(q)}(\mathsf{flip}(p))) \approx \mathsf{flip}(p; q)$ with $p : a \rightsquigarrow b$ and $q : b \rightsquigarrow c$:

$$\begin{tikzcd} a \arrow[r, "p"] & b \arrow[d, "\mathsf{flip}(p)"'] \arrow[r, "q"] & c \arrow[d, "\mathsf{flip}(q)"'] \\ & F^{\mathsf{k}(p)}(a) & F^{\mathsf{k}(q)}(b) \arrow[d, "\mathsf{mon}^{\mathsf{k}(q)}(\mathsf{flip}(p))"'] \\ & & F^{\mathsf{k}(p)+\mathsf{k}(q)}(a) \end{tikzcd} \approx \begin{tikzcd} c \arrow[d, "\mathsf{flip}(p;q)"] & a \arrow[l, "(p;q)"] \\ F^{\mathsf{k}(p;q)}(a) & \end{tikzcd}$$

4. (E_9) $\mathsf{flip}(\mathsf{mon}(p)) \approx \mathsf{mon}(\mathsf{flip}(p))$ with $p : a \rightsquigarrow b$:

$$\begin{tikzcd} F(a) \arrow[r, "\mathsf{mon}(p)"'] & F(b) \arrow[d, "\mathsf{flip}(\mathsf{mon}(p))"'] \\ & F^{1+\mathsf{k}(\mathsf{mon}(p))}(a) \end{tikzcd} \approx \begin{tikzcd} F(b) \arrow[d, "\mathsf{mon}(\mathsf{flip}(p))"] & b \arrow[d, "\mathsf{flip}(p)"] & a \arrow[l, "p"'] \\ F^{1+\mathsf{k}(p)}(a) & F^{\mathsf{k}(p)}(a) & \end{tikzcd}$$

Next for reduction paths we define a reduction relation $\Longrightarrow$ based on $\mathcal{E}$, and then show that the system is terminating and confluent. Hence, any reduction path can be reduced to a unique normal path with the same source and target. We write $\Longrightarrow^*$ for the reflexive and transitive closure of $\Longrightarrow$.

Definition 3 (Reduction relation $\Longrightarrow$ for reduction paths)

(R_1) $\mathsf{flip}(p) \Longrightarrow \mathsf{flip}(q)$, $\mathsf{mon}(p) \Longrightarrow \mathsf{mon}(q)$, $(p; s) \Longrightarrow (q; s)$ *and* $(s; p) \Longrightarrow (s; q)$ *where* $p \Longrightarrow q$.

(R_2) $(\mathsf{id}_a; p) \Longrightarrow p$ *and* $(p; \mathsf{id}_b) \Longrightarrow p$ *for a path* $p : a \rightsquigarrow b$.

(R_3) $((p; q); s) \Longrightarrow (p; (q; s))$.

(R_4) $\mathsf{mon}_{a,a}(\mathsf{id}_a) \Longrightarrow \mathsf{id}_{F(a)}$.

(R_5) $\mathsf{mon}(p;q) \Longrightarrow (\mathsf{mon}(p);\mathsf{mon}(q))$.

(R_6) $\mathsf{flip}(\mathsf{id}_a) \Longrightarrow \mathsf{id}_a$.

(R_7) $\mathsf{flip}^2(\alpha) \Longrightarrow \mathsf{mon}_{a,b}(\alpha)$ *for an atomic path* $\alpha : a \rightsquigarrow b$.

(R_8) $\mathsf{flip}(p;q) \Longrightarrow (\mathsf{flip}(q);\mathsf{mon}^{\mathsf{k}(q)}(\mathsf{flip}(p)))$ *for paths* $p : a \rightsquigarrow b$ *and* $q : b \rightsquigarrow c$.

(R_9) $\mathsf{flip}(\mathsf{mon}(p)) \Longrightarrow \mathsf{mon}(\mathsf{flip}(p))$.

Proposition 2 *The path reduction* $\Longrightarrow$ *is terminating and confluent.*

Proof. Let Σ be the finite set of symbols $\{\mathsf{flip}, \mathsf{mon}, ;, \mathsf{id}\}$, and $>$ be the strict order such that $\{\mathsf{flip} > \mathsf{mon} > \, ; > \mathsf{id}\}$. Here, for termination it is enough to consider $\mathsf{id}, \mathsf{mon}, \mathsf{flip}$ instead of id_a, $\mathsf{mon}_{a,b}$, $\mathsf{flip}_{a,b}$, respectively. Then consider the well-known lexicographic path order $>_{lpo}$ on the set Σ over a countable set of variables induced by $>$ (e.g., [2], Section 5.4.2.). See [11] for the details.

Next, from Newman's Lemma, it is enough to verify that all critical pairs such as $\mathsf{flip}((p;q);r))$ and $\mathsf{flip}(\mathsf{mon}(p;q))$ are joinable. See also [11] for the details. □

Proposition 3 (Subject reduction) *If* $\vec{\Delta} \vdash p : a \rightsquigarrow b$ *and* $p \Longrightarrow q$*, then* $\vec{\Delta} \vdash q : a \rightsquigarrow b$.

Proof. By induction on the derivation of $p \Longrightarrow q$, see also the diagrams for the intuitive meaning of $\mathcal{E}$. □

We introduce the following grammar for a syntax of normal paths.

Definition 4 (Normal paths NP) *Let* $a \in \Delta_0$*,* $\alpha \in \Delta_1$*, and* $i \geq 0$.

$$\begin{array}{rcl} \mathsf{NP} & ::= & \mathsf{id}_{F^i(a)} \mid atom \mid np \\ atom & ::= & \alpha \mid \mathsf{flip}(\alpha) \mid \mathsf{mon}(atom) \\ np & ::= & (atom; np) \end{array}$$

Now, for any path we have a normal path with the desired property.

Theorem 2 (Normal reduction paths) *If* $\vec{\Delta} \vdash p : a \rightsquigarrow b$ *then there exists a unique normal path* $q \in \mathsf{NP}$ *such that* $p \Longrightarrow^* q$ *and* $\vec{\Delta} \vdash q : a \rightsquigarrow b$.

