An Envy-free and Truthful Mechanism for the Cake-cutting Problem

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Abstract

Alijani, Farhadi, Ghodsi, Seddighin, and Tajik considered a restricted version of the cake-cutting problem and proposed a mechanism based on the expansion process with unlocking [1, 6]. They claimed that their mechanism uses a small number of cuts, and that it is envy-free and truthful. We first show that it is not actually envy-free and truthful. Then, for the same cake-cutting problem, we give a new envy-free and truthful mechanism with a small number of cuts, which is not based on their expansion process with unlocking.

1 Introduction

The problem of dividing a cake among players in a fair manner has been widely studied since it was first defined by Steinhaus [7]. Prcocaccia has claimed in his survey paper [5] as follows: insight from the study of cake-cutting problem can be applied to the allocation of computational resources, and designing cake-cutting algorithms that are computationally efficient and immune to manipulation is a challenge for computer scientists. Recently, the cake-cutting problem has been studied by computer scientists, not only from the viewpoint of computational complexity [3], but also from the game theoretical point of view [2].

Alijani, Farhadi, Ghodsi, Seddighin, and Tajik considered the following cake-cutting problem from the game theoretical point of view [1, 6]:

Given a divisible heterogeneous cake $C = (0, 1] = \{x \mid 0 < x \leq 1\}$, a set of n strategic players $N = \{1, 2, \ldots, n\}$ and the valuation intervals $\mathcal{T} = \{C_1, C_2, \ldots, C_n\}$ with valuation interval $C_i = (\alpha_i, \beta_i] = \{x \mid 0 \leq \alpha_i < x \leq \beta_i \leq 1\} \subseteq C$ of each player $i \in N$, find a mechanism (that is, a polynomial time algorithm) for dividing the cake into pieces and allocating pieces of the cake to n players to meet the following conditions (Figure 1):

(i) the mechanism is envy-free, i.e., each player (weakly) prefers his/her own allocated piece to any other player's allocated piece,

(ii) the mechanism is strategy-proof (truthful), i.e., each player's dominant strategy is to reveal his/her own true valuation interval over the cake (i.e., making a lie will not lead to a better result), and

(iii) the number of cuts made on the cake is small.

They proposed an expansion process with unlocking and gave a mechanism for the above cake-cutting problem based on the expansion process with unlocking [1, 6]. They

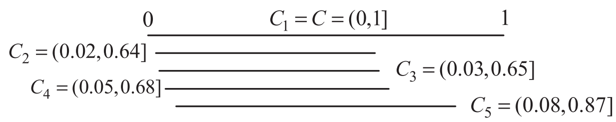

Figure 1: An input example for the cake-cutting problem ($n = 5$). Player 1 is allocated $(0, 0.07] \cup (0.87, 1]$, player 2 is allocated $(0.07, 0.27]$, player 3 is allocated $(0.27, 0.47]$, player 4 is allocated $(0.47, 0.67]$ and player 5 is allocated $(0.67, 0.87]$.

claimed that their mechanism satisfies the above three conditions, i.e., it is envy-free, truthful and the number of cuts made on the cake is at most $2(n-1)$.

In this paper, we first show that the mechanism based on the expansion process with unlocking proposed by Alijani, Farhadi, Ghodsi, Seddighin, and Tajik in the paper [1, 6] uses a small number of cuts, but is not actually envy-free and truthful.

Then, for the same cake-cutting problem, we give an alternative envy-free and truthful mechanism which is not based on the expansion process with unlocking.

2 Mechanism Proposed by Alijani et al.

In this section, we explain the mechanism based on the expansion process with unlocking proposed by Alijani, Farhadi, Ghodsi, Seddighin, and Tajik by borrowing their description in the paper [1, 6]. Since our description is the same as their description, although we changed a little such as notation, we would like to express our sincere gratitude to them.

As mentioned before, in the cake-cutting problem, we are given a divisible heterogeneous cake $C = (0, 1] = \{x \mid 0 < x \leq 1\}$, a set of n strategic players $N = \{1, 2, \ldots, n\}$ and the valuation intervals $\mathcal{T} = \{C_1, C_2, \ldots, C_n\}$ with valuation interval $C_i = (\alpha_i, \beta_i] = \{x \mid 0 \leq \alpha_i < x \leq \beta_i \leq 1\} \subseteq C$ of each player $i \in N$.

A *piece* of C is a set of mutually disjoint intervals in C. For an interval $I = (a, b] \subseteq C$, the length of I, denoted by $len(I)$, is defined by $len(I) = b - a$. Thus, for a piece $A = \{I_1, I_2, \ldots, I_{|A|}\}$, the *length* of A, denoted by $len(A)$, is defined by the total length of the intervals in A, i.e., $len(A) = \sum_{k=1}^{|A|} len(I_k)$. The *value* of an interval $I = (a, b]$ to player $i \in N$, denoted by $V_i(I)$, is defined by the length of interval $I \cap C_i$, i.e.,

$$V_i(I) = len(I \cap C_i).$$

Thus, for a piece $A = \{I_1, I_2, \ldots, I_{|A|}\}$, the *value* of A to player $i \in N$, denoted by $V_i(A)$, is defined by the total value of the intervals in A to player $i \in N$, i.e.,

$$V_i(A) = \sum_{k=1}^{|A|} V_i(I_k) = \sum_{k=1}^{|A|} len(I_k \cap C_i).$$

A *division* of the cake C among n players $N = \{1, 2, \ldots, n\}$ is a set $D = \{A_1, A_2, \ldots, A_n\}$ of pieces, with each piece $A_i = \{I_{i,1}, I_{i,2}, \ldots, I_{i,|A_i|}\}$ to player $i \in N$ with the following two properties:

(i) every pair of pieces are mutually disjoint, i.e., for all A_i, A_j $(1 \leq i < j \leq n)$, $A_i \cap A_j = \emptyset$, and

(ii) no piece of the cake is left behind, i.e., $\bigcup_{i \in N} A_i = C$.

The number of cuts in division $D = \{A_1, A_2, \ldots, A_n\}$ is $(\sum_{i \in N} |A_i|) - 1$. A division $D = \{A_1, A_2, \ldots, A_n\}$ is *envy-free*, if, for every player $i \in N$ and every piece $A_j \in D$, the inequality $V_i(A_i) \geq V_i(A_j)$ holds. In this setting, the envy-free notion for division $D = \{A_1, A_2, \ldots, A_n\}$ with each piece $A_i = \{I_{i,1}, I_{i,2}, \ldots, I_{i,|A_i|}\}$ to player $i \in N$ can be written as follows: for each player $i \in N$ and each $j \in N$,

$$V_i(A_i) = \sum_{k=1}^{|A_i|} V_i(I_{i,k}) = \sum_{k=1}^{|A_i|} len(I_{i,k} \cap C_i) \geq V_i(A_j) = \sum_{k=1}^{|A_j|} V_i(I_{j,k}) = \sum_{k=1}^{|A_j|} len(I_{j,k} \cap C_i).$$

For a set of valuation intervals $T \subseteq \mathfrak{T}$, let $\mathrm{DOM}(T)$ be the minimal interval that includes all members of T as sub-intervals. Thus, for a set $T \subseteq \mathfrak{T}$,

$$\mathrm{DOM}(T) = (\min_{C_j \in T} \alpha_j, \max_{C_i \in T} \beta_i].$$

Furthermore, the *density* of T, denoted by $\Phi(T)$, is defined by

$$\Phi(T) = \frac{\lambda(T)}{|T|},$$

where $\lambda(T)$ is the total length of $\mathrm{DOM}(T)$ that is covered by at least one interval in T. Thus, $\lambda(T) \leq len(\mathrm{DOM}(T))$ holds. A set of valuation intervals $T \subseteq \mathfrak{T}$ is called *solid*, if for every point $x \in \mathrm{DOM}(T)$, there exists a valuation interval C_i in T such that $x \in C_i$. They assume that every piece of the cake is valuable for at least one player. Thus they assume that the valuation intervals $\mathfrak{T} = \{C_1, C_2, \ldots, C_n\}$ is solid, i.e.,

$$\bigcup_{i \in N} C_i = C.$$

This implies that $\mathrm{DOM}(\mathfrak{T}) = C$ and $\lambda(\mathfrak{T}) = len(\mathrm{DOM}(\mathfrak{T})) = 1$.

2.1 Expansion process

The main tool in their mechanisms in [1, 6] for dividing the cake is a procedure they call *expansion process*. The expansion process expands some associated intervals to the players, inside their desired areas (i.e., valuation intervals). They use $exp(T)$ to refer to the expansion process on set $T \subseteq \mathfrak{T}$ of valuation intervals. They initiate the expansion process for T by associating a zero-length interval $I_i = (a_i, b_i]$ at the beginning of its corresponding valuation interval $C_i = (\alpha_i, \beta_i] \in T$, i.e., $I_i = (a_i = \alpha_i, b_i = \alpha_i]$. Denote by $S(T) = \{I_i \mid C_i \in T\}$, the set of these intervals. They expand the intervals in $S(T)$ concurrently, all from the endpoint. The expansion is performed in a way that maintains two invariants:

(i) The expansion has the same speed for all the intervals in $S(T)$ so as the lengths of the intervals in $S(T)$ remain the same, and

(ii) each $I_i = (a_i, b_i] \in S(T)$ always remains within $C_i = (\alpha_i, \beta_i] \in T$.

During the expansion, the right endpoint b_i of an interval $I_i = (a_i, b_i] \in S(T)$ may collide with the starting point a_j of another interval $I_j = (a_j, b_j] \in S(T)$. In this case, I_i pushes the starting point of I_j forward during the expansion. The push continues to the end of the process. If I_i pushes I_j, they say I_i is *stuck* in I_j. Note that by the way they initiate the process, the intervals $I_i = (a_i, b_i] \in S(T)$ remain sorted according to the corresponding $\alpha_i^{\prime}s$. In the special case of equal α_i for two players, the one with smaller β_i comes first.

Definition 2.1 (Definition 1 in [1, 6]) During the expansion, an interval $I_i = (a_i, b_i]$ in $S(T)$ becomes *locked*, if the endpoint b_i of I_i reaches the right endpoint β_i of $C_i = (\alpha_i, \beta_i]$.

Definition 2.2 (Definition 2 in [1, 6]) A *chain* is a sequence $I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ of intervals in $S(T)$, with the property that, for $1 \le i < k$, I_{σ_i} is stuck in $I_{\sigma_{i+1}}$. A chain $I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ is *locked*, if I_{σ_k} is locked.

The size of a chain is the number of intervals in that chain. By definition, a single interval is a chain of size 1. The expansion ends when an interval in $S(T)$ becomes locked. The termination condition ensures that the second invariant is always preserved.

Definition 2.3 (Definition 3 in [1, 6]) The expansion process for T is *perfect*, if the associated intervals in $S(T)$ cover the entire DOM(T). If the process terminates due to a locked interval before entirely covering DOM(T), the process is *imperfect*.

Note that if an expansion process on T ends perfectly, then $len(I_i) = \Phi(T)$ for every associated interval I_i in $S(T)$.

Observation 2.1 (Observation 1 in [1], Observation 31 in [6]) During the expansion process, every interval I_i in $S(T)$ is either being pushed by another interval in $S(T)$, or its starting point is still on α_i.

2.2 Expansion Process with Unlocking

They introduce a more general form of the expansion process. The basic idea is the fact that during the expansion process, there might be some cases that a locked chain becomes unlocked by re-permuting some of its intervals, without violating the expansion invariants.

Definition 2.4 (Definition 4 in [1, 6]) Let $\mathcal{C} = I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ be a maximal locked chain. A permutation $I_{\delta_1}, I_{\delta_2}, \ldots, I_{\delta_r}$ of the intervals in $\mathcal{C}$ is said to be $\mathcal{C}$-*unlocking*, if the following conditions hold.

(i) All the intervals of the permutation are members of the locked chain, i.e., $I_{\delta_i} \in \mathcal{C}$ for all $i = 1, 2, \ldots, r$, and the last interval I_{δ_r} of the permutation is the locked interval (i.e., $\delta_r = \sigma_k$).

(ii) For every $j < r$, the share $I_{\delta_j} = (a_{\delta_j}, b_{\delta_j}]$ associated to player δ_j is totally within the valuation interval $C_{\delta_{j+1}} = (\alpha_{\delta_{j+1}}, \beta_{\delta_{j+1}}]$ of player δ_{j+1} (with its right endpoint b_{δ_j} of $I_{\delta_j} = (a_{\delta_j}, b_{\delta_j}]$ strictly less than the right endpoint $\beta_{\delta_{j+1}}$ of the valuation interval $C_{\delta_{j+1}} = (\alpha_{\delta_{j+1}}, \beta_{\delta_{j+1}}]$), i.e., $a_{\delta_j} \ge \alpha_{\delta_{j+1}}$ and $b_{\delta_j} < \beta_{\delta_{j+1}}$ for all j with $1 \le j \le r-1$.

(iii) The share $I_{\delta_r} = (a_{\delta_r}, b_{\delta_r}]$ associated to player δ_r is within the valuation interval $C_{\delta_1} = (\alpha_{\delta_1}, \beta_{\delta_1}]$ of player δ_1 (with its right endpoint b_{δ_r} of $I_{\delta_r} = (a_{\delta_r}, b_{\delta_r}]$ strictly less than the right endpoint β_{δ_1} of the valuation interval $C_{\delta_1} = (\alpha_{\delta_1}, \beta_{\delta_1}]$), i.e., $\alpha_{\delta_1} \le a_{\delta_r}$ and $\beta_{\delta_1} > b_{\delta_r}$.

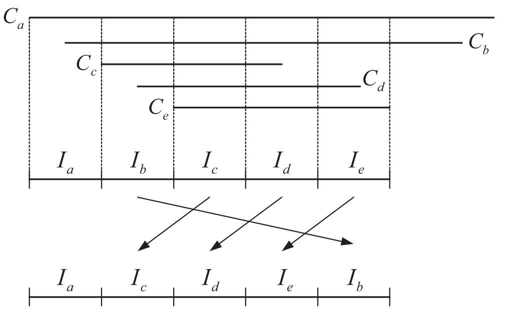

Figure 2: From a $\mathcal{C}$-unlocking permutation I_b, I_c, I_d, I_e in a maximal locked chain $\mathcal{C} = I_a, I_b, I_c, I_d, I_e$, we can obtain a chain $\mathcal{C}' = I_a, I_c, I_d, I_e, I_b$ which is no longer locked. From another $\mathcal{C}$-unlocking permutation I_b, I_c, I_e, we can also obtain a chain $\mathcal{C}'' = I_a, I_c, I_e, I_d, I_b$ which is no longer locked.