Proof. By induction on the derivation of $\vec{\Delta} \vdash p : a \rightsquigarrow b$ with Propositions 2 and 3. □

5 Representation of Reduction Paths

The formal system of reduction paths provides a graphical representation, denoted by $\mathcal{G}(\vec{\Delta})$, of a reduction path generated from $\vec{\Delta} = \langle \Delta_0, \Delta_1, s, t \rangle$ such that $\mathcal{G}(\vec{\Delta}) = \langle V, E \rangle$, where $V = \bigcup_{i \geq 0} F^i(\Delta_0)$ is the set of vertices (points) and $E = \{p \in \mathsf{RP} \mid \vec{\Delta} \vdash p : a \rightsquigarrow b$ for some $a, b \in V\}$ is the set of edges (arrows). Here, following theorem 2, we consider graphs consisting only of normal reduction paths.

Definition 5 (Reduction graph) *For $\vec{\Delta} = \langle \Delta_0, \Delta_1, s, t\rangle$, the graph $\mathcal{G}(\vec{\Delta})$ with normal paths is defined by the following digraph $\langle V, E\rangle$:*

$$V = \bigcup_{b\in\Delta_0} \{F^i(b) \mid i \geq 0\},\ E = \bigcup_{\alpha\in\Delta_1} \{\mathsf{mon}^i(\mathsf{flip}^j(\alpha)) \mid i \geq 0, j = 0, 1\}.$$

First, we show some examples of such graphs with normal paths. For conversion sequence, we consider a special type of quiver, called a (simply laced) Dynkin diagram of type $\mathbb{A}_{n+1}$ $(n \geq 0)$ [1], such that $\Delta_0 = \{b_0, b_1, \ldots, b_n\}$, $\Delta_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and either $\alpha_i : b_{i-1} \to b_i$ or $\alpha_i : b_i \to b_{i-1}$ $(i = 1, 2, \ldots, n)$. If $\alpha_i : b_{i-1} \to b_i$ then α_i is called a right arrow, and α_i is a left arrow otherwise. We employ the notation $\sharp(\leftarrow)[i, j]$ to denote the number of occurrences of a left arrow between two points b_i and b_j $(0 \leq i \leq j \leq n)$. We also write $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$, called a conversion sequence, for type $\mathbb{A}_{n+1}$ with $\Delta_0 = \{b_0, b_1, \ldots, b_n\}$, $l = \sharp(\leftarrow)[0, n]$, and $r = n - l$.

1. Case of $\mathcal{G}(b_0 \overset{\langle 1,0\rangle}{\leftrightsquigarrow} b_1) = \langle V, E\rangle$ with type $\mathbb{A}_2$:

$$V = \bigcup_{i\in\{0,1\}} \{F^n(b_i) \mid n \geq 0\},\ E = \{\mathsf{mon}^m(\mathsf{flip}^n(\alpha_1)) \mid m \geq 0, n = 0, 1\}.$$

   From now we draw no identity paths id for each vertex of the graph.

2. Case of $\mathcal{G}(b_0 \overset{\langle 1,1\rangle}{\leftrightsquigarrow} b_2) = \langle V, E\rangle$ with type $\mathbb{A}_3$:

$$V = \bigcup_{i\in\{0,1,2\}} \{F^n(b_i) \mid n \geq 0\}, E = \bigcup_{i\in\{1,2\}} \{\mathsf{mon}^m(\mathsf{flip}^n(\alpha_i)) \mid m \geq 0, n = 0, 1\}.$$

Next, we investigate the structure of the graph $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n)$, where the graph itself is infinite although the sequence $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$ is finite. For this, we show that $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n)$ is a plane graph, and that there exists a unique path from b_0 to $F^l(b_n)$ within the graph. It is indeed possible to apply the well-known method of the use of adjacency matrices[4] in order to count reduction paths. However, here we adopt a simple method to show the fundamental properties on the graph.

Let $\sharp(b_i \rightsquigarrow b_j)$ be the number of paths from b_i to b_j. Then in the graph $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_{n+1})$ where $l = L(n + 1)$ and $r = n + 1 - l$, the number of paths from b_0 to $F^{L(n+1)}(b_{n+1})$ is given by the following summation, exactly like the multiplication of adjacency matrices.

$$\sharp(b_0 \rightsquigarrow F^{L(n+1)}(b_{n+1})) = \sum_{i\geq 0} \sharp(b_0 \rightsquigarrow F^i(b_n)) \cdot \sharp(F^i(b_n) \rightsquigarrow F^{L(n+1)}(b_{n+1})),$$

[4]See section 7 for encoding a conversion sequence by adjacency matrices, which makes it possible to represent the structure of a finite fragment of the graph as an augmented matrix elegantly.

provided that $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\rightsquigarrow} b_{n+1})$ is planar, i.e., there exist no paths from b_0 to $F^{L(n+1)}(b_{n+1})$, which never pass through any $F^i(b_n)$. Using this idea, we can prove the following statement by induction on n.

Theorem 3 (Planar and unique path properties) *The reduction graph $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n)$ $(n \geq 1)$ is a plane graph such that there exists a unique path from b_0 to $F^l(b_n)$ and that there exist no paths from b_0 to $F^i(b_n)$ for $0 \leq i < l$.*

Proof. By induction on n, see [11] for the details. □

6 Unique Path and Universal Common-Reduct Properties

Based on Proposition 1 and Theorem 3, a path from b_0 to $F^l(b_n)$ is called a right main path, and a path from b_n to $F^r(b_0)$ is called a left main path, see also the example of section 2.

Proposition 4 (Unique main paths) *In the graph $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n)$ of type $\mathbb{A}_{n+1}$ $(n \geq 1)$, for every natural number $i \geq 0$ we have the unique path property such that there exists a unique path from $F^i(b_0)$ to $F^{i+l}(b_n)$, and that there exists a unique path from $F^i(b_n)$ to $F^{i+r}(b_0)$, where the length of the paths is n.*

Consider the conversion sequence $b \overset{\langle i+l,r\rangle}{\leftrightsquigarrow} b_n$ that is obtained by the concatenation of two sequences $b \overset{\langle i,0\rangle}{\leftrightsquigarrow} b_0$ and $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$ for some b. Then from Theorem 3, there exists a unique path from b to $F^{i+l}(b_n)$, which consists of the two unique paths from b to $F^i(b_0)$ and from $F^i(b_0)$ to $F^{i+l}(b_n)$. Similarly we have a unique path from $F^i(b_n)$ to $F^{i+r}(b_0)$.

Proposition 5 (Unique crossing point) *In the graph $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n)$ of type $\mathbb{A}_{n+1}$ $(n \geq 1)$, for every natural number i with $0 \leq i \leq l$, the right main path from b_0 to $F^l(b_n)$ has a unique crossing point with each path from $F^i(b_n)$ to $F^{i+r}(b_0)$.*

We have a crossing point $F^{L(r+i)}(b_{r+i})$ of the two paths by Proposition 1 [9, 10]. If the paths had more than one crossing point, then this would give more than one path from b_0 to $F^l(b_n)$, which contradicts Proposition 4.

Theorem 4 (Universal common-reduct) *In the graph $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n)$ of type $\mathbb{A}_{n+1}$ $(n \geq 1)$, the common reduct $F^k(b_r)$ has the universal common-reduct property: For any common reduct c of b_0 and b_n, there exists a reduction path from $F^k(b_r)$ to c where $k = \sharp(\leftarrow)[0,r]$.*

Proof. The two main paths divide the plane graph $\mathcal{G}(b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n)$ into the following four regions R_1, R_2, R_3, R_4.

1. The region R_1 consists of all points which lie to the right of the right main path and simultaneously to the left of the left main path including the boundary except $F^k(b_r)$.

2. The region R_2 consists of all points which lie to the left of the left main path and simultaneously to the left of the right main path excluding the boundary.

3. The region R_3 is the symmetric case of R_2.

4. The region R_4 consists of all points which lies to the right of the left main path and simultaneously to the left of the right main path including the boundary.

Then neither R_1, R_2, nor R_3 contains a common reduct of b_1 and b_n by Proposition 4. Only the bottom region R_4 can contain common reducts of b_0, b_n. The point $F^k(b_r)$ is indeed a common reduct of b_0, b_n by Proposition 5, from which we have a reduction path leading to every point in this region. Therefore, $F^k(b_r)$ is the universal common reduct of b_0 and b_n. □

Now we have unique paths to the common reduct $F^k(b_r)$, so that a set of rules with an application strategy can transform conversion sequences into reduction paths, leading to the universal common reduct. Let $p : b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$ of type $\mathbb{A}_{n+1}$ $(n \geq 0)$. Then applying the following transformation rule $\Longrightarrow_l$ to the left-hand side of $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$ generates the left main path. And applying the rule $\Longrightarrow_r$ to the right-hand side of $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$ does the right main path, as follows.

1. FromLeft-Transformation rule $\Longrightarrow_l$:
   - Start rules:
     (a) $\xrightarrow{\alpha} \quad \Longrightarrow_l \quad \xrightarrow{\langle \alpha,0\rangle}$
     (b) $\xleftarrow{\alpha} \quad \Longrightarrow_l \quad \xrightarrow{\langle \mathsf{flip}(\alpha),1\rangle}$
   - Step rules:
     (a) $\xrightarrow{\langle p,n\rangle};\xrightarrow{\alpha} \quad \Longrightarrow_l \quad \xrightarrow{\langle p,n\rangle};\xrightarrow{\langle \mathsf{mon}^n(\alpha),n\rangle}$
     (b) $\xrightarrow{\langle p,n\rangle};\xleftarrow{\alpha} \quad \Longrightarrow_l \quad \xrightarrow{\langle p,n\rangle};\xrightarrow{\langle \mathsf{mon}^n(\mathsf{flip}(\alpha)),n+1\rangle}$

2. FromRight-Transformation rule $\Longrightarrow_r$:
   - Start rules:
     (a) $\xleftarrow{\alpha} \quad \Longrightarrow_r \quad \xleftarrow{\langle \alpha,0\rangle}$
     (b) $\xrightarrow{\alpha} \quad \Longrightarrow_r \quad \xleftarrow{\langle \mathsf{flip}(\alpha),1\rangle}$
   - Step rules:
     (a) $\xleftarrow{\alpha};\xleftarrow{\langle p,n\rangle} \quad \Longrightarrow_r \quad \xleftarrow{\langle \mathsf{mon}^n(\alpha),n\rangle};\xleftarrow{\langle p,n\rangle}$
     (b) $\xrightarrow{\alpha};\xleftarrow{\langle p,n\rangle} \quad \Longrightarrow_r \quad \xleftarrow{\langle \mathsf{mon}^n(\mathsf{flip}(\alpha)),n+1\rangle};\xleftarrow{\langle p,n\rangle}$

3. FromBoth-Transformation rule $\Longrightarrow_{lr}$:
   The rule of FromLeft-Transformation is applied r times to the left-hand side of a given conversion sequence $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$, and that of FromRight-Transformation is applied l times from to right-hand side, simultaneously.