The intuition behind the definition of unlocking permutation is as follows:

Let $I_{\delta_1}, I_{\delta_2}, \ldots, I_{\delta_r}$ be a $\mathcal{C}$-unlocking permutation, where $\mathcal{C} = I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$. Then, the order of the intervals in $\mathcal{C}$ can be changed by placing $I_{\delta_{j+1}}$ in the location I_{δ_j} for each j with $1 \le j < r$ and placing I_{δ_1} in the location I_{δ_r}. By the definition of unlocking permutation, after such operations, $I_{\delta_r} = I_{\sigma_k}$ is no longer locked. Thus, I_{σ_k} is not a barrier for the expansion process and the expansion can be continued.

It is worthwhile to mention that there may be multiple locked intervals in a moment. In such case, they separately try to unlock each interval. For a set T of valuation intervals, they use *U-exp*(T) to refer to the expansion process with unlocking for T. See Figure 2 for an example of this process.

Definition 2.5 (Definition 5 in [1, 6]) A maximal locked chain $\mathcal{C} = I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ is *strongly locked*, if $\mathcal{C}$ admits no unlocking permutation.

Definition 2.6 (Definition 6 in [1, 6]) An expansion process with unlocking *U-exp*$(\cdot)$ is *strongly locked*, if at least one of its maximal locked chains is strongly locked.

Definition 2.7 (Definition 7 in [1, 6]) A *permutation graph* for a maximal locked chain $\mathcal{C} = I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ is a directed graph $G_{\mathcal{C}}(V, E)$ defined as follows: For every interval $I_{\sigma_i} \in \mathcal{C}$, there is a vertex v_{σ_i} in V. The edges in E are in two types E_l and E_r, i.e., $E = E_l \cup E_r$. The edges in E_l and E_r are determined as follows:

(i) For each I_{σ_i} and I_{σ_j}, the edge $(v_{\sigma_i}, v_{\sigma_j})$ is in E_l, if $i > j$ and $\alpha_{\sigma_i} \le a_{\sigma_j}$.

(ii) For each I_{σ_i} and I_{σ_j}, the edge $(v_{\sigma_i}, v_{\sigma_j})$ is in E_r, if $i < j$ and $b_{\sigma_j} < \beta_{\sigma_i}$.

An example of permutation graph $G_{\mathcal{C}}(V, E)$ is shown in Figure 3. Note that, if there is an edge $(v_{\sigma_i}, v_{\sigma_j})$ in E_l, then I_{σ_i} can be moved to the place where I_{σ_j} is, since $I_{\sigma_j} = (a_{\sigma_j}, b_{\sigma_j}] \subseteq C_{\sigma_i} = (\alpha_{\sigma_i}, \beta_{\sigma_i}]$ (i.e., $\alpha_{\sigma_i} \le a_{\sigma_j} < b_{\sigma_j} \le a_{\sigma_i} < b_{\sigma_i} \le \beta_{\sigma_i}$). Similarly, if there is an edge $(v_{\sigma_i}, v_{\sigma_j})$ in E_r, then I_{σ_i} can be moved to the place where I_{σ_j} is, since $I_{\sigma_j} = (a_{\sigma_j}, b_{\sigma_j}] \subset C_{\sigma_i} = (\alpha_{\sigma_i}, \beta_{\sigma_i}]$ and $b_{\sigma_j} < \beta_{\sigma_i}$ (i.e., $\alpha_{\sigma_i} \le a_{\sigma_i} < b_{\sigma_i} \le a_{\sigma_j} < b_{\sigma_j} < \beta_{\sigma_i}$).

A trivial necessary and sufficient condition for a maximal locked chain $\mathcal{C}$ to be strongly locked is that $G_{\mathcal{C}}$ contains no cycle including v_{σ_k}. Thus, $\mathcal{C} = I_1, I_2, I_3, I_4$ in Figure 3(a) is a strongly locked chain.

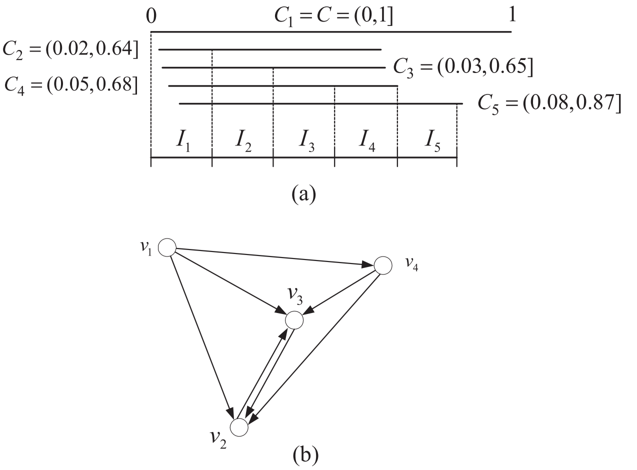

Figure 3: (a) A maximal locked chain $\mathcal{C} = I_1, I_2, I_3, I_4$ of chain I_1, I_2, I_3, I_4, I_5. (b) Permutation graph $G_{\mathcal{C}}(V, E)$.

2.3 Description of their mechanism

Their mechanism for finding a proper allocation is based on the expansion process with unlocking. Generally speaking, they iteratively run *U-exp*$(\cdot)$ on the remaining players' shares. This process allocates the entire cake or stops in a strongly locked situation. They prove some desirable properties for this situation and leverage these properties to allocate a piece of the cake to the players in the strongly locked chain. Next, they remove the satisfied players and shrink the allocated piece (as defined in Definition 2.8 below) and solve the problem recursively for the remaining players and the remaining part of the cake.

Lemma 2.1 (Lemma 4 in [1, 6]) Assume *U-exp*$(\cdot)$ stops in a strongly locked situation. Let $\mathcal{C} = I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ be a maximal strongly locked chain and let $G_{\mathcal{C}}(V, E)$ be the permutation graph of the chain $\mathcal{C}$. Let ℓ be the minimum index such that there is a directed path from v_{σ_k} to v_{σ_ℓ} using only edges in E_l. Then there is a directed path from v_{σ_k} to every vertex $v_{\sigma_{\ell'}}$ with $\ell' > \ell$ using only edges in E_l.

(In a strongly locked chain $\mathcal{C} = I_1, I_2, I_3, I_4$ in Figure 3, $v_{\sigma_k} = v_4$ and $\sigma_\ell = 2$.)

Definition 2.8 (Definition 9 in [1, 6]) Let C be a cake and $I = (I_s, I_e] \subset C$ be an interval. By the term *shrinking* of I, they mean removing I from C and gluing the pieces to the left and right of I together. More formally, every valuation interval $(\alpha_i, \beta_i]$ turns into $(f(\alpha_i), f(\beta_i)]$ by shrinking of I, where

$$f(x) = \begin{cases} x & (x < I_s) \\ I_s & (I_s \le x \le I_e) \\ x - I_e + I_s & (x > I_e). \end{cases} \tag{1}$$

Definition 2.9 (defined in Lemma 5 in [1, 6]) Let T be a set of valuation intervals. Then T is called *irreducible* if $\Phi(T') > \Phi(T)$ holds for every $T' \subset T$.

Lemma 2.2 (Lemma 5 in [1, 6]) Let T be an irreducible set of valuation intervals. Then DOM(T) can be divided into at most $2|T| - 1$ intervals each of which is associated to a valuation interval in T such that:

(i) The total length of the intervals associated to any valuation interval in T is exactly $\Phi(T)$.

(ii) The intervals associated to any valuation interval in T are totally within that valuation interval.

They proved this lemma by induction on $|T|$ using the following lemmas.

Lemma 2.3 (Lemma 6 in [1, 6]) Let $\mathcal{C} = I_{\sigma_1}, I_{\sigma_2}, \ldots, I_{\sigma_k}$ be a maximal strongly locked chain after running *U-exp*(T) and let ℓ be the minimum index such that there is a directed path from v_{σ_k} to v_{σ_ℓ} in the permutation graph $G_{\mathcal{C}}$ for $\mathcal{C}$ using only edges in E_l. Then, $\ell > 1$ holds.

Lemma 2.4 (Lemma 7 in [1, 6]) Let

$$x = \beta_{\sigma_k} - (k - \ell + 1)\Phi(T), \tag{2}$$

where ℓ is the minimum index such that there is a directed path from v_{σ_k} to v_{σ_ℓ}. Then, $a_{\sigma_{\ell-1}} < x < a_{\sigma_\ell}$ holds.

By this lemma, they break DOM(T) into two pieces both of which, they claim, preserve the properties defined in Lemma 2.2. More specifically, they claim, the piece of cake $(x, \beta_{\sigma_k}]$ can be allocated to players $\sigma_\ell, \sigma_{\ell+1}, \ldots, \sigma_k$ using $2(k - \ell + 1) - 2$ cuts. For this, they consider the valuation intervals $T' = \{C'_{\sigma_\ell}, C'_{\sigma_{\ell+1}}, \ldots, C'_{\sigma_k}\}$ such that:

$$C'_{\sigma_i} = (\max\{x, \alpha_{\sigma_i}\}, \beta_{\sigma_i}]$$

for all i with $\ell \leq i \leq k$. Note that DOM(T') $= (x, \beta_{\sigma_k}]$ and hence,

$$\Phi(T') = \frac{\beta_{\sigma_k} - x}{k - \ell + 1} = \frac{b_{\sigma_k} - x}{k - \ell + 1}. \tag{3}$$

Regarding Equation (2), $\Phi(T') = \Phi(T)$.

Lemma 2.5 (Lemma 8 in [1, 6]) T' is irreducible, i.e., for all $T'' \subset T'$, $\Phi(T'') > \Phi(T')$ hold.

Lemma 2.5 above states that the set of intervals in T' admits the properties described in Lemma 2.2. Furthermore, regarding Lemma 2.3, T' is a subset of T. By induction hypothesis, they know that one can cut DOM(T') into at most $2(k - \ell + 1) - 2$ disjoint intervals and allocate them to players $\sigma_\ell, \sigma_{\ell+1}, \ldots, \sigma_k$ such that both the properties in Lemma 2.2 are satisfied. Denote by N_T, the players with valuation intervals in T. Regarding Equation (2), Lemma 2.6 assures that the conditions in Lemma 2.2 hold for the remaining cake and the remaining players.

Lemma 2.6 (Lemma 9 in [1, 6]) Let T'' be the intervals related to the players in $N_{T''} = N_T \setminus \{\sigma_\ell, \sigma_{\ell+1}, \ldots, \sigma_k\}$ after shrinking of DOM(T'). Then T'' is irreducible and $\Phi(T'') = \Phi(T)$.

Based on Lemma 2.2, they introduce EFGISM as follows: among all subsets of $\mathcal{T}$, they find a subset T of minimum density (and the set with minimum size, if there were multiple options). Let $N(T)$ be the set of players whose valuation intervals are in T, i.e., $N(T) = \{i \in N \mid C_i \in T\}$. In Lemma 2.7 below, they show that T (and consequently $N(T)$) can be found in polynomial time.

Lemma 2.7 (Lemma 10 in [1, 6]) Let T be a subset of $\mathcal{T}$ of minimum density (and the set with minimum size, if there were multiple options). Then T can be found in polynomial time.

The mechanism EFGISM proposed in [1, 6] first finds such T. Since T has the minimum possible density, T is irreducible. Hence, EFGISM allocates to every player in $N(T)$, a piece from DOM(T) with the properties defined in Lemma 2.2. Afterwards, EFGISM removes the players in $N(T)$ from N and shrinks DOM(T) from C. Next, by recursively calling EFGISM, it allocates the remaining pieces of the cake to the remaining players.

Theorem 2.2 (Theorem 4 in [1], Theorem 3 in [6]) EFGISM is envy-free, truthful, and uses at most $2(n-1)$ cuts.

2.4 Counter Example of Lemma 2.6

Consider the input example in Figure 1 (also in Figure 3(a)) of the cake-cutting problem. Thus, the valuation intervals $\mathcal{T} = \{C_1, C_2, C_3, C_4, C_5\}$ are

$$C_1 = (0, 1],\ C_2 = (0.02, 0.64],\ C_3 = (0.03, 0.65],\ C_4 = (0.05, 0.68],\ C_5 = (0.08, 0.87].$$

It is easy to see that the valuation intervals $\mathcal{T} = \{C_1, C_2, C_3, C_4, C_5\}$ is irreducible by Definition 2.9 (defined in Lemma 5 in [1, 6]), since $\Phi(T') > \Phi(\mathcal{T}) = 0.2$ holds for every $T' \subset \mathcal{T}$. For this set $\mathcal{T}$ of valuation intervals, if the expansion process with unlocking for $\mathcal{T}$ (i.e., *U-exp*($\mathcal{T}$)) is applied, then a maximal locked chain $\mathcal{C} = I_1, I_2, I_3, I_4$ of chain I_1, I_2, I_3, I_4, I_5 in Figure 3(a) is obtained, where

$$I_1 = (0, 0.17],\ I_2 = (0.17, 0.34],\ I_3 = (0.34, 0.51],\ I_4 = (0.51, 0.68],\ I_5 = (0.68, 0.85].$$

It is easy to see that $\mathcal{C} = I_1, I_2, I_3, I_4$ is a maximal strongly locked chain, since the permutation graph $G_{\mathcal{C}}(V, E)$ in Figure 3(b) contains no cycle including v_4.

Thus, by Lemma 2.3 (Lemma 6 in [1, 6]), the minimum index ℓ such that there is a directed path from v_4 to v_ℓ in $G_{\mathcal{C}}(V, E)$ using only edges in E_l becomes $\ell = 2 > 1$. By Eq. (2) in Lemma 2.4 (Lemma 7 in [1, 6]) and $\Phi(\mathcal{T}) = 0.2$,

$$x = \beta_4 - (4 - \ell + 1)\Phi(\mathcal{T}) = 0.68 - 3 \times 0.2 = 0.08,$$

and $a_{\ell-1} = a_1 = 0 < x = 0.08 < a_\ell = a_2 = 0.17$ holds.

Then, we have $N_{T'} = \{2, 3, 4\}$, $T' = \{C'_2, C'_3, C'_4\}$ with

$$\begin{aligned} C'_2 &= (\max\{x, \alpha_2\}, \beta_2] = (0.08, 0.64], \\ C'_3 &= (\max\{x, \alpha_3\}, \beta_3] = (0.08, 0.65], \\ C'_4 &= (\max\{x, \alpha_4\}, \beta_4] = (0.08, 0.68] \end{aligned}$$

and DOM(T') $= \beta_4 - x = 0.68 - 0.08 = 0.6$ and

$$\Phi(T') = \frac{\beta_4 - x}{4 - \ell + 1} = \frac{0.6}{3} = 0.2 = \Phi(\mathcal{T}).$$

Thus, by Lemma 2.5 (Lemma 8 in [1, 6]), $T' = \{C'_2, C'_3, C'_4\}$ is irreducible.

Thus, we apply the mechanism based on the expansion process with unlocking in [1, 6] for the valuation intervals $T' = \{C'_2, C'_3, C'_4\}$ and obtain

$$A_2 = (0.08, 0.28],\ A_3 = (0.28, 0.48],\ A_4 = (0.48, 0.68].$$

Then, in the mechanism based on the expansion process with unlocking in [1, 6], they apply shrinking of $\mathrm{DOM}(T') = (0.08.0.68]$ and removing of $N_{T'} = \{2, 3, 4\}$ and obtain the remaining players $N_{T''} = \{1, 5\}$ and the valuation intervals

$$T'' = \{C''_1 = (0, 0.08] \cup (0.68, 1],\ C''_5 = (0.68, 0.87]\}$$

where we consider $0.08 = 0.68$ since shrinking of $\mathrm{DOM}(T') = (0.08.0.68]$ is done

(in the form of Definition 2.8, the remaining cake is $C'' = (0, 0.4]$, and the valuation intervals $C''_1 = (0, 0.4]$, $C''_5 = (0.08, 0.27]$.)

Note that

$$\Phi(T'') = \frac{1 - 0.68 + 0.08 - 0}{2} = 0.2 = \Phi(\mathcal{T}) \quad (\Phi(T'') = \frac{0.4 - 0}{2} = 0.2 = \Phi(\mathcal{T})).$$

However, the valuation intervals

$$T'' = \{C''_1 = (0, 0.08] \cup (0.68, 1],\ C''_5 = (0.68, 0.87]\}$$

($T'' = \{C''_1 = (0, 0.4],\ C''_5 = (0.08, 0.27]\}$ in Definition 2.8) is not irreducible, since $C''_5 = (0.68, 0.87] \subset T''$ ($C''_5 = (0.08, 0.27] \subset T''$ in Definition 2.8) is of density

$$\Phi(C''_5) = 0.87 - 0.68 = 0.19 < 0.2 = \Phi(T'') = \Phi(\mathcal{T})$$

($\Phi(C''_5) = 0.27 - 0.08 = 0.19 < 0.2 = \Phi(T'') = \Phi(\mathcal{T})$ in Definition 2.8).

This implies that Lemma 2.6 (Lemma 9 in [1, 6]) does not hold. Thus, we cannot apply the mechanism based on the expansion process with unlocking in [1, 6] for the valuation intervals $T'' = \{C''_1 = (0, 0.08] \cup (0.68, 1], C''_5 = (0.68, 0.87]\}$ ($T'' = \{C''_1 = (0, 0.4],\ C''_5 = (0.08, 0.27]\}$ in Definition 2.8) recursively, since T'' is not irreducible.

If we insist on applying the mechanism based on the expansion process with unlocking in [1, 6] for the valuation intervals T'' which is not irreducible, then we obtain a strongly locked chain $I''_1 = (0, 0.08] \cup (0.68, 0.735]$, $I''_5 = (0.735, 0.87]$ (in the form of Definition 2.8, it is the strongly locked chain $I''_1 = (0, 0.135]$, $I''_5 = (0.135, 0.27]$ in the remaining cake $C'' = (0, 0.4]$ with $I''_1 = (0, 0.135] \subset C''_1 = (0, 0.4]$ and $I''_5 = (0.135, 0.27] \subset C''_5 = (0.08, 0.27]$). If we use $\Phi(T'') = 0.2$, then $x = \beta_5 - (5 - 5 + 1)\Phi(T'') = 0.87 - (0.68 - 0.08) - 0.2 = 0.07$ ($x = \beta_5 - (5 - 5 + 1)\Phi(T'') = 0.27 - 0.2 = 0.07$ in Definition 2.8) since shrinking of $\mathrm{DOM}(T') = (0.08.0.68]$ is done and we consider $0.68 = 0.08$. Thus, we have $A_5 = (0.07, 0.08] \cup (0.68, 0.87]$ ($A_5 = (0.07, 0.27]$ in Definition 2.8) and $A_1 = (0, 0.07] \cup (0.87, 1]$ ($A_1 = (0, 0.07] \cup (0.27, 0.4]$ in Definition 2.8).

In this division $D = \{A_1, A_2, A_3, A_4, A_5\}$ of the cake $C = (0, 1]$ with each piece A_i allocated to player $i = 1, 2, 3, 4, 5$ is not envy-free since, player 5 would envy the piece $A_2 = (0.08, 0.28]$ allocated to player 2, since the value of A_2 for player 5 is the length of $(0.08, 0.28] \cap C_5 = (0.08, 0.28]$ and is $0.2 = 0.28 - 0.08$, while the value of $A_5 = (0.07, 0.08] \cup (0.68, 0.87]$ allocated to player 5 is the length of $((0.07, 0.08] \cup (0.68, 0.87]) \cap C_5 = (0.68, 0.87]$ and is $0.19 = 0.87 - 0.68 < 0.2$. Note that $(0.07, 0.08]$ is not contained in $C_5 = (0.08, 0.87]$.

Since T'' contains valuation intervals $\{C_5''\}$ with minimum density $\Phi(\{C_5''\}) = 0.19$, if we apply the mechanism based on the expansion process with unlocking in [1, 6] for the valuation intervals $\{C_5''\}$ then we obtain $A_5 = (0.68, 0.87]$. Finally, if we apply the mechanism based on the expansion process with unlocking in [1, 6] for the valuation intervals $\{C_1''\}$ then we obtain $A_1 = (0, 0.08] \cup (0.87, 1]$.

In this division $D = \{A_1, A_2, A_3, A_4, A_5\}$ of the cake $C = (0, 1]$ with each piece A_i allocated to player $i = 1, 2, 3, 4, 5$ is not envy-free. Actually, player 5 would envy the piece $A_2 = (0.08, 0.28]$ allocated to player 2, since $(0.08, 0.28] \cap C_5 = (0.08, 0.28] = A_2$ and its value for player 5 is $0.2 = 0.28 - 0.08$, while the value of $A_5 = (0.68, 0.87]$ allocated to player 5 is the length of $(0.68, 0.87] \cap C_5 = (0.68, 0.87]$ and is $0.19 = 0.87 - 0.68 < 0.2$.

3 Notation in Our Mechnism

In this section, we give notation which will be used in the rest of this paper. It is almost the same as in the paper by Alijani, Farhadi, Ghodsi, Seddighin, and Tajik [1, 6].

We are given a divisible heterogeneous cake $C = (0, 1] = \{x \mid 0 < x \leq 1\}$, a set of n strategic players $N = \{1, 2, \ldots, n\}$ with valuation interval $C_i = (\alpha_i, \beta_i] = \{x \mid 0 \leq \alpha_i < x \leq \beta_i \leq 1\} \subseteq C$ of each player $i \in N$. We denote by $\mathcal{C}_N$ the set of valuation intervals of all the players N, i.e., $\mathcal{C}_N = \{C_i \mid i \in N\}$.

The valuation intervals $\mathcal{C}_N$ is called *solid*, if for every point $x \in C$, there is a valuation interval $C_i \in \mathcal{C}_N$ containing x. We assume that the valuation intervals $\mathcal{C}_N$ is solid in this paper, i.e.,

$$\bigcup_{C_i \in \mathcal{C}_N} C_i = C. \tag{4}$$

A *piece* of the cake C is a set of mutually disjoint intervals in C. Thus, $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ is a piece of C if and only if each A_{i_j} $(j = 1, 2, \ldots, k_i)$ is an interval of C and any two distinct A_{i_j} and $A_{i_{j'}}$ $(1 \leq j < j' \leq k_i)$ are disjoint (i.e., $A_{i_j} \cap A_{i_{j'}} = \emptyset$). For a piece $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ of C for each $i \in N$, let $A_i = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_{k_i}}$. A union A_i of mutual disjoint sets $A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}$ is called a *direct sum* of $A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}$ and is denoted by $A_i = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}}$.

Let $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ be a piece of the cake C for each $i \in N$ and let $A_i = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}}$. Then $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ is called an *allocation* of the cake C to n players N if any two distinct A_i and A_j $(1 \leq i < j \leq n)$ are disjoint and $\sum_{i \in N} A_i = A_i + A_2 + \cdots + A_n = C$. For each $i \in N$, $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ (also, $A_i = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}}$) is called an *allocated piece* of the cake C to player i in $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ (Figure 4).

For an interval $X = (x', x'']$ of C, the *size* of X, denoted by $\mathrm{csize}(X)$, is defined by $x'' - x'$. For a direct sum $X = X_1 + X_2 + \cdots + X_k$, i.e., a union of mutual disjoint intervals X_j $(j = 1, 2, \ldots, k)$ of C, the *size* of X, denoted by $\mathrm{csize}(X)$, is defined by the total sum of $\mathrm{csize}(X_j)$ (Figure 4). Thus, $\mathrm{csize}(X) = \mathrm{csize}(X_1) + \mathrm{csize}(X_2) + \cdots + \mathrm{csize}(X_k)$.

Let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ be a piece of C and let $X = X_1 + X_2 + \cdots + X_k$. For each $i \in N$ and valuation interval C_i of player i, the *utility* of $\mathcal{X}$ for player i, denoted by $\mathrm{ut}_i(\mathcal{X})$, is the total sum of $\mathrm{csize}(X_j \cap C_i)$ for all $X_j \in \mathcal{X}$, i.e.,

$$\mathrm{ut}_i(\mathcal{X}) = \mathrm{csize}(X_1 \cap C_i) + \mathrm{csize}(X_2 \cap C_i) + \cdots + \mathrm{csize}(X_k \cap C_i). \tag{5}$$

We sometimes use $\mathrm{ut}_i(X)$ in place of $\mathrm{ut}_i(\mathcal{X})$. Thus, $\mathrm{ut}_i(X) = \mathrm{csize}(X_1 \cap C_i) + \mathrm{csize}(X_2 \cap C_i) + \cdots + \mathrm{csize}(X_k \cap C_i)$.

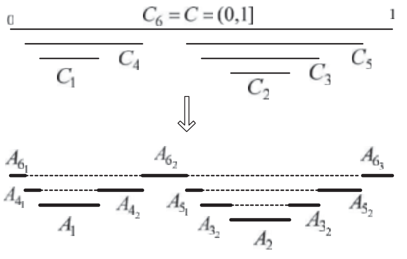

Figure 4: An allocated piece to player 5 is $\{A_{5_1}, A_{5_2}\}$ ($n = 6$). $\text{csize}(A_6) = \text{csize}(A_{6_1}) + \text{csize}(A_{6_2}) + \text{csize}(A_{6_3})$ for $A_6 = A_{6_1} + A_{6_2} + A_{6_3}$.

Let $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ be an allocation of the cake C to n players N and let $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ be an allocated piece of C to player $i \in N$. If

$$\text{ut}_i(\mathcal{A}_i) \geq \text{ut}_i(\mathcal{A}_j) \quad \text{for all } j \in N \setminus \{i\}, \tag{6}$$

then the allocated piece $\mathcal{A}_i$ is called *envy-free* for player i. If, for every player $i \in N$, the allocated piece $\mathcal{A}_i$ is envy-free for player i, then the allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ of the cake C to n players N is called *envy-free*.

Let $\mathcal{M}$ be a mechanism for the cake-cutting problem. For an arbitrary input $\mathcal{C}_N = \{C_1, C_2, \ldots, C_n\}$ to the mechanism $\mathcal{M}$, let $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ be an allocation of the cake C to n players N obtained by $\mathcal{M}$ with $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ for each $i \in N$. If the allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ for every input $\mathcal{C}_N = \{C_1, C_2, \ldots, C_n\}$ to the mechanism $\mathcal{M}$ is envy-free, then the mechanism $\mathcal{M}$ is called *envy-free*.

Now, for each player $i \in N$, assume that only player i makes a lie and gives a false valuation interval C'_i. Thus, let

$$\mathcal{C}'_N(i) = \{C_1, C_2, \ldots, C_{i-1}, C'_i, C_{i+1}, \ldots, C_n\} \tag{7}$$

be an input to the mechanism $\mathcal{M}$ and let an allocation of the cake C to n players N obtained by $\mathcal{M}$ be

$$\mathcal{A}'_N(i) = \{\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_{i-1}, \mathcal{A}'_i, \mathcal{A}'_{i+1}, \ldots, \mathcal{A}'_n\} \tag{8}$$

with $\mathcal{A}'_j = \{A'_{j_1}, A'_{j_2}, \ldots, A'_{j_{k'_j}}\}$ for each $j \in N$. The utilities of $\mathcal{A}_i$ and $\mathcal{A}'_i$ for player i are

$$\text{ut}_i(\mathcal{A}_i) = \sum_{j=1}^{k_i} \text{csize}(A_{i_j} \cap C_i), \quad \text{ut}_i(\mathcal{A}'_i) = \sum_{j=1}^{k'_i} \text{csize}(A'_{i_j} \cap C_i) \tag{9}$$

(note that $\text{ut}_i(\mathcal{A}'_i) \neq \sum_{j=1}^{k'_i} \text{csize}(A'_{i_j} \cap C'_i)$). If $\text{ut}_i(\mathcal{A}_i) \geq \text{ut}_i(\mathcal{A}'_i)$, then player i does not want to tell a lie and player i will report the true valuation interval C_i to the mechanism $\mathcal{M}$ (this implies that to report the true valuation interval C_i is a *dominant strategy* of player i). For each player $i \in N$, if this holds, then no player wants to tell a lie and all players want to report true valuation intervals to the mechanism $\mathcal{M}$. In this case, the mechanism $\mathcal{M}$ is called *truthful* (allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ of C to n players N obtained by $\mathcal{M}$ is also called *truthful*).