For instance, take a simple example $\overset{\alpha_1}{\longleftarrow}; \overset{\alpha_2}{\longrightarrow}; \overset{\alpha_3}{\longleftarrow}; \overset{\alpha_4}{\longrightarrow}; \overset{\alpha_5}{\longrightarrow}$ from $b_0 \overset{\langle 2,3\rangle}{\leftrightsquigarrow} b_5$. Then we obtain the common reduct $F^2(b_3)$ of b_0 and b_5 as follows:

$$
\begin{array}{ll}
 & \overset{\alpha_1}{\longleftarrow}; \overset{\alpha_2}{\longrightarrow}; \overset{\alpha_3}{\longleftarrow}; \overset{\alpha_4}{\longrightarrow}; \overset{\alpha_5}{\longrightarrow} \\
\Longrightarrow_{lr} & \xrightarrow{\langle \mathsf{flip}(\alpha_1),1\rangle}; \overset{\alpha_2}{\longrightarrow}; \overset{\alpha_3}{\longleftarrow}; \overset{\alpha_4}{\longrightarrow}; \xleftarrow{\langle \mathsf{flip}(\alpha_5),1\rangle} \\
\Longrightarrow_{lr} & \xrightarrow{\langle \mathsf{flip}(\alpha_1),1\rangle}; \xrightarrow{\langle \mathsf{mon}(\alpha_2),1\rangle}; \overset{\alpha_3}{\longleftarrow}; \xleftarrow{\langle \mathsf{mon}(\mathsf{flip}(\alpha_4)),2\rangle}; \xleftarrow{\langle \mathsf{flip}(\alpha_5),1\rangle} \\
\Longrightarrow_{l} & \xrightarrow{\langle \mathsf{flip}(\alpha_1),1\rangle}; \xrightarrow{\langle \mathsf{mon}(\alpha_2),1\rangle}; \xrightarrow{\langle \mathsf{mon}(\mathsf{flip}(\alpha_3)),2\rangle}; \xleftarrow{\langle \mathsf{mon}(\mathsf{flip}(\alpha_4)),2\rangle}; \xleftarrow{\langle \mathsf{flip}(\alpha_5),1\rangle}
\end{array}
$$

which provides the right main path $(\mathsf{flip}(\alpha_1); \mathsf{mon}(\alpha_2); \mathsf{mon}(\mathsf{flip}(\alpha_3))) : b_0 \rightsquigarrow F^2(b_3)$ and the left main path $(\mathsf{flip}(\alpha_5); \mathsf{mon}(\mathsf{flip}(\alpha_4))) : b_5 \rightsquigarrow F^2(b_3)$.

Next, we introduce a path matrix to represent all arrows in a finite subset of E.

7 Path Matrices

If a quiver is finite, then generated reduction paths are represented by an adjacency matrix. Given a conversion sequence, then based on Definition 5, we construct such adjacency matrices $\mathcal{P}$, called path matrices, as block matrices. The number of reduction paths of length m can be counted by operating on the matrix power $\mathcal{P}^m$. Let $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$ be a conversion sequence of type $\mathbb{A}_{n+1}$. Then the adjacency matrices are defined in the following.

Definition 6 (Atomic matrix) *An atomic matrix $A = (a_{i,j})$ with $1 \leq i, j \leq n+1$ is defined from $b_0 \overset{\langle l,r\rangle}{\leftrightsquigarrow} b_n$, as follows*:

1. $a_{i+1,i} = 1$ *if* $b_{i-1} \leftarrow b_i$ $(1 \leq i \leq n)$.
2. $a_{i,i+1} = 1$ *if* $b_{i-1} \rightarrow b_i$ $(1 \leq i \leq n)$.
3. $a_{i,j} = 0$ *otherwise.*

In other words, the exclusive or (disjunction) of $a_{i,(i+1)}$ and $a_{(i+1),i}$ is 1 for $i = 1, 2, \ldots, n$, and $a_{i,j} = 0$ otherwise.

Definition 7 (Path matrix) *Let A be an atomic matrix, and ${}^tA$ be its transposed matrix, then a path matrix $\mathcal{P}$ with type $((n+1)^2, (n+1)^2)$ is defined by using the tensor product*[5] $\otimes$ *as follows*:

$$\mathcal{P} = I_{n+1} \otimes A + \sum_{i=1}^{n} (E_{i,i+1} \otimes {}^tA)$$

[5] From the definition, the tensor product $X \otimes Y$ provides the matrix that has every element of X, scalar multiplied with Y, i.e., $x_{i,j} \cdot Y$ for $X = (x_{i,j})$.

where the identity matrix I_{n+1} has type $(n+1, n+1)$. The element matrix $E_{p,q} = (e_{i,j})$ with $(n+1, n+1)$ denotes the matrix such that $e_{i,j} = 1$ for $(i,j) = (p,q)$, and $e_{i,j} = 0$ otherwise.

It should be remarked that the transposed matrix ${}^tA$ represents an effect of an application of flip to each atomic path. That is, flipped paths that are obtained from atomic paths are coded by ${}^tA$. An application of mon with the point constructor F increases the type of the matrix, which is represented by the use of the tensor product. It should be clear that by the definition, the path matrix $\mathcal{P}$ has the following form:

$$I_{n+1} \otimes A + \sum_{i=1}^{n} (E_{i,i+1} \otimes {}^tA) = \begin{pmatrix} A & {}^tA & O & O & \cdots & O & O \\ O & A & {}^tA & O & \cdots & O & O \\ O & O & A & {}^tA & \vdots & O & O \\ O & O & O & A & {}^tA & \vdots & O \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & O \\ O & O & O & O & \vdots & A & {}^tA \\ O & O & O & O & \cdots & O & A \end{pmatrix}$$

where the atomic matrix A is assigned to every diagonal position of the block matrix, and the transposed ${}^tA$ is assigned to each position to the right of A.

Let $\mathcal{P}^m = (p_{i,j})$ with $1 \leq i,j \leq (n+1)^2$. Then the element $p_{1,j}$ represents the number of paths with m arrows (length m) from b_0 to $F^q(b_k)$, where $j = (n+1) \cdot q + k + 1$ and $0 \leq k \leq n$, see also the forthcoming example in this section.

For natural numbers n, m and square matrices X, Y with the same type, define the following matrix $\mathsf{C}(X^n, Y^m)$. The matrix represents all paths of length $(n+m)$, which are obtained by choosing n paths that are coded by X and m paths that are coded by Y.

Definition 8 *1.* $\mathsf{C}(X^0, Y^0) = I$ *where I is the identity matrix.*

2. $\mathsf{C}(X^0, Y^{m+1}) = Y^{m+1}$.

3. $\mathsf{C}(X^{n+1}, Y^0) = X^{n+1}$.

4. $\mathsf{C}(X^{n+1}, Y^{m+1}) = \mathsf{C}(X^n, Y^{m+1}) \cdot X + \mathsf{C}(X^{n+1}, Y^m) \cdot Y$.

Then we have $(X+Y)^k = \sum_{i=0}^{k} \mathsf{C}(X^{k-i}, Y^i)$ by induction on k. Now, for $1 \leq k < n$, the matrix power $\mathcal{P}^k$ can be expressed by the small matrices A and ${}^tA$ as follows, which can be verified by induction on k:

$$\begin{pmatrix} A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots & \mathsf{C}(A, {}^tA^{k-1}) & {}^tA^k & O & \cdots & O \\ O & A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots & \mathsf{C}(A, {}^tA^{k-1}) & {}^tA^k & O & \cdots \\ O & O & A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots & \cdots & \cdots & \cdots \\ O & O & O & A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots \\ O & O & O & O & O & \cdots & O & A^k \end{pmatrix}$$

For $k \geq n$, $\mathcal{P}^k$ can be expressed similarly as follows:

$$\begin{pmatrix} A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots & \cdots & \cdots & \mathsf{C}(A^{k-n}, {}^tA^n) \\ O & A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots & \cdots & \mathsf{C}(A^{k-n+1}, {}^tA^{n-1}) \\ O & O & A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots & \cdots \\ O & O & O & A^k & \mathsf{C}(A^{k-1}, {}^tA) & \cdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots \\ O & O & O & \cdots & O & A^k \end{pmatrix}$$

That is to say, $(1, i)$-matrix of $\mathcal{P}^m$ $(i = 1, \ldots, n+1)$ is described as follows.

1. Case of $m < n$:

$$((1, i)\text{-matrix of } \mathcal{P}^m) = \begin{cases} \mathsf{C}(A^{m-i+1}, ({}^tA)^{i-1}) & \text{for } i = 1, 2, \ldots, m+1, \\ O & \text{for } i = m+2, \ldots, n+1. \end{cases}$$

2. Case of $m \geq n$:

$$((1, i)\text{-matrix of } \mathcal{P}^m) = \mathsf{C}(A^{m-(i-1)}, ({}^tA)^{i-1}), \quad \text{for } i = 1, 2, \ldots, n+1.$$

From the definition of the adjacency matrix, we should remark the following facts. The points b_0 and $F^q(b_i)$ $(q \geq 0)$ cannot be connected by using m arrows with $m \leq i - 1$, so that $((1, (n+1) \cdot q + i + 1)$-element of $\mathcal{P}^m) = 0$ for $m < i$. That is, $((1, i+1)$-element of $\mathsf{C}(A^{m-q}, ({}^tA)^q)) = 0$ for $m < i$.

Recall $L(i) = \sharp(\leftarrow)[0, i]$, and let $L(i) \geq 1$. We have no paths from b_0 to b_i trivially, if $L(i) = 1$. Similarly, we have no paths from b_0 to $F^q(b_i)$ with $q < L(i)$, so that $((1, (n+1) \cdot q + i + 1)$-element of $\mathcal{P}^m) = 0$ for $q < L(i)$. That is, $((1, i+1)$-element of $\mathsf{C}(A^{m-q}, ({}^tA)^q)) = 0$ for $q < L(i)$.

Remark 1 *1.* $((1, i+1)$*-element of* $\mathsf{C}(A^{m-q}, ({}^tA)^q)) = 0$*, for* $m < i$*.*

2. $((1, i+1)$*-element of* $\mathsf{C}(A^{m-q}, ({}^tA)^q)) = 0$*, for* $q < L(i)$*.*

For the unique path property we have the following main theorem.

Theorem 5 (Unique path property) *Given* $b_0 \overset{\langle l,r \rangle}{\leftrightsquigarrow} b_n$ *with type* $\mathbb{A}_{n+1}$*. Then for each* $i = 1, 2, \ldots, n$*,*

$$((1, (n+1) \cdot L(i) + i + 1)\text{-element of } \mathcal{P}^m) = \delta_{m,i}$$

where $L(i) = \sharp(\leftarrow)[0, i]$ *and* $\delta_{m,i}$ *is the Kronecker delta. In other words, for each* $i = 1, 2, \ldots, n$*, there exists a unique path from* b_0 *to* $F^{L(i)}(b_i)$ *where the length of the path is* i*.*

Proof. From the form of $\mathcal{P}^m$ and the remark above, it is enough to show that

$$((1, i+1)\text{-element of } \mathsf{C}(A^{m-L(i)}, ({}^tA)^{L(i)})) = \delta_{m,i}$$

with $m \geq i$ for each $i = 1, 2, \ldots, n$ by induction on i. See Appendix A for the details. $\square$

Take a simple example $b_0 \leftarrow b_1 \rightarrow b_2 \leftarrow b_3$ from $b_0 \overset{\langle 2,1\rangle}{\rightsquigarrow} b_3$ of type $\mathbb{A}_4$. Then we have the following atomic matrix A and the transposed ${}^tA$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{pmatrix} \qquad \text{and} \qquad {}^tA = \begin{pmatrix} 0 & a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{23} & 0 & a_{43} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $a_{21} = a_{23} = a_{43} = 1$. The path matrix $\mathcal{P}$ with type $(4^2, 4^2)$ is obtained as follows.

$$\mathcal{P} = \begin{pmatrix} A & {}^tA & O & O \\ O & A & {}^tA & O \\ O & O & A & {}^tA \\ O & O & O & A \end{pmatrix}$$

From $\mathcal{P}^k$ $(2 \leq k \leq 7)$, we obtain the number of reduction paths from b_i to $F^m(b_j)$ $(1 \leq m \leq 3)$ with length k for each $0 \leq i, j \leq 3$. For instance, the number of reduction paths of $b_0 \rightsquigarrow F^3(b_3)$ with length 5 is obtained as 3 by $(1, 16)$-element of $\mathcal{P}^5$, and that of $b_1 \rightsquigarrow F^3(b_2)$ with length 7 is obtained as 21 by $(2, 15)$-element of $\mathcal{P}^7$. See also $\mathcal{G}(b_0 \overset{\langle 2,1\rangle}{\rightsquigarrow} b_3)$ below for the instance $b_0 \leftarrow b_1 \rightarrow b_2 \leftarrow b_3$, where all the arrows in this graph are coded by $\mathcal{P}$.