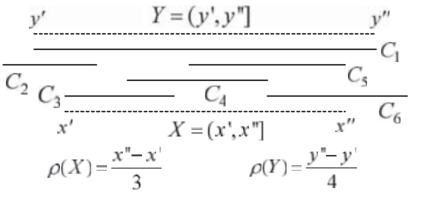

Figure 5: Example of 6 valuation intervals $C_1, C_2, \ldots, C_6$ (solid line). An interval $X = (x', x'']$ is of size $x'' - x'$ and three valuation intervals C_3, C_4, C_5 are contained in X (thus, $N(X) = \{3, 4, 5\}$). Thus, the density of the interval X is $\rho(X) = \frac{x''-x'}{3}$. The interval $(x', y'']$ is not a minimal interval with respect to density, since no valuation interval within $(x', y'']$ contains y'' as endpoint. However, the density of $(x', y'']$ is defined by $\rho((x', y'']) = \frac{y''-x'}{3}$.

For an interval $X = (x', x'']$ of C, let $N(X)$ be the set of players in N whose valuation intervals are entirely contained in X and let $\mathcal{C}_{N(X)}$ be the set of valuation intervals in $\mathcal{C}_N$ which are entirely contained in X. Let n_X be the cardinality of $N(X)$ ($\mathcal{C}_{N(X)}$) . Thus,

$$N(X) = \{i \in N \mid C_i \subseteq X, C_i \in \mathcal{C}_N\}, \tag{10}$$
$$\mathcal{C}_{N(X)} = \{C_i \in \mathcal{C}_N \mid i \in N(X)\}, \tag{11}$$
$$n_X = |N(X)| = |\mathcal{C}_{N(X)}|. \tag{12}$$

As we defined the solidness of the valuation intervals $\mathcal{C}_N$ in the cake C in Eq.(4), we define the solidness of the valuation intervals $\mathcal{C}_{N(X)}$. For an interval $X = (x', x'']$ of C, the valuation intervals $\mathcal{C}_{N(X)}$ is called *solid*, if for every point $x \in X$, there is a valuation interval $C_i \in \mathcal{C}_{N(X)}$ containing x, i.e.,

$$\bigcup_{C_i \in \mathcal{C}_{N(X)}} C_i = X. \tag{13}$$

Definition 3.1 For an interval $X = (x', x'']$ of C, the *density* of the interval X, denoted by $\rho(X)$, is defined by

$$\rho(X) = \frac{\mathrm{csize}(X)}{|\mathcal{C}_{N(X)}|} = \frac{x'' - x'}{n_X}. \tag{14}$$

Definition 3.2 For an interval $X = (x', x'']$ of C, if there are valuation intervals $C_i = (\alpha_i, \beta_i]$ and $C_j = (\alpha_j, \beta_j]$ in $\mathcal{C}_{N(X)} = \{C_k \in \mathcal{C}_N \mid k \in N(X)\}$ such that $x' = \alpha_i$ and $x'' = \beta_j$, then $X = (x', x'']$ is called a *minimal interval with respect to density.*

Figure 5 shows densities of some intervals. Note that, if $X \neq \emptyset$ (i.e., $\mathrm{csize}(X) \neq 0$) and $n_X = 0$ then $\rho(X) = \infty$. Note also that, there are at most n^2 minimal intervals $X = (x', x'']$ with respect to density, since x' is a left endpoint of a valuation interval, x'' is a right endpoint of a valuation interval and there are n valuation intervals.

Let $\mathcal{X}$ be the set of all nonempty intervals in C. Let $\rho_{\min}$ be the minimum density among the densities of all nonempty intervals in C, i.e.,

$$\rho_{\min} = \min_{X \in \mathcal{X}} \rho(X). \tag{15}$$

Let $\mathcal{X}_{\min}$ be the set of all intervals in C of minimum density, i.e.,

$$\mathcal{X}_{\min} = \{X \in \mathcal{X} \mid \rho(X) = \rho_{\min}\}. \tag{16}$$

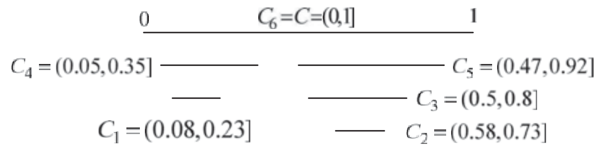

Figure 6: Example of the valuation intervals $C_1, C_2, \ldots, C_6$. The minimum density is $\rho_{\min} = 0.15$ and the intervals of minimum density are C_1, C_2, C_3, C_4, C_5. Among them, C_1 and C_2 are the minimal intervals of minimum density and C_4 and C_5 are the maximal intervals of minimum density and interval C_6 is of density $\rho(C_6) = \frac{1}{6} = 0.1666...$

Definition 3.3 An interval $X \in \mathcal{X}_{\min}$ is called an *interval of minimum density.* An interval X of minimum density is called a *minimal interval of minimum density* if X contains no other interval of minimum density properly. Similarly, an interval X of minimum density is called a *maximal interval of minimum density* if no other interval of minimum density contains X properly (Figure 6).

Lemma 3.1 Let $X = (x', x'']$ be a minimal interval with respect to density in the cake C. Suppose that $\rho(Y) \geq \rho(X)$ holds for each nonempty interval $Y = (y', y'']$ properly contained in X (i.e., $\emptyset \neq Y \subset X$). Then $\mathcal{C}_{N(X)}$ is solid (i.e., $\bigcup_{C_i \in \mathcal{C}_{N(X)}} C_i = X$).

Proof: We show that $\mathcal{C}_{N(X)}$ is solid, i.e., for each point x in $X = (x', x'']$ ($x' < x \leq x''$), there is a valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ containing x.

Suppose contrarily that no valuation interval in $\mathcal{C}_{N(X)}$ contains x. Thus, each valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ satisfies $\beta_i < x$ or $x \leq \alpha_i$. Since $X = (x', x'']$ is a mininal interval with respect to density, there are valuation intervals $C_j = (\alpha_j, \beta_j]$ and $C_k = (\alpha_k, \beta_k]$ in $\mathcal{C}_{N(X)}$ with $\alpha_j = x' < x$ and $\beta_k = x'' \geq x$. Thus, we have $\alpha_j < \beta_j < x$ and $x \leq \alpha_k < \beta_k$. Let y be the largest right endpoint among valuation intervals in $\mathcal{C}_{N(X)}$ whose right endpoints are smaller than x. Similarly, let z be the smallest left endpoint among valuation intervals in $\mathcal{C}_{N(X)}$ whose left endpoints are larger than or equal to x. Thus $\epsilon = x - y > 0$ and $\delta = z - x \geq 0$. Let $Y = (x', y]$ and $Z = (z, x'']$. Then $C_j = (\alpha_j, \beta_j] \subseteq Y \subset X$, $n_Y > 0$, $C_k = (\alpha_k, \beta_k] \subseteq Z \subset X$, $n_Z > 0$, and $Y \cap Z = \emptyset$. Thus, each valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ satisfies $\beta_i \leq y < x$ or $x \leq z \leq \alpha_i$ i.e., each valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ is either in $\mathcal{C}_{N(Y)}$ or in $\mathcal{C}_{N(Z)}$. Therefore, $\mathcal{C}_{N(X)} = \mathcal{C}_{N(Y)} + \mathcal{C}_{N(Z)}$ and $n_X = n_Y + n_Z$. Since $\mathrm{csize}(Y) = \rho(Y) n_Y$, $\rho(Y) \geq \rho(X)$, $\mathrm{csize}(Z) = \rho(Z) n_Z$, and $\rho(Z) \geq \rho(X)$, we have

$$\begin{aligned}
\rho(X) &= \frac{\mathrm{csize}(X)}{n_X} = \frac{x'' - x'}{n_X} = \frac{x'' - z + z - x + x - y + y - x'}{n_Z + n_Y} \\
&= \frac{\mathrm{csize}(Z) + \delta + \epsilon + \mathrm{csize}(Y)}{n_Z + n_Y} = \frac{\rho(Z) n_Z + \rho(Y) n_Y + \delta + \epsilon}{n_Z + n_Y} \\
&> \frac{\rho(Z) n_Z + \rho(Y) n_Y}{n_Z + n_Y} \geq \frac{\rho(X) n_Z + \rho(X) n_Y}{n_Z + n_Y} = \rho(X),
\end{aligned}$$

a contradiction. Thus, we have $\bigcup_{C_i \in \mathcal{C}_{N(X)}} C_i = X$, i.e., $\mathcal{C}_{N(X)}$ is solid. □

For an interval $X = (x', x'']$ of minimum density, we have the following lemma using Lemma 3.1. It is almost clear, so we omit a proof.

Lemma 3.2 Let $X = (x', x'']$ be an interval of minimum density $\rho_{\min}$. Then it is a minimal interval with respect to density and the valuation intervals $\mathcal{C}_{N(X)}$ is solid.

4 Structures of Intervals of Minimum Density

In this paper, we will give a mechanism in later section, for a given input of a cake $C = (0, 1]$, a set of n players $N = \{1, 2, \ldots, n\}$, and solid valuation intervals $\mathcal{C}_N = \{C_i \mid i \in N\}$ with valuation interval $C_i = (\alpha_i, \beta_i]$ of each player $i \in N$ and $\cup_{C_i \in \mathcal{C}_N} C_i = C$, which finds an allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ to players N with $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ $(A_i = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}})$ for each player $i \in N$ satisfying the following.

(a) The mechanism is envy-free.

(b) The mechanism is truthful.

(c) $A_i \subseteq C_i$ for each $i \in N$.

(d) $\sum_{i \in N} A_i = C$.

In this section, we discuss structures of intervals of minimum density which play a central role in our mechanism.

Lemma 4.1 Let $X_j = (x'_j, x''_j]$ be a minimal interval in the cake C with respect to density. Let $X_i = (x'_i, x''_i]$ be another minimal interval in C with respect to density such that $X_i \cap X_j \neq \emptyset$. If $\rho(X_i) \geq \rho(X_j)$ and $\rho(X_i \cap X_j) \geq \rho(X_j)$, then $\rho(X_i \cup X_j) \leq \rho(X_i)$.

Proof: If $X_i \setminus X_j = \emptyset$, then $X_i \cup X_j = X_j$ and $\rho(X_i \cup X_j) = \rho(X_j) \leq \rho(X_i)$ holds. Similarly, if $X_j \setminus X_i = \emptyset$, then $X_i \cup X_j = X_i$ and $\rho(X_i \cup X_j) = \rho(X_i) \leq \rho(X_i)$ holds.

Thus, we assume $X_i \setminus X_j \neq \emptyset$ and $X_j \setminus X_i \neq \emptyset$ below. By symmetry we can assume $x'_i < x'_j < x''_i < x''_j$ since $X_i \cap X_j \neq \emptyset$, $X_i \setminus X_j \neq \emptyset$, and $X_j \setminus X_i \neq \emptyset$ (Figure 7). Let

$$Y = X_i \cap X_j = (y', y''], \quad Z = X_i \cup X_j = (z', z''].$$

Thus, $y' = x'_j, \;\; y'' = x''_i, \;\; z' = x'_i, \;\; z'' = x''_j$. By Definition 3.1,

$$N(X_i) = \{k \in N \mid C_k \subseteq X_i, C_k \in \mathcal{C}_N\}, \;\; \mathcal{C}_{N(X_i)} = \{C_k \in \mathcal{C}_N \mid k \in N(X_i)\},$$

$$N(X_j) = \{k \in N \mid C_k \subseteq X_j, C_k \in \mathcal{C}_N\}, \;\; \mathcal{C}_{N(X_j)} = \{C_k \in \mathcal{C}_N \mid k \in N(X_j)\},$$

$$N(Y) = \{k \in N \mid C_k \subseteq Y, C_k \in \mathcal{C}_N\}, \;\; \mathcal{C}_{N(Y)} = \{C_k \in \mathcal{C}_N \mid k \in N(Y)\},$$

$$N(Z) = \{k \in N \mid C_k \subseteq Z, C_k \in \mathcal{C}_N\}, \;\; \mathcal{C}_{N(Z)} = \{C_k \in \mathcal{C}_N \mid k \in N(Z)\},$$

$$\mathcal{C}_{N(Y)} = \mathcal{C}_{N(X_i)} \cap \mathcal{C}_{N(X_j)}.$$

Note that, $Z = X_i \cup X_j = (z', z'']$ is a minimal interval with respect to density, but $Y = X_i \cap X_j = (y', y'']$ may not be a minimal interval with respect to density, since $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ are two distinct minimal intervals in C with respect to density and a valuation interval $C_k = (\alpha_k, \beta_k] \in \mathcal{C}_{N(X_i)} \cup \mathcal{C}_{N(X_j)}$ with $y' = x'_j = \alpha_k$ (or with $y'' = x''_i = \beta_k$) may not be contained in Y if $\beta_k > y''$ (or if $\alpha_k < y'$). Furthermore, since a valuation interval $C_k = (\alpha_k, \beta_k] \in \mathcal{C}_{N(X_i)}$ with $x'_i = \alpha_k < x'_j$ is not contained in $\mathcal{C}_{N(X_j)}$ (and is not in $\mathcal{C}_{N(Y)}$) and a valuation interval $C_k = (\alpha_k, \beta_k] \in \mathcal{C}_{N(X_j)}$ with $x''_j = \beta_k > x''_i$ is not contained in $\mathcal{C}_{N(X_i)}$ (and is not $\mathcal{C}_{N(Y)}$), we have $\mathcal{C}_{N(Y)} \subset \mathcal{C}_{N(X_i)} \subset \mathcal{C}_{N(Z)}$ and

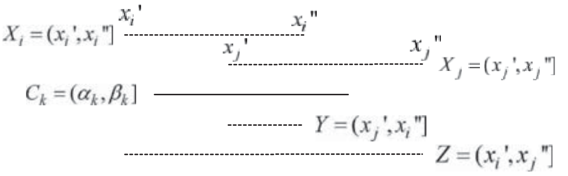

Figure 7: Two intervals $X_i = (x_i', x_i'']$ and $X_j = (x_j', x_j'']$ in Proof of Lemma 4.1. A valuation interval $C_k = (\alpha_k, \beta_k]$ is not in X_i nor X_j, but in $Z = X_i \cup X_j$.