$$\begin{array}{ccccccc}
b_0 & \leftarrow & b_1 & \rightarrow & b_2 & \leftarrow & b_3 \\
 & \searrow & & \swarrow & & \searrow & \\
F(b_0) & \leftarrow & F(b_1) & \rightarrow & F(b_2) & \leftarrow & F(b_3) \\
 & \searrow & & \swarrow & & \searrow & \\
F^2(b_0) & \leftarrow & F^2(b_1) & \rightarrow & F^2(b_2) & \leftarrow & F^2(b_3) \\
 & \searrow & & \swarrow & & \searrow & \\
F^3(b_0) & \leftarrow & F^3(b_1) & \rightarrow & F^3(b_2) & \leftarrow & F^3(b_3)
\end{array}$$

8 Concluding Remarks

From the motivation of quantitative analysis of reduction systems we introduced a category-like structure induced from a quiver, together with the three path constructors based on the well-known triangle property. In section 2, we started with β-reduction. However, it should be remarked that the property of Proposition 1 still holds for parallel $\beta\eta$-reduction and moreover holds even for the parallel reductions defined for other reduction systems such as Girard's system **F** and Gödel's system **T** as well [22]. This might mean that the formal system of reduction paths extracts some abstract and common property from such reduction systems that satisfy the condition (A). Next, the equational theory and reduction rules for reduction paths are introduced to obtain normal reduction paths. Based on the normal paths, simple and planar reduction graphs are constructed from a conversion sequence. To count reduction paths, it is natural to define path matrices by adjacency matrices induced from a finite fragment of the reduction graphs consisting of normal paths. Then the unique path property (UPP) and universal common-reduct property (UCR) are established. As a result, the common reduct $F^k(P)$ in Proposition 1 can be considered as the optimum one among those which are generated by the condition (c) of (A) (triangle property). It is now straightforward to define transformation rules from conversion sequences to reduction paths leading to the universal common reduct.

Our formal system of reduction paths simply regards terms as points, and a finite fragment of a reduction graph can be represented by a path matrix. From the augmented matrix, we can obtain an adjacency matrix representing a quiver which is generated from a given quiver of type $\mathbb{A}_{n+1}$. Quivers are well known as representations of algebras [1]. It would be worthwhile to investigate this viewpoint further, which might induce an algebraic semantics of lambda-calculus such that lambda-terms are interpreted as vector spaces and reduction paths as linear maps. One possible way to make a higher order extension is to introduce a Groupoid-like structure as in [6]. This subject should be investigated further.

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A Proof of Theorem 5

We recall the statement of **Theorem 5**:

Given $b_0 \overset{\langle l,r \rangle}{\rightsquigarrow} b_n$ of type $\mathbb{A}_{n+1}$. Then for each $i = 1, 2, \ldots, n$, we have

$$((1, (n+1) \cdot L(i) + i + 1)\text{-element of } \mathcal{P}^m) = \delta_{m,i}$$

where $L(i) = \sharp(\leftarrow)[0, i]$ and $\delta_{m,i}$ is the Kronecker delta. That is, for each $i = 1, 2, \ldots, n$, there exists a unique path from b_0 to $F^{L(i)}(b_i)$.

Proof. From the form of $\mathcal{P}^m$ and Remark 1, it is enough to show that

$$((1, i+1)\text{-element of } \mathsf{C}(A^{m-L(i)}, ({}^tA)^{L(i)})) = \delta_{m,i}$$

with $m \geq i$ for each $i = 1, 2, \ldots, n$ by induction on i.

1. Case of $i = 1$ and $L(1) = 1$, i.e., $b_0 \leftarrow b_1$:

   From the definition of A, we have $a_{2,1} = 1$, and $(1, 2)$-element of ${}^tA$ is indeed $a_{2,1} = 1$.

   Next, $(1, 2)$-element of $\mathsf{C}(A^{m-1}, {}^tA)$ is 0 for $m \geq 2$. In fact, for $m \geq 2$,

$$\begin{aligned} & \mathsf{C}(A^{m-1}, {}^tA) \\ = \;& ({}^tA)(A^{m-1}) + A({}^tA)(A^{m-2}) + \cdots + (A^{m-2})({}^tA)A + A^{m-1}({}^tA) \\ = \;& ({}^tA)(A^{m-1}) + A(({}^tA)(A^{m-2}) + \cdots + (A^{m-3})({}^tA)A + A^{m-2}({}^tA)), \end{aligned}$$

   where $((1, 2)\text{-element of } ({}^tA)(A^{m-1})) = 0$ for $m \geq 2$, and the first raw of A with $a_{1,2} = 0$ consists only of 0, which implies that $((1, 2)\text{-element of } A \cdot B) = 0$ for any B.

2. Case of $i = 1$ and $L(1) = 0$, i.e., $b_0 \rightarrow b_1$:

   From the definition of A, we have $a_{1,2} = 1$.

   In fact, $((1, 2)\text{-element of } \mathsf{C}(A^m, ({}^tA)^0)) = 0$ for $m \geq 2$.

3. Case of $L(i) = \sharp(\leftarrow)[0, i] = \sharp(\leftarrow)[0, i+1] = L(i+1)$ with $i + 2 \leq n$, i.e., $b_i \rightarrow b_{i+1}$ where $a_{i+1,i+2} = 1$:

   (a) First, we will show that for $m = i + 1$,

$$((1, i+2)\text{-element of } \mathsf{C}(A^{m-L(i+1)}, ({}^tA)^{L(i+1)})) = 1.$$

   Here, for $L(i) \geq 1$ we have

$$\begin{aligned} & \mathsf{C}(A^{m-L(i+1)}, ({}^tA)^{L(i+1)}) \\ = \;& \mathsf{C}(A^{m-L(i+1)}, ({}^tA)^{L(i+1)-1}) \cdot ({}^tA) + \mathsf{C}(A^{m-1-L(i+1)}, ({}^tA)^{L(i+1)}) \cdot A. \end{aligned}$$

   For $L(i) = 0$, we have $\mathsf{C}(A^m, ({}^tA)^0) = A^m$, which is the same as the base case.