$\mathcal{C}_{N(Y)} \subset \mathcal{C}_{N(X_j)} \subset \mathcal{C}_{N(Z)}$. Thus, $n_Z = |N(Z)| > n_{X_i} = |N(X_i)| > n_Y = |N(Y)| \geq 0$ and $n_Z = |N(Z)| > n_{X_j} = |N(X_j)| > n_Y = |N(Y)| \geq 0$. Let

$$\mathcal{C}_W = \mathcal{C}_{N(Z)} \setminus (\mathcal{C}_{N(X_i)} \cup \mathcal{C}_{N(X_j)}), \quad n_W = |\mathcal{C}_W|.$$

Note that a valuation interval $C_k = (\alpha_k, \beta_k] \in \mathcal{C}_N$ with $x_i' < \alpha_k < x_j'$ and $x_i'' < \beta_k < x_j''$ is in $\mathcal{C}_W = \mathcal{C}_{N(Z)} \setminus (\mathcal{C}_{N(X_i)} \cup \mathcal{C}_{N(X_j)})$ (Figure 7). Thus, by the inclusion-exclusion principle, we have

$$n_Z = n_{X_i} + n_{X_i} - n_Y + n_W$$

and the density $\rho(Z)$ of interval $Z = X_i \cup X_j$ is

$$\rho(Z) = \frac{\text{csize}(Z)}{n_Z} = \frac{x_j'' - x_i'}{n_{X_i} + n_{X_j} - n_Y + n_W} = \frac{x_j'' - x_j' + x_i'' - x_i' - (x_i'' - x_j')}{n_{X_i} + n_{X_j} - n_Y + n_W}.$$

(i) We first discuss the case of $n_Y > 0$. Since $n_W \geq 0$ and by the definition of density of an interval in Definition 3.1, we have

$$n_{X_i}\, \rho(X_i) = x_i'' - x_i', \quad n_{X_j}\, \rho(X_j) = x_j'' - x_j', \quad n_Y\, \rho(Y) = x_i'' - x_j', \text{ and}$$

$$\begin{aligned}
\rho(Z) &= \frac{x_j'' - x_j' + x_i'' - x_i' - (x_i'' - x_j')}{n_{X_i} + n_{X_j} - n_Y + n_W} \\
&\leq \frac{x_j'' - x_j' + x_i'' - x_i' - (x_i'' - x_j')}{n_{X_i} + n_{X_j} - n_Y} = \frac{n_{X_i}\, \rho(X_i) + n_{X_j}\, \rho(X_j) - n_Y\, \rho(Y)}{n_{X_i} + n_{X_j} - n_Y}.
\end{aligned}$$

Note that,

$$\rho(Z) = \frac{n_{X_i}\, \rho(X_i) + n_{X_j}\, \rho(X_j) - n_Y\, \rho(Y)}{n_{X_i} + n_{X_j} - n_Y}$$

if and only if $n_W = 0$. Since $\rho(Y) = \rho(X_i \cap X_j) \geq \rho(X_j)$, we have

$$\rho(Z) \leq \frac{n_{X_i}\, \rho(X_i) + n_{X_j}\, \rho(X_j) - n_Y\, \rho(Y)}{n_{X_i} + n_{X_j} - n_Y} \leq \frac{n_{X_i}\, \rho(X_i) + n_{X_j}\, \rho(X_j) - n_Y\, \rho(X_j)}{n_{X_i} + n_{X_j} - n_Y}.$$

Furthermore, since $\rho(X_i) \geq \rho(X_j)$ and $n_{X_j} > n_Y$, we have

$$\begin{aligned}
\rho(Z) &\leq \frac{n_{X_i}\, \rho(X_i) + n_{X_j}\, \rho(X_j) - n_Y\, \rho(X_j)}{n_{X_i} + n_{X_j} - n_Y} = \frac{n_{X_i}\, \rho(X_i) + (n_{X_j} - n_Y)\, \rho(X_j)}{n_{X_i} + n_{X_j} - n_Y} \\
&\leq \frac{(n_{X_i} + n_{X_j} - n_Y)\, \rho(X_i)}{n_{X_i} + n_{X_j} - n_Y} = \rho(X_i).
\end{aligned}$$

By the argument above, when $n_Y > 0$, we have $\rho(Z) = \rho(X_i)$ if and only if $n_W = 0$ and $\rho(Y) = \rho(X_j) = \rho(X_i)$.

(ii) We next discuss the case of $n_Y = 0$. In this case, we have

$$\begin{aligned}
\rho(Z) &= \frac{x''_j - x'_j + x''_i - x'_i - (x''_i - x'_j)}{n_{X_i} + n_{X_j} - n_Y + n_W} \\
&\leq \frac{x''_j - x'_j + x''_i - x'_i - (x''_i - x'_j)}{n_{X_i} + n_{X_j} - n_Y} = \frac{x''_j - x'_j + x''_i - x'_i - (x''_i - x'_j)}{n_{X_i} + n_{X_j}} \\
&< \frac{x''_j - x'_j + x''_i - x'_i}{n_{X_i} + n_{X_j}}
\end{aligned}$$

since $n_W \geq 0$ and $x''_i - x'_j > 0$ by $Y = X_i \cap X_j \neq \emptyset$. Thus, we have

$$\begin{aligned}
\rho(Z) &< \frac{x''_j - x'_j + x''_i - x'_i}{n_{X_i} + n_{X_j}} = \frac{n_{X_i}\,\rho(X_i) + n_{X_j}\,\rho(X_j)}{n_{X_i} + n_{X_j}} \\
&\leq \frac{(n_{X_i} + n_{X_j})\;\rho(X_i)}{n_{X_i} + n_{X_j}} = \rho(X_i)
\end{aligned}$$

since $\rho(X_i) \geq \rho(X_j)$.

Thus, when $n_Y = 0$, we have $\rho(Y) = \frac{x''_i - x'_j}{n_Y} = \infty$ and $\rho(Z) < \rho(X_i)$. □

By Lemma 4.1, the following corollaries can be easily obtained. We omit proofs.

Collorary 4.1 Let $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ be two distinct intervals in C of minimum density $\rho_{\min}$. If $X_i \cap X_j \neq \emptyset$ then both $Y = X_i \cap X_j$ and $Z = X_i \cup X_j$ are intervals of minimum density $\rho_{\min}$.

Collorary 4.2 If $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ are two distinct minimal intervals of minimum density $\rho_{\min}$, then $X_i \cap X_j = \emptyset$. Furthermore, if $X_i = (x'_i, x''_i]$ lies to the left of $X_j = (x'_j, x''_j]$ then $x''_i \leq x'_j$. In this case, if $x''_i = x'_j$ then $Z = X_i \cup X_j = (x'_i, x''_j]$ is an interval of minimum density and there is no valuation interval $C_k = (x'_k, x''_k] \in \mathcal{C}_N$ such that $x'_i \leq x'_k < x''_i = x'_j < x''_k \leq x''_j$.

Similarly, if $X_i = (x'_i, x''_i]$ and $X_j = (x'_j, x''_j]$ are two distinct maximal intervals of minimum density $\rho_{\min}$, then $X_i \cap X_j = \emptyset$. Furthermore, if $X_i = (x'_i, x''_i]$ lies to the left of $X_j = (x'_j, x''_j]$ then $x''_i < x'_j$.

5 Our Mechanism

We first give a brief outline of our mechanism.

Let $H_1 = (h'_1, h''_1]$, $H_2 = (h'_2, h''_2], \ldots,$ $H_L = (h'_L, h''_L]$ be the maximal intervals of minimum density $\rho_{\min}$ in the cake $C = (0, 1]$. We first cut $C = (0, 1]$ at both endpoints of each maximal interval H_i of minimum density $\rho_{\min}$. By Corollary 4.2, two distinct maximal intervals H_i, H_j of minimum density are disjoint and we can cut the cake at both endpoints of each maximal interval of minimum density, independently. By these cuts, we can reduce the original cake-cutting problem into two types of cake-cutting subproblems of type (i) and type (ii) as follows (Figure 8):

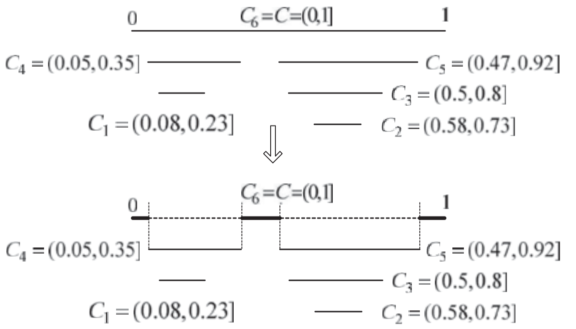

Figure 8: The cake-cutting problem can be reduced into two types of cake-cutting subproblems by cutting the cake $C = (0, 1]$ at both endpoints of each maximal interval of minimum density: (i) one within each maximal interval of minimum density (players $R_1 = \{1, 4\}$ and players $R_2 = \{2, 3, 5\}$]), and (ii) one with all valuations obtained by deleting all the valuation intervals contained in all maximal intervals of minimum density (players $P = \{6\}$).

(i) the cake-cutting problem within each maximal interval $H_i = (h'_i, h''_i]$ of minimum density (which consists of the cake H_i, the players $N(H_i)$ whose valuation intervals in H_i and valuations $\mathcal{C}_{N(H_i)}$); and

(ii) the cake-cutting problem with all valuations obtained by deleting all the valuation intervals contained in all the maximal intervals $H_1 = (h'_1, h''_1]$, $H_2 = (h'_2, h''_2], \ldots,$ $H_L = (h'_L, h''_L]$ of minimum density.

Note that the cake-cutting problem of type (i) is almost the same as the original cake-cutting problem, since the cake H_i is a single interval, each valuation $C_k \in \mathcal{C}_{N(H_i)}$ is also a single interval, and the valuation intervals $\mathcal{C}_{N(H_i)}$ is solid by Lemma 3.2 (i.e., $\bigcup_{C_k \in \mathcal{C}_{N(H_i)}} C_k = H_i$).

On the other hand, the cake-cutting problem of type (ii) is different from the original cake-cutting problem, because the resulting cake may become a set of two or more disjoint intervals and a resulting valuation may also become a set of two or more disjoint intervals. However, the cake-cutting problem of type (ii) has a nice property as described below.

Let $X_\ell = (x'_\ell, x''_\ell]$ be an interval of the cake C. Then, by cutting the cake at both endpoints of X_ℓ and deleting X_ℓ, we have the cake-cutting problem of type (ii) for the cake $C \setminus X_\ell$, players $N \setminus N(X_\ell)$ and valuations

$$\mathcal{C}_N(C \setminus X_\ell) = \{C_k \setminus X_\ell \mid \ C_k \in \mathcal{C}_N \setminus \mathcal{C}_{N(X_\ell)}\}. \tag{17}$$

Note that $C_k \setminus X_\ell \neq \emptyset$ for each $C_k \in \mathcal{C}_N \setminus \mathcal{C}_{N(X_\ell)}$, since $C_k \setminus X_\ell = \emptyset$ would imply $C_k \subseteq X_\ell$ and thus $C_k \in \mathcal{C}_{N(X_\ell)}$. Furthermore, for each point $x \in C \setminus X_\ell$, there is a valuation interval $C_k \in \mathcal{C}_N$ containing x by the solidness of the valuation intervals $\mathcal{C}_N$ (i.e., $\bigcup_{C_i \in \mathcal{C}_N} = C$ in Eq.(4)), and the valuation interval C_k is not contained in $\mathcal{C}_{N(X_\ell)}$ since C_k contains $x \in C \setminus X_\ell$. Thus, the valuation $C_k \setminus X_\ell \in \mathcal{C}_N(C \setminus X_\ell)$ contains x. This implies

$$\bigcup_{C_k \setminus X_\ell \in \mathcal{C}_N(C \setminus X_\ell)} (C_k \setminus X_\ell) \ = \ C \setminus X_\ell. \tag{18}$$

Note also that there are three types of valuation intervals $C_k = (\alpha_k, \beta_k] \in \mathcal{C}_N \setminus \mathcal{C}_{N(X_\ell)}$ according to the valuations $C_k \setminus X_\ell$.

(a) $C_k \cap X_\ell = \emptyset$, i.e., $\alpha_k < \beta_k \leq x'_\ell$ or $x''_\ell \leq \alpha_k < \beta_k$. In this type, $C_k \setminus X_\ell = C_k$ is a single interval.

(b) $\alpha_k < x'_\ell < x''_\ell < \beta_k$. In this type, $X_\ell \subset C_k$ and $C_k \setminus X_\ell$ is a set of two disjoint intervals, i.e., $C_k \setminus X_\ell = (\alpha_k, x'_\ell] + (x''_\ell, \beta_k]$.

(c) $\alpha_k < x'_\ell < \beta_k \leq x''_\ell$ (in this case $C_k \setminus X_\ell$ is a single interval $C_k \setminus X_\ell = (\alpha_k, x'_\ell]$) or $x'_\ell \leq \alpha_k < x''_\ell < \beta_k$ (in this case $C_k \setminus X_\ell$ is a single interval $C_k \setminus X_\ell = (x''_\ell, \beta_k]$).

Similarly, there are three types of intervals $X = (x', x'']$ in C such that $X \not\subseteq X_\ell$.

(a) $X \cap X_\ell = \emptyset$, i.e., $x' < x'' \leq x'_\ell$ or $x''_\ell \leq x' < x''$. In this type, $X \setminus X_\ell = X$ is a single interval.

(b) $x' < x'_\ell < x''_\ell < x''$. In this type, $X_\ell \subset X$ and $X \setminus X_\ell$ is a set of two disjoint intervals, i.e., $X \setminus X_\ell = (x', x'_\ell] + (x''_\ell, x'']$.

(c) $x' < x'_\ell < x'' \leq x''_\ell$ (in this case $X \setminus X_\ell$ is a single interval $X \setminus X_\ell = (x', x'_\ell]$) or $x'_\ell \leq x' < x''_\ell < x''$ (in this case $X \setminus X_\ell$ is a single interval $X \setminus X_\ell = (x''_\ell, x'']$).