From the induction hypothesis (I-H), for $m = i$ we have

$$((1, i+1)\text{-element of } \mathsf{C}(A^{m-L(i)}, ({}^tA)^{L(i)})) = 1.$$

Now, $((1, i+2)\text{-element of } \mathsf{C}(A^{i+1-L(i+1)}, ({}^tA)^{L(i+1)})) = 1$ is obtained as follows:

$$\begin{aligned}
& ((1, i+2)\text{-element of } \mathsf{C}(A^{i+1-L(i+1)}, {}^tA^{L(i+1)})) \\
= \; & \sum_{j=1}^{n+1} ((1, j)\text{-el. of } \mathsf{C}(A^{i+1-L(i+1)}, {}^tA^{L(i+1)-1})) \cdot ((j, i+2)\text{-el. of } {}^tA) \\
& \quad + \sum_{j=1}^{n+1} ((1, j)\text{-el. of } \mathsf{C}(A^{i-L(i+1)}, {}^tA^{L(i+1)})) \cdot ((j, i+2)\text{-el. of } A) \\
= \; & ((1, i+1)\text{-el. of } \mathsf{C}(A^{i+1-L(i)}, {}^tA^{L(i)-1})) \cdot ((i+1, i+2)\text{-el. of } {}^tA) \\
& + ((1, i+3)\text{-el. of } \mathsf{C}(A^{i+1-L(i)}, {}^tA^{L(i)-1})) \cdot ((i+3, i+2)\text{-el. of } {}^tA) \\
& \quad + ((1, i+1)\text{-el. of } \mathsf{C}(A^{i-L(i)}, {}^tA^{L(i)})) \cdot ((i+1, i+2)\text{-el. of } A) \\
& \quad + ((1, i+3)\text{-el. of } \mathsf{C}(A^{i-L(i)}, {}^tA^{L(i)})) \cdot ((i+3, i+2)\text{-el. of } A) \\
= \; & ((1, i+1)\text{-element of } \mathsf{C}(A^{i+1-L(i)}, {}^tA^{L(i)-1})) \cdot 0 \quad \because a_{i+2,i+1} = 0 \\
& + 0 \cdot a_{i+2,i+3} \qquad \because \text{Remark1-1} \\
& \quad + 1 \cdot 1 \qquad \because \text{I-H and } a_{i+1,i+2} = 1 \\
& \quad + 0 \cdot a_{i+3,i+2} \qquad \because \text{Remark1-1} \\
= \; & 1,
\end{aligned}$$

where we have $a_{j,i+2} = 0$ except for $j = i+1, i+3$.

(b) Next, we will show that for $m \geq i+2$,

$$((1, i+2)\text{-element of } \mathsf{C}(A^{m-L(i+1)}, ({}^tA)^{L(i+1)})) = 0.$$

The induction hypothesis (I-H) gives that for $m \geq i+1$, we have

$$((1, i+1)\text{-element of } \mathsf{C}(A^{m-L(i)}, ({}^tA)^{L(i)})) = 0.$$

$$\begin{aligned}
& ((1,i+2)\text{-element of } \mathsf{C}(A^{m-L(i+1)}, {}^tA^{L(i+1)})) \\
= & \sum_{j=1}^{n+1}((1,j)\text{-el. of } \mathsf{C}(A^{m-L(i+1)}, {}^tA^{L(i+1)-1})) \cdot ((j,i+2)\text{-el. of } {}^tA) \\
& \quad + \sum_{j=1}^{n+1}((1,j)\text{-el. of } \mathsf{C}(A^{m-L(i+1)-1}, {}^tA^{L(i+1)})) \cdot ((j,i+2)\text{-el. of } A) \\
= & ((1,i+1)\text{-element of } \mathsf{C}(A^{m-L(i)}, {}^tA^{L(i)-1})) \cdot a_{i+2,i+1} \\
& +((1,i+3)\text{-element of } \mathsf{C}(A^{m-L(i)}, {}^tA^{L(i)-1})) \cdot a_{i+2,i+3} \\
& \quad + ((1,i+1)\text{-el. of } \mathsf{C}(A^{(m-1)-L(i)}, {}^tA^{L(i)})) \cdot a_{i+1,i+2} \\
& \quad + ((1,i+3)\text{-el. of } \mathsf{C}(A^{m-1-L(i)}, {}^tA^{L(i)})) \cdot a_{i+3,i+2} \\
= & 0 \quad \because a_{i+2,i+1} = 0 \\
& +((1,i+3)\text{-el. of } \mathsf{C}(A^{m-1-(L(i)-1)}, {}^tA^{L(i)-1})) \cdot a_{i+2,i+3} \\
& \quad + 0 \ \because \text{I-H with } (m-1) \geq i+1 \\
& \quad + ((1,i+3)\text{-el. of } \mathsf{C}(A^{m-1-L(i)}, {}^tA^{L(i)})) \cdot a_{i+3,i+2} \\
= & 0 \ \because \text{Remark1-2 with } L(i)-1 < L(i) = L(i+2) \text{ for } a_{i+2,i+3} = 1 \\
& +0 \ \because L(i) < L(i)+1 = L(i+2) \text{ for } a_{i+3,i+2} = 1 \\
= & 0.
\end{aligned}$$

4. Case of $L(i+1) = \sharp(\leftarrow)[0,i+1] = L(i)+1$ with $i+2 \leq n$, i.e., $b_i \leftarrow b_{i+1}$ where $a_{i+2,i+1} = 1$:

(a) We will show that for $m = i+1$,

$$((1,i+2)\text{-element of } \mathsf{C}(A^{m-L(i+1)}, ({}^tA)^{L(i+1)})) = 1.$$

Here, we have

$$\begin{aligned}
& \mathsf{C}(A^{m-L(i+1)}, ({}^tA)^{L(i+1)}) \\
= & \mathsf{C}(A^{m-1-L(i)}, ({}^tA)^{L(i)}) \cdot ({}^tA) + \mathsf{C}(A^{m-L(i)-2}, ({}^tA)^{L(i)+1}) \cdot A.
\end{aligned}$$

From the induction hypothesis, for $m = i$ we have

$$((1,i+1)\text{-element of } \mathsf{C}(A^{m-L(i)}, ({}^tA)^{L(i)})) = 1.$$

Now, $((1,i+2)\text{-element of } \mathsf{C}(A^{i+1-L(i+1)}, ({}^tA)^{L(i+1)})) = 1$ is obtained as follows:

$$
\begin{aligned}
& ((1,i+2)\text{-element of } \mathsf{C}(A^{i+1-L(i+1)}, {}^tA^{L(i+1)})) \\
= \; & \sum_{j=1}^{n+1} ((1,j)\text{-el. of } \mathsf{C}(A^{i-L(i)}, {}^tA^{L(i)})) \cdot ((j,i+2)\text{-el. of } {}^tA) \\
& \quad + \sum_{j=1}^{n+1} ((1,j)\text{-el. of } \mathsf{C}(A^{i-(L(i)+1)}, {}^tA^{L(i)+1})) \cdot ((j,i+2)\text{-el. of } A) \\
= \; & ((1,i+1)\text{-el. of } \mathsf{C}(A^{i-L(i)}, {}^tA^{L(i)})) \cdot ((i+1,i+2)\text{-el. of } {}^tA) \\
& +((1,i+3)\text{-el. of } \mathsf{C}(A^{i-L(i)}, {}^tA^{L(i)})) \cdot ((i+3,i+2)\text{-el. of } {}^tA) \\
& \; + ((1,i+1)\text{-el. of } \mathsf{C}(A^{i-L(i)-1}, {}^tA^{L(i)+1})) \cdot ((i+1,i+2)\text{-el. of } A) \\
& \; + ((1,i+3)\text{-el. of } \mathsf{C}(A^{i-L(i)-1}, {}^tA^{L(i)+1})) \cdot ((i+3,i+2)\text{-el. of } A) \\
= \; & 1 \cdot 1 \qquad \because \text{I-H and } a_{i+2,i+1} = 1 \\
& + 0 \cdot a_{i+2,i+3} \qquad \because \text{Remark1-1} \\
& \quad + ((1,i+1)\text{-el. of } \mathsf{C}(A^{i-L(i)-1}, {}^tA^{L(i)+1})) \cdot 0 \;\; \because a_{i+1,i+2} = 0 \\
& \quad + 0 \cdot a_{i+3,i+2} \qquad \because \text{Remark1-1} \\
= \; & 1,
\end{aligned}
$$

where we have $a_{j,i+2} = 0$ except for $j = i+1, i+3$ with $i+2 \leq n$.

(b) Next, we will show that for $m \geq i+2$,

$$((1,i+2)\text{-element of } \mathsf{C}(A^{m-L(i+1)}, ({}^tA)^{L(i+1)})) = 0.$$

The induction hypothesis (I-H) gives that for $m \geq i+1$, we have

$$((1,i+1)\text{-element of } \mathsf{C}(A^{m-L(i)}, ({}^tA)^{L(i)})) = 0.$$

$$
\begin{aligned}
& ((1,i+2)\text{-element of } \mathsf{C}(A^{m-L(i+1)}, {}^tA^{L(i+1)})) \\
= \; & \sum_{j=1}^{n+1}((1,j)\text{-el. of } \mathsf{C}(A^{m-L(i+1)}, {}^tA^{L(i+1)-1})) \cdot ((j,i+2)\text{-el. of } {}^tA) \\
& \quad + \sum_{j=1}^{n+1}((1,j)\text{-el. of } \mathsf{C}(A^{m-L(i+1)-1}, {}^tA^{L(i+1)})) \cdot ((j,i+2)\text{-el. of } A) \\
= \; & ((1,i+1)\text{-element of } \mathsf{C}(A^{(m-1)-L(i)}, {}^tA^{L(i)})) \cdot a_{i+2,i+1} \\
& +((1,i+3)\text{-element of } \mathsf{C}(A^{m-1-L(i)}, {}^tA^{L(i)})) \cdot a_{i+2,i+3} \\
& \quad + ((1,i+1)\text{-element of } \mathsf{C}(A^{m-L(i)-2}, {}^tA^{L(i)+1})) \cdot a_{i+1,i+2} \\
& \quad + ((1,i+3)\text{-element of } \mathsf{C}(A^{m-L(i)-2}, {}^tA^{L(i)+1})) \cdot a_{i+3,i+2} \\
= \; & 0 \qquad \because \text{I-H with } (m-1) \geq i+1 \\
& +((1,i+3)\text{-el. of } \mathsf{C}(A^{m-1-L(i)}, {}^tA^{L(i)})) \cdot a_{i+2,i+3} \\
& \quad + 0 \qquad \because a_{i+1,i+2} = 0 \\
& \quad + ((1,i+3)\text{-el. of } \mathsf{C}(A^{m-1-(L(i)+1)}, ({}^tA)^{L(i)+1})) \cdot a_{i+3,i+2} \\
= \; & 0 \because \text{Remark1-2 with } L(i) < L(i+1) = L(i+2) \text{ for } a_{i+2,i+3} = 1 \\
& +0 \because L(i)+1 = L(i+1) < L(i+1)+1 = L(i+2) \text{ for } a_{i+3,i+2} = 1 \\
= \; & 0.
\end{aligned}
$$

5. Following the similar pattern, for $m = n$ we can show

$$((1,n+1)\text{-element of } \mathsf{C}(A^{m-L(n)}, ({}^tA)^{L(n)})) = 1.$$

Similarly, for $m \geq n+1$ we can also show

$$((1,n+1)\text{-element of } \mathsf{C}(A^{m-L(n)}, ({}^tA)^{L(n)})) = 0.$$

□