For the cake-cutting problem of type (ii) for the cake $C \setminus X_\ell$, players $N \setminus N(X_\ell)$ and valuations $\mathcal{C}_N(C \setminus X_\ell)$ defined by Eq.(17), let $X \not\subseteq X_\ell$ be an interval $X = (x', x'']$ of the cake C and let

$$\mathcal{C}_N(X \setminus X_\ell) = \{C_k \setminus X_\ell \in \mathcal{C}_N(C \setminus X_\ell) \mid C_k \setminus X_\ell \subseteq X \setminus X_\ell\}, \tag{19}$$

$$n_{C\setminus X_\ell}(X \setminus X_\ell) = |\mathcal{C}_N(X \setminus X_\ell)|, \tag{20}$$

and we define the density $\rho_{C\setminus X_\ell}(X \setminus X_\ell)$ of $X \setminus X_\ell$ by

$$\rho_{C\setminus X_\ell}(X \setminus X_\ell) = \frac{\mathrm{csize}(X \setminus X_\ell)}{n_{C\setminus X_\ell}(X \setminus X_\ell)}. \tag{21}$$

Note that, we defined the density $\rho_{C\setminus X_\ell}(X \setminus X_\ell)$ even for interval $X = (x', x''] \not\subseteq X_\ell$ of type (b) (i.e., $x' < x'_\ell < x''_\ell < x''$) such that $X \setminus X_\ell$ is a set of two disjoint intervals $X \setminus X_\ell = (x', x'_\ell] + (x''_\ell, x''])$. In general, $\mathcal{C}_N(X \setminus X_\ell) \neq \mathcal{C}_{N(X\setminus X_\ell)} = \{C_k \in \mathcal{C}_N \mid C_k \subseteq X \setminus X_\ell\}$, and $n_{C\setminus X_\ell}(X \setminus X_\ell) \neq n_{X\setminus X_\ell} = |\mathcal{C}_{N(X\setminus X_\ell)}|$. Furthermore, $C_k = (\alpha_k, \beta_k]$ and $C_{k'} = (\alpha_{k'}, \beta_{k'}]$ of type (c) with $\alpha_k \leq \alpha_{k'} < x'_\ell < \beta_k < \beta_{k'} \leq x''_\ell$ become intervals $C_k \setminus X_\ell = (\alpha_k, x'_\ell]$ and $C_{k'} \setminus X_\ell = (\alpha_{k'}, x'_\ell]$ in $\mathcal{C}_N(C \setminus X_\ell)$ in Eq.(17). Thus, for an interval $X = (x', x'']$ with $x' \leq \alpha_k \leq \alpha_{k'} < x'_\ell < \beta_k < \beta_{k'} \leq x''_\ell$ and $x'_\ell < x'' \leq x''_\ell$, the interval $X \setminus X_\ell = (x', x'_\ell]$ contains $C_k \setminus X_\ell = (\alpha_k, x'_\ell]$ and $C_{k'} \setminus X_\ell = (\alpha_{k'}, x'_\ell]$, i.e.,

$$C_k \setminus X_\ell, C_{k'} \setminus X_\ell \in \mathcal{C}_N(X \setminus X_\ell),$$

even if $x'' < \beta_k < \beta_{k'}$ (i.e., even if $X = (x', x'']$ does not contain C_k nor $C_{k'}$). Actually, $C_k \setminus X_\ell \subseteq X \setminus X_\ell$ if and only if $C_k \subseteq X \cup X_\ell$. Thus,

$$\mathcal{C}_N(X \setminus X_\ell) = \{C_k \setminus X_\ell \mid C_k \in \mathcal{C}_N \setminus \mathcal{C}_N(X_\ell),\ C_k \subseteq X \cup X_\ell\} \tag{22}$$

and in general, $n_{C\setminus X_\ell}(X \setminus X_\ell)$ does not equal to $n_X - n_{X_\ell}$.

We will discuss the density $\rho_{C\setminus X_\ell}(X \setminus X_\ell)$ of $X \setminus X_\ell$ using the following definition.

Definition 5.1 For an arbitrary interval $X = (x', x'']$, the interval $[x', x'']$, denoted by $\mathrm{cl}(X)$, is called the *closure* of X.

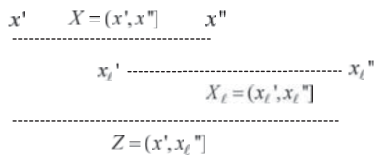

Figure 9: Two intervals $X = (x', x'']$ and $X_\ell = (x'_\ell, x''_\ell]$ in Proof of Lemma 5.1.

Let $X = (x', x'']$ and $X_\ell = (x'_\ell, x''_\ell]$ be two intervals in C such that $\mathrm{cl}(X) \cap \mathrm{cl}(X_\ell) = \emptyset$ ($[x', x''] \cap [x'_\ell, x''_\ell] = \emptyset$). Then $x' < x'' < x'_\ell$ or $x''_\ell < x' < x''$. Thus, $X \cap X_\ell = \emptyset$, $X \cup X_\ell$ is not a single interval, and each $C_k \in \mathcal{C}_{N(X \cup X_\ell)}$ is $C_k \subseteq X$ or $C_k \subseteq X_\ell$ (i.e., $C_k \setminus X_\ell = C_k$ or $C_k \setminus X_\ell = \emptyset$). This implies that if $\mathrm{cl}(X) \cap \mathrm{cl}(X_\ell) = \emptyset$, then $\mathcal{C}_N(X \setminus X_\ell) = \mathcal{C}_{N(X)}$ by Eq.(22) and $n_{C \setminus X_\ell}(X \setminus X_\ell) = n_X$ and thus $\rho_{C \setminus X_\ell}(X \setminus X_\ell) = \rho(X)$ hold.

On the other hand, if $\mathrm{cl}(X) \cap \mathrm{cl}(X_\ell) \neq \emptyset$, then $X \cup X_\ell$ is a single interval in C, and $\rho_{C \setminus X_\ell}(X \setminus X_\ell) = \rho(X)$ may not hold and we have the following lemma.

Lemma 5.1 Let $X_\ell = (x'_\ell, x''_\ell]$ be a minimal interval with respect to density in the cake C. Let $X = (x', x''] \not\subseteq X_\ell$ be an interval of C such that $\mathrm{cl}(X) \cap \mathrm{cl}(X_\ell) \neq \emptyset$. Let $Z = X \cup X_\ell$. Suppose that $\rho(X) \geq \rho(X_\ell)$ and $\rho(Z) \geq \rho(X_\ell)$. Then,

$$\rho_{C \setminus X_\ell}(Z \setminus X_\ell) = \frac{\mathrm{csize}(Z \setminus X_\ell)}{n_{C \setminus X_\ell}(Z \setminus X_\ell)} \geq \rho(X_\ell). \tag{23}$$

Furthermore, if $\rho(Z) > \rho(X_\ell)$ then $\rho_{C \setminus X_\ell}(Z \setminus X_\ell) > \rho(X_\ell)$, and if $\rho(Z) = \rho(X_\ell)$ then $\rho_{C \setminus X_\ell}(Z \setminus X_\ell) = \rho(X_\ell)$.

Proof: Clearly, $X \setminus X_\ell = X \setminus (X \cap X_\ell) = (X \cup X_\ell) \setminus X_\ell = Z \setminus X_\ell$ and $\rho_{C \setminus X_\ell}(Z \setminus X_\ell) = \rho_{C \setminus X_\ell}(X \setminus X_\ell)$. We divide the case into two subcases: (i) $X_\ell \setminus X \neq \emptyset$ and (ii) $X_\ell \setminus X = \emptyset$.

(i) $X_\ell \setminus X \neq \emptyset$: Since $X \setminus X_\ell \neq \emptyset$ (by $X = (x', x''] \not\subseteq X_\ell$) and $\mathrm{cl}(X) \cap \mathrm{cl}(X_\ell) \neq \emptyset$ ($Z = X \cup X_\ell$ is a single interval in C), we have $x' < x'_\ell \leq x'' < x''_\ell$ or $x''_\ell < x' \leq x''_\ell < x''$. By symmetry we can assume $x' < x'_\ell \leq x'' < x''_\ell$ (Figure 9). Note that

$$n_{C \setminus X_\ell}(X \setminus X_\ell) = n_{C \setminus X_\ell}(Z \setminus X_\ell) = n_Z - n_{X_\ell} \geq 0$$

since $X_\ell \subset Z = X \cup X_\ell$, $N(X_\ell) \subseteq N(Z)$, $\mathcal{C}_{N(X_\ell)} \subseteq \mathcal{C}_{N(Z)}$, $n_Z = |\mathcal{C}_{N(Z)}|$, $n_{X_\ell} = |\mathcal{C}_{N(X_\ell)}|$, $n_{C \setminus X_\ell}(Z \setminus X_\ell) = |\mathcal{C}_N(Z \setminus X_\ell)| = n_{C \setminus X_\ell}(X \setminus X_\ell)$, and $\mathcal{C}_N(X \setminus X_\ell) = \mathcal{C}_N(Z \setminus X_\ell)$ in Eq.(19) by $X \setminus X_\ell = Z \setminus X_\ell$. Furthermore, by $X_\ell \subset Z = X \cup X_\ell$ and $X \cap X_\ell \subset X$, we have

$$\mathrm{csize}(Z \setminus X_\ell) = \mathrm{csize}(Z) - \mathrm{csize}(X_\ell) = \mathrm{csize}(X) - \mathrm{csize}(X \cap X_\ell) = x'_\ell - x' > 0$$

and

$$\begin{aligned} \rho_{C \setminus X_\ell}(Z \setminus X_\ell) &= \frac{\mathrm{csize}(Z \setminus X_\ell)}{n_{C \setminus X_\ell}(Z \setminus X_\ell)} = \frac{x'_\ell - x'}{n_{C \setminus X_\ell}(X \setminus X_\ell)} = \frac{x''_\ell - x' - (x''_\ell - x'_\ell)}{n_Z - n_{X_\ell}} \\ &= \frac{n_Z\, \rho(Z) - n_{X_\ell}\, \rho(X_\ell)}{n_Z - n_{X_\ell}}. \end{aligned}$$

Thus, if $n_Z = n_{X_\ell}$, then, since $X_\ell \subset Z = X \cup X_\ell$ and $X_\ell = (x'_\ell, x''_\ell]$ is a minimal interval with respect to density in C, we have

$$\rho(Z) = \frac{\mathrm{csize}(Z)}{n_Z} > \frac{\mathrm{csize}(X_\ell)}{n_{X_\ell}} = \rho(X_\ell),$$

$$\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\frac{n_Z\,\rho(Z)-n_{X_\ell}\,\rho(X_\ell)}{n_Z-n_{X_\ell}}=\frac{\mathrm{csize}(Z\setminus X_\ell)}{0}=\infty>\rho(X_\ell).$$

Otherwise (i.e., if $n_Z>n_{X_\ell}$), by $\rho(Z)\geq\rho(X_\ell)$, we have,

$$\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\frac{n_Z\,\rho(Z)-n_{X_\ell}\,\rho(X_\ell)}{n_Z-n_{X_\ell}}\geq\frac{(n_Z-n_{X_\ell})\,\rho(X_\ell)}{n_Z-n_{X_\ell}}=\rho(X_\ell).$$

Note that, $\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\rho(X_\ell)$ if and only if $\rho(Z)=\rho(X_\ell)$. Actually, if $\rho(Z)>\rho(X_\ell)$ (regardless $n_Z=n_{X_\ell}$ or $n_Z>n_{X_\ell}$) then $\rho_{C\setminus X_\ell}(Z\setminus X_\ell)>\rho(X_\ell)$, and if $\rho(Z)=\rho(X_\ell)$ then $\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\rho(X_\ell)$ and $n_Z>n_{X_\ell}$.

(ii) $X_\ell\setminus X=\emptyset$ (i.e., $X_\ell\subset X$ and $Z=X$): by symmetry, we can assume

$$x'<x'_\ell<x''_\ell<x''\quad\text{or}\quad x'=x'_\ell<x''_\ell<x''\quad\text{or}\quad x'<x'_\ell<x''_\ell=x'',$$

since $X\setminus X_\ell\neq\emptyset$. Note that, if $x'<x'_\ell<x''_\ell<x''$ then $X\setminus X_\ell=(x',x'_\ell]+(x''_\ell,x'']$ is a direct sum of two disjoint intervals $(x',x'_\ell]$ and $(x''_\ell,x'']$. Otherwise (i.e., if $x'=x'_\ell<x''_\ell<x''$ or $x'<x'_\ell<x''_\ell=x''$), $X\setminus X_\ell$ is a single interval. Thus, in either case, $Z\setminus X_\ell=X\setminus X_\ell$,

$$\mathrm{csize}(Z\setminus X_\ell)=x''-x'-(x''_\ell-x'_\ell)>0,\quad n_{C\setminus X_\ell}(Z\setminus X_\ell)=n_Z-n_{X_\ell}\geq 0.$$

Thus, by an argumet similar to one above, if $n_Z=n_{X_\ell}$, then we have $\rho(Z)>\rho(X_\ell)$ and

$$\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\frac{n_Z\,\rho(Z)-n_{X_\ell}\,\rho(X_\ell)}{n_Z-n_{X_\ell}}=\frac{\mathrm{csize}(Z\setminus X_\ell)}{0}=\infty>\rho(X_\ell).$$

Otherwise (if $n_Z>n_{X_\ell}$), by $\rho(X)=\rho(Z)\geq\rho(X_\ell)$, we have

$$\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\frac{n_Z\,\rho(Z)-n_{X_\ell}\,\rho(X_\ell)}{n_Z-n_{X_\ell}}\geq\frac{(n_Z-n_{X_\ell})\,\rho(X_\ell)}{n_Z-n_{X_\ell}}=\rho(X_\ell).$$

Note that, $\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\rho(X_\ell)$ if and only if $\rho(X)=\rho(Z)=\rho(X_\ell)$.

By the argument above, we have the following: If $\rho(Z)>\rho(X_\ell)$ then $\rho_{C\setminus X_\ell}(Z\setminus X_\ell)>\rho(X_\ell)$, and if $\rho(Z)=\rho(X_\ell)$ then $\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\rho(X_\ell)$. □

As mentioned before, if $\mathrm{cl}(X)\cap\mathrm{cl}(X_\ell)=\emptyset$ (i.e., $x'<x''<x'_\ell$ or $x''_\ell<x'<x''$ and $Z=X\cup X_\ell$ is not a single interval in C), then deletion of X_ℓ gives no effect on the interval X, and the density of $X=Z\setminus X_\ell$ remains the same, i.e., $\rho_{C\setminus X_\ell}(Z\setminus X_\ell)=\rho_{C\setminus X_\ell}(X)=\rho(X)$.

Now, if we choose X_ℓ as an interval of minimum density $\rho_{\min}$ in Lemma 5.1, then $\rho(X)\geq\rho(X_\ell)$ for each interval X in C and we have the following corollary.

Collorary 5.1 Let $X_\ell=(x'_\ell,x''_\ell]\subset C$ be an interval of the cake C of minimum density $\rho(X_\ell)=\rho_{\min}$. Then, by cutting C at both endpoints of X_ℓ and deleting X_ℓ, we have the cake-cutting problem of type (ii) for the cake $C'=C\setminus X_\ell\neq\emptyset$, players $N'=N\setminus N(X_\ell)$, valuations $\mathcal{C}'_{N'}=\mathcal{C}_N(C\setminus X_\ell)=\{C'_k=C_k\setminus X_\ell\mid\ C_k\in\mathcal{C}_N\setminus\mathcal{C}_{N(X_\ell)}\}$ in Eq.(17) where $\bigcup_{k\in N'}C'_k=C'$ by Eq.(18) and the density $\rho'=\rho_{C\setminus X_\ell}$ defined in Eq.(21) satisfies the following:

For an interval $X=(x',x'']$ in C with $X\not\subseteq X_\ell$ and $Z=X\cup X_\ell$, the density $\rho'(Z\setminus X_\ell)$ of $Z\setminus X_\ell=X\setminus X_\ell$ satisfies $\rho'(Z\setminus X_\ell)=\rho_{C\setminus X_\ell}(Z\setminus X_\ell)\geq\rho_{\min}$. Furthermore, if $\mathrm{cl}(X)\cap\mathrm{cl}(X_\ell)=\emptyset$ then $\rho'(Z\setminus X_\ell)=\rho(X)$, and otherwise (i.e., if $\mathrm{cl}(X)\cap\mathrm{cl}(X_\ell)\neq\emptyset$), $\rho(Z)>\rho_{\min}$ if and only if $\rho'(Z\setminus X_\ell)>\rho_{\min}$.

The cake-cutting problem of type (ii) for the cake $C' = C \setminus X_\ell \neq \emptyset$, players $N' = N \setminus N(X_\ell)$, valuations $\mathcal{C}'_{N'} = \mathcal{C}_N(C \setminus X_\ell) = \{C'_k = C_k \setminus X_\ell \mid C_k \in \mathcal{C}_N \setminus \mathcal{C}_{N(X_\ell)}\}$ in Eq.(17) with $\bigcup_{k \in N'} C'_k = C'$ by Eq.(18) and the density $\rho' = \rho_{C \setminus X_\ell}$ defined in Eq.(21) can be solved in almost the same way by using an idea proposed by Alijani et al. in paper [1, 6]: shrink $X_\ell = (x'_\ell, x''_\ell]$ and virtually consider $x'_\ell = x''_\ell$. In this paper we will also call it *shrinking* of $X_\ell = (x'_\ell, x''_\ell]$.

By shrinking of X_ℓ, the cake $C' = C \setminus X_\ell$ becomes a single interval $C'^{(S)}$, players $N' = N \setminus N(X_\ell)$ remains the same, each valuation $C'_k \in \mathcal{C}'_{N'}$ becomes a single interval $C'^{(S)}_k$ of $C'^{(S)}$, and $\bigcup_{k \in N'} C'^{(S)}_k = C'^{(S)}$. Note that, by shrinking of X_ℓ, the size of an interval (the size of a set of disjoint intervals) remains the same and the density of the interval (the density of the set of disjoint intervals) also remains the same, since X_ℓ is already deleted and the empty (hollow) piece X_ℓ is of size 0.

Thus, by shrinking of X_ℓ, the cake-cutting problem of type (ii) for the cake $C' = C \setminus X_\ell$, players $N' = N \setminus N(X_\ell)$, and valuations $\mathcal{C}'_{N'}$ with $\bigcup_{k \in N'} C'_k = C'$ and the density ρ' can be reduced to the cake-cutting problem of type (i) for the cake $C'^{(S)}$, players $N' = N \setminus N(X_\ell)$, and solid valuation intervals $\mathcal{C}'^{(S)}_{N'} = \{C'^{(S)}_k \mid C'_k \in \mathcal{C}'_{N'}\}$ with $\bigcup_{k \in N'} C'^{(S)}_k = C'^{(S)}$ and the same density $\rho'^{(S)} = \rho'$.

From an allocation $\mathcal{A}'^{(S)}_{N'} = \{\mathcal{A}'^{(S)}_i \mid i \in N'\}$ to players N' such that, for each $i \in N'$, $\mathcal{A}'^{(S)}_i = \{A'^{(S)}_{i_1}, A'^{(S)}_{i_2}, \ldots, A'^{(S)}_{i_{k_i}}\}$ is the allocated piece of the cake $C'^{(S)}$ to player i with $A'^{(S)}_i = A'^{(S)}_{i_1} + A'^{(S)}_{i_2} + \cdots + A'^{(S)}_{i_{k_i}} \subseteq C'^{(S)}_i$, and that $\sum_{i \in N'} A'^{(S)}_i = C'^{(S)}$, we obtain an allocation $\mathcal{A}'_{N'} = \{\mathcal{A}'_i \mid i \in N'\}$ to players N' such that, for each $i \in N'$, $\mathcal{A}'_i = \{A'_{i_1}, A'_{i_2}, \ldots, A'_{i_{k_i}}\}$ is the allocated piece of the cake C' to player i with $A'_i = A'_{i_1} + A'_{i_2} + \cdots + A'_{i_{k_i}} \subseteq C'_i$, and that $\sum_{i \in N'} A'_i = C'$ as follows:

First find a player $i \in N'$ such that $A'^{(S)}_i = A'^{(S)}_{i_1} + A'^{(S)}_{i_2} + \cdots + A'^{(S)}_{i_{k_i}} \subseteq C'^{(S)}_i$ contains the point $x'_\ell = x''_\ell$ obtained by shrinking of $X_\ell = (x'_\ell, x''_\ell]$. By symmetry, we can assume that $A'^{(S)}_{i_1} = (a'_{i_1}, a''_{i_1}]$ contains $x'_\ell = x''_\ell$.

Then set $A'_{i_1} = (a'_{i_1}, x'_\ell] + (x''_\ell, a''_{i_1}] \subseteq C'_i$ if $a'_{i_1} < x'_\ell < x''_\ell < a''_{i_1}$ in the original world before shrinking. Otherwise (i.e., $a'_{i_1} = x'_\ell$ or $x''_\ell = a''_{i_1}$), if $a'_{i_1} = x'_\ell$ then set $A'_{i_1} = (x''_\ell, a''_{i_1}] \subseteq C'_i$, and if $x''_\ell = a''_{i_1}$ then set $A'_{i_1} = (a'_{i_1}, x'_\ell] \subseteq C'_i$.

Finally, set $A'_{i_j} = A'^{(S)}_{i_j}$ for all $j = 2, 3, \ldots, k_i$ and set $\mathcal{A}'_i = \{A'_{i_1}, A'_{i_2}, \ldots, A'_{i_{k_i}}\}$. Set also $\mathcal{A}'_j = \mathcal{A}'^{(S)}_j$ for all $j \in N' \setminus \{i\}$.

We will call this *inverse shrinking* of X_ℓ. Thus, we can obtain the desired allocation $\mathcal{A}'_{N'} = \{\mathcal{A}'_i \mid i \in N'\}$ to players N' in the cake-cutting problem of type (ii) for the cake $C' = C \setminus X_\ell \neq \emptyset$, players $N' = N \setminus N(X_\ell)$, valuations $\mathcal{C}'_{N'}$ with $\bigcup_{k \in N'} C'_k = C'$ as follows:

First obtain the reduced cake-cutting problem of type (i) for the cake $C'^{(S)}$, players $N' = N \setminus N(X_\ell)$, and valuation intervals $\mathcal{C}'^{(S)}_{N'} = \{C'^{(S)}_k \mid C'_k \in \mathcal{C}'_{N'}\}$ with $\bigcup_{k \in N'} C'^{(S)}_k = C'^{(S)}$ by shrinking of X_ℓ.

Then obtain an allocation $\mathcal{A}'^{(S)}_{N'} = \{\mathcal{A}'^{(S)}_i \mid i \in N'\}$ to players N' in the reduced cake-cutting problem of type (i) above.

Finally, obtain the allocation $\mathcal{A}'_{N'} = \{\mathcal{A}'_i \mid i \in N'\}$ to players N' in the cake-cutting problem of type (ii) from the allocation $\mathcal{A}'^{(S)}_{N'} = \{\mathcal{A}'^{(S)}_i \mid i \in N'\}$ to players N' in the reduced cake-cutting problem of type (i) by inverse shrinking of X_ℓ.

Thus, we have the following corollary.

Collorary 5.2 Let $X_\ell = (x'_\ell, x''_\ell] \subset C$ be an interval of the cake C of minimum density $\rho(X_\ell) = \rho_{\min}$. Then, by cutting C at both endpoints of X_ℓ and deleting X_ℓ, we can reduce the original cake-cutting problem into two types of cake-cutting subproblems:

(i) the cake-cutting problem of type (i) for the cake $X_\ell = (x'_\ell, x''_\ell] \subset C$ of minimum density $\rho_{\min}$ which consists of the players $N(X_\ell)$ and solid valuation intervals $\mathcal{C}_{N(X_\ell)} = \{C_k \in \mathcal{C}_N \mid \ C_k \subseteq X_\ell\}$ with $\bigcup_{C_k \in \mathcal{C}_{N(X_\ell)}} C_k = X_\ell$;

(ii) the cake-cutting problem of type (ii) for the cake $C' = C \setminus X_\ell \neq \emptyset$, players $N' = N \setminus N(X_\ell)$, valuations $\mathcal{C}'_{N'} = \mathcal{C}_N(C \setminus X_\ell) = \{C'_k = C_k \setminus X_\ell \mid \ C_k \in \mathcal{C}_N \setminus \mathcal{C}_{N(X_\ell)}\}$ in Eq.(17) with $\bigcup_{k \in N'} C'_k = C'$ by Eq.(18) and the density $\rho' = \rho_{C \setminus X_\ell}$ defined in Eq.(21).

Furthermore, the cake-cutting problem of type (i) can be solved in the same way as the original cake-cutting problem.

The cake-cutting problem of type (ii) can be solved by shrinking and inverse shrinking of X_ℓ and the minimum density $\rho'_{\min}$ of intervals of this cake-cutting problem of type (ii) satisfies $\rho'_{\min} \geq \rho_{\min}$. Furthermore, if X_ℓ is a maximal interval of minimum density in C and there is no other maximal interval of minimum density in C, then $\rho'_{\min} > \rho_{\min}$.

By Corollary 5.2, using an interval $X_\ell = (x'_\ell, x''_\ell] \subset C$ of the cake C of minimum density $\rho(X_\ell) = \rho_{\min}$, we can solve the original cake-cutting problem by reducing into two types of cake-cutting subproblems.

The proposition in Corollary 5.2 can be extended to hold for two or more disjoint intervals of minimum density and we have the following lemma using the argument above repeatedly. Note that, two distinct maximal intervals of minimum density $\rho_{\min}$ in the cake C are disjoint by Corollary 4.2.

Lemma 5.2 For the cake-cutting problem for the cake $C = (0, 1]$, a set of n players $N = \{1, 2, \ldots, n\}$, solid valuation intervals $\mathcal{C}_N$ with valuation interval $C_i = (\alpha_i, \beta_i]$ of each player $i \in N$ and $\bigcup_{C_i \in \mathcal{C}_N} C_i = C$, let $\rho(C) > \rho_{\min}$ and let all the maximal intervals of minimum density $\rho_{\min}$ be $H_1 = (h'_1, h''_1]$, $H_2 = (h'_2, h''_2]$, $\ldots$, $H_L = (h'_L, h''_L]$. Then by cutting the cake at both endpoints of $H_\ell = (h'_\ell, h''_\ell]$ ($\ell = 1, 2, \ldots, L$) we can reduce the original cake-cutting problem into two types of cake-cutting subproblems:

(i) the cake-cutting problem within each maximal interval $H_\ell = (h'_\ell, h''_\ell]$ ($\ell = 1, 2, \ldots, L$) of minimum density which consists of the players $N(H_\ell)$ and solid valuation intervals

$$\mathcal{C}_{N(H_\ell)} = \{C_k \in \mathcal{C}_N \mid \ C_k \subseteq H_\ell\} \ \text{ with } \bigcup_{C_k \in \mathcal{C}_{N(H_\ell)}} C_k = H_\ell;$$

(ii) the cake-cutting problem for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$ with players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations

$$\mathcal{D}_P = \{D_i = C_i \setminus \sum_{\ell=1}^{L} H_\ell \mid C_i \in \mathcal{C}_N \setminus \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\} \text{ with } \bigcup_{D_i \in \mathcal{D}_P} D_i = D.$$

Furthermore, the minimum density of intervals in each cake-cutting problem of type (i) is equal to $\rho_{\min}$. On the other hand, the minimum density of intervals in the cake-cutting problem of type (ii) is greater than $\rho_{\min}$.

Similarly, two distinct minimal intervals of minimum density $\rho_{\min}$ in the cake C are disjoint by Corollary 4.2 and we have the following lemma.

Lemma 5.3 For the cake-cutting problem for the cake $C = (0, 1]$, a set of n players $N = \{1, 2, \ldots, n\}$, solid valuation intervals $\mathcal{C}_N$ with valuation interval $C_i = (\alpha_i, \beta_i]$ of each player $i \in N$ and $\bigcup_{C_i \in \mathcal{C}_N} C_i = C$, let $\rho(C) = \rho_{\min}$ and let all the minimal intervals of minimum density $\rho_{\min}$ be $X_1 = (x'_1, x''_1]$, $X_2 = (x'_2, x''_2]$, $\ldots$, $X_K = (x'_K, x''_K]$. Then by cutting the cake at both endpoints of each $X_k = (x'_k, x''_k]$ we can reduce the original cake-cutting problem into two types of cake-cutting subproblems:

(i) the cake-cutting problem within each minimal interval $X_k = (x'_k, x''_k]$ $(k = 1, 2, \ldots, K)$ of minimum density which consists of the players $N(X_k)$ and solid valuation intervals

$$\mathcal{C}_{N(X_k)} = \{C_i \in \mathcal{C}_N \mid \ C_i \subseteq X_k\} \ \text{ with } \bigcup_{C_i \in \mathcal{C}_{N(X_k)}} C_i = X_k;$$

(ii) the cake-cutting problem for the cake $D = C \setminus \sum_{k=1}^{K} X_k$ with players $P = N \setminus \sum_{k=1}^{K} N(X_k)$ and valuations

$$\mathcal{D}_P = \{D_i = C_i \setminus \sum_{k=1}^{K} X_k \mid C_i \in \mathcal{C}_N \setminus \sum_{k=1}^{K} \mathcal{C}_{N(X_k)}\} \text{ with } \bigcup_{D_i \in \mathcal{D}_P} D_i = D.$$

Furthermore, the minimum density of intervals in each cake-cutting problem of type (i) is equal to $\rho_{\min}$. The minimum density of intervals in the cake-cutting problem of type (ii) is also equal to $\rho_{\min}$.

We denote, by Procedure CutCake$(P, D, \mathcal{D}_P)$, a method for solving the cake-cutting problem for the cake D which is a single interval, players P and solid valuation intervals $\mathcal{D}_P$ (where each valuation $D_i \in \mathcal{D}_P$ for $i \in P$ is a single interval in D) and $\bigcup_{i \in P} D_i = D$. The original cake-cutting problem for the cake C, players N and solid valuation intervals $\mathcal{C}_N$ with $\bigcup_{C_i \in \mathcal{C}_N} C_i = C$ can be solved by setting $P = N$, $D = C$ and $\mathcal{D}_P = \mathcal{C}_N$, and calling Procedure CutCake$(N, C, \mathcal{C}_N)$. Thus, we can write our mechanism as follows.

Mechanism 5.1 Our cake-cutting mechanism.

Input: A cake $C = (0, 1]$, a set of n players $N = \{1, 2, \ldots, n\}$, and solid valuation intervals $\mathcal{C}_N$ with valuation interval $C_i = (\alpha_i, \beta_i]$ of each player $i \in N$ and $\bigcup_{C_i \in \mathcal{C}_N} C_i = C$.

Output: Allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ to players N.

Algorithm {
  for each $i \in N$ **do** $D_i = C_i$;
  $P = N$; $D = C$; $\mathcal{D}_P = \{D_i \mid i \in P\}$;
  CutCake$(P, D, \mathcal{D}_P)$;
}

As mentioned before, the cake-cutting problem of type (i) within each maximal interval $H_\ell = (h'_\ell, h''_\ell]$ $(\ell = 1, 2, \ldots, L)$ of minimum density can be solved similarly. However, we use a slightly different method for solving the cake-cutting problem of type (i) with the cake $H = H_\ell$, players $R = N(H_\ell) = \{i \in N \mid C_i \subseteq H_\ell\}$ and solid valuation intervals

$\mathcal{D}_R = \mathcal{C}_{N(H_\ell)} = \{C_i \in \mathcal{C}_N \mid i \in N(H_\ell)\}$ with $\cup_{C_i \in \mathcal{D}_R} C_i = H_\ell$, since H is a maximal interval of minimum density. We call it Procedure CutMaxInterval$(R, H, \mathcal{D}_R)$.

Based on shrinking and inverse shrinking in Lemma 5.2, we can solve the cake-cutting problem of type (ii) for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{D}_P = \{D_i = C_i \setminus \sum_{\ell=1}^{L} H_\ell \mid C_i \in \mathcal{C}_N \setminus \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\}$ with $\cup_{C_i \in \mathcal{D}_P} C_i = D$ in the same way as the original cake-cutting problem, since all the maximal intervals $H_1 = (h'_1, h''_1],\ H_2 = (h'_2, h''_2], \ldots,\ H_L = (h'_L, h''_L]$ of minimum density $\rho_{\min}$ in the cake C are mutually disjoint by Corollary 4.2. We call this Procedure CutCakeType(ii)$(P, D, \mathcal{D}_P)$.

We first give a detailed description of CutCake$(P, D, \mathcal{D}_P)$ for the cake D, players P and solid valuation intervals $\mathcal{D}_P$ with valuation interval D_i of each player $i \in P$ and $\bigcup_{i \in P} D_i = D$.

Procedure 5.1 CutCake$(P, D, \mathcal{D}_P)$ {
Find all the maximal intervals of minimum density $\rho_{\min}$ in the cake-cutting
    problem with cake D, players P and solid valuation intervals $\mathcal{D}_P$;
Let $H_1 = (h'_1, h''_1]$, $H_2 = (h'_2, h''_2]$, $\ldots$, $H_L = (h'_L, h''_L]$ be all the maximal intervals
    of minimum density $\rho_{\min}$;
    // $H_1, H_2, \ldots, H_L$ are mutually disjoint by Corollary 4.2
for $\ell = 1$ **to** L **do**
    cut the cake at both endpoints h'_ℓ, h''_ℓ of H_ℓ;
    $R_\ell = \{i \in P \mid D_i \subseteq H_\ell, D_i \in \mathcal{D}_P\}$; $\mathcal{D}_{R_\ell} = \{D_i \in \mathcal{D}_P \mid i \in R_\ell\}$;
    CutMaxInterval$(R_\ell, H_\ell, \mathcal{D}_{R_\ell})$;
$P' = P$; $D' = D$;
for $\ell = 1$ **to** L **do** $P' = P' \setminus R_\ell$; $D' = D' \setminus H_\ell$;
// $P' = P \setminus \sum_{\ell=1}^{L} R_\ell$ and $D' = D \setminus \sum_{\ell=1}^{L} H_\ell$
if $P' \neq \emptyset$ **then**
    $\mathcal{D}'_{P'} = \emptyset$;
    for each $D_i \in \mathcal{D}_P$ with $i \in P'$ **do** $D'_i = D_i \setminus \sum_{\ell=1}^{L} H_\ell$; $\mathcal{D}'_{P'} = \mathcal{D}'_{P'} + \{D'_i\}$;
    // CutCakeType(ii)$(P', D', \mathcal{D}'_{P'})$
    Perform shrinking of all $H_1, H_2, \ldots, H_L$;
    Let $D^{(S)}$, $D_i^{(S)} \in \mathcal{D}_{P'}^{(S)}$, and $\mathcal{D}_{P'}^{(S)}$ be obtained from
        D', $D'_i \in \mathcal{D}'_{P'}$, and $\mathcal{D}'_{P'}$ by shrinking of all $H_1, H_2, \ldots, H_L$, respectively;
    CutCake$(P', D^{(S)}, \mathcal{D}_{P'}^{(S)})$;
    Perform inverse shrinking of all $H_1, H_2, \ldots, H_L$;
}

Note that, if $P' \neq \emptyset$ after the deletion of $H_1, H_2, \ldots, H_L$ and CutCake$(P', D^{(S)}, \mathcal{D}_{P'}^{(S)})$ is recursively called, then the minimum density $\rho'_{\min}$ in CutCake$(P', D^{(S)}, \mathcal{D}_{P'}^{(S)})$ is strictly larger than the minimum density $\rho_{\min}$ in CutCake$(P, D, \mathcal{D}_P)$ by Lemma 5.2.

Next, we give a detailed description of Procedure CutMaxInterval$(R, H, \mathcal{D}_R)$ for the cake H of maximal interval of minimum density $\rho_{\min}$, players R and solid valuation intervals $\mathcal{D}_R$, based on Lemma 5.3 and Procedure CutMinInterval$(S, X, \mathcal{D}_S)$ which is a method for solving the cake-cutting problem of type (i) where the cake is a minimal interval X of minimum density $\rho_{\min}$ in maximal interval $H = H_\ell$ of minimum density $\rho_{\min}$, players $S = R(X) = \{i \in R \mid D_i \in \mathcal{D}_R,\ D_i \subseteq X\}$ and solid valuation intervals $\mathcal{D}_S = \mathcal{D}_{R(X)} = \{D_i \in \mathcal{D}_R \mid i \in S\}$ with $\cup_{D_i \in \mathcal{D}_S} D_i = X$.

Procedure 5.2 CutMaxInterval($R, H, \mathcal{D}_R$) {
Let $X_1 = (x'_1, x''_1],\ X_2 = (x'_2, x''_2], \ldots,\ X_K = (x'_K, x''_K]$ be all the minimal intervals of density $\rho_{\min}$ in H;
// $X_1, X_2, \ldots, X_K$ are mutually disjoint by Corollary 4.2
for $k = 1$ **to** K **do**
cut the cake at both endpoints x'_k, x''_k of X_k;
$S_k = \{i \in R \mid D_i \subseteq X_k, D_i \in \mathcal{D}_R\}$; $\mathcal{D}_{S_k} = \{D_i \in \mathcal{D}_R \mid i \in S_k\}$;
CutMinInterval($S_k, X_k, \mathcal{D}_{S_k}$);
$R' = R$; $H' = H$;
for $k = 1$ **to** K **do** $R' = R' \setminus S_k$; $H' = H' \setminus X_k$;
// $R' = R \setminus \sum_{k=1}^{K} S_k$ and $H' = H \setminus \sum_{k=1}^{K} X_k$
if $R' \neq \emptyset$ **then**
$\mathcal{D}'_{R'} = \emptyset$;
for each $D_i \in \mathcal{D}_R$ with $i \in R'$ **do** $D'_i = D_i \setminus \sum_{k=1}^{K} X_k$; $\mathcal{D}'_{R'} = \mathcal{D}'_{R'} + \{D'_i\}$;
Perform shrinking of all $X_1, X_2, \ldots, X_K$;
Let $H^{(S)}$, $D_i^{(S)} \in \mathcal{D}_{R'}^{(S)}$, and $\mathcal{D}_{R'}^{(S)}$ be obtained from
H', $D'_i \in \mathcal{D}'_{R'}$, and $\mathcal{D}'_{R'}$ by shrinking of all $X_1, X_2, \ldots, X_K$, respectively;
CutMaxInterval($R', H^{(S)}, \mathcal{D}_{R'}^{(S)}$);
Perform inverse shrinking of all $X_1, X_2, \ldots, X_K$;
}

Note that if $R' \neq \emptyset$ after deletion of $X_1, X_2, \ldots, X_K$ and CutMaxInterval($R', H^{(S)}, \mathcal{D}_{R'}^{(S)}$) is recursively called, then the minimum density $\rho'_{\min}$ in CutMaxInterval($R', H^{(S)}, \mathcal{D}_{R'}^{(S)}$) is the same as the minimum density $\rho_{\min}$ in CutMaxInterval($R, H, \mathcal{D}_R$) by Lemma 5.3.

Note also that, the cake-cutting problem of type (i) within each minimal interval $X_k = (x'_k, x''_k]$ of minimum density $\rho_{\min}$ (which consists of the players $S_k = R(X_k)$ and solid valuation intervals $\mathcal{D}_{S_k} = \{D_i \in \mathcal{D}_R \mid i \in S_k\}$ with $\cup_{D_i \in \mathcal{D}_{S_k}} D_i = X_k$) is solved by CutMinInterval($S_k, X_k, \mathcal{D}_{S_k}$), as mentioned above.

In order to give a detailed description of Procedure CutMinInterval($S, X, \mathcal{D}_S$) for the cake-cutting problem of type (i) where the cake is a minimal interval X of minimum density $\rho_{\min}$ in maximal interval $H = H_\ell$ of minimum density $\rho_{\min}$, players $S = R(X) = \{i \in R \mid D_i \in \mathcal{D}_R,\ D_i \subseteq X\}$ and solid valuation intervals $\mathcal{D}_S = \mathcal{D}_{R(X)} = \{D_i \in \mathcal{D}_R \mid i \in S\}$ with $\cup_{D_i \in \mathcal{D}_S} D_i = X$, we need some more definitions and notations.

Definition 5.2 Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$. A minimal interval $Y = (y', y'']$ with respect to density which is properly contained in X (i.e., $Y \subset X$) is called a *separable interval* of X, if csize(Y) is less than $(n_Y + 1)\,\rho_{\min}$, where n_Y is the number of players whose valuation intervals are entirely contained in Y (Figure 10).

If there is no separable interval of $X = (x', x'']$, then X is called *nonseparable*.

Note that there are at most n^2 separable intervals in X, since a separable interval is a minimal interval with respect to density and there are at most n^2 minimal intervals with respect to density as mentioned before.

We first consider the case when a minimal interval X of minimum density $\rho_{\min}$ is nonseparable. This has a nice property.

Lemma 5.4 Let $X = (x', x'']$ be a nonseparable minimal interval of minimum density $\rho_{\min}$. For simplicity, we assume $X = (0, 1] = C$, $N(X) = N = \{1, 2, \ldots, n\}$, $\mathcal{C}_{N(X)} = \mathcal{C}_N$. Let $I_j = ((j-1)\rho_{\min}, j\rho_{\min}]$ for each $j \in N$, and let $\mathcal{I}_N = \{I_1, I_2, \ldots, I_n\}$ ($\sum_{i \in N} I_i = X$). Let $G = (\mathcal{C}_N, \mathcal{I}_N, E)$ be a bipartite graph with vertex set $\mathcal{C}_N + \mathcal{I}_N$ and edge set E where $(C_i, I_j) \in E$ if and only if $I_j \subseteq C_i$. Then G has a perfect matching $M = \{(C_i, I_{\pi(i)}) \mid i \in N\} \subseteq E$ ($\pi : N \to N$ is a permutation on N).

Lemma 5.4 can be proved by Hall's Theorem [4]: for all positive integers $k \leq n$ and for all k subsets $\{C_{i_1}, C_{i_2}, \ldots, C_{i_k}\} \subseteq \mathcal{C}_N$, if the union $C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_k}$ contains at least k intervals in $\mathcal{I}_N = \{I_1, I_2, \ldots, I_n\}$ (that is, $C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_k}$ contains $\ell \geq k$ intervals $I_{j_1}, I_{j_2}, \ldots, I_{j_\ell}$), then the bipartite graph $G = (\mathcal{C}_N, \mathcal{I}_N, E)$ has a perfect matching.

Let $M = \{(C_i, I_{\pi(i)}) \mid i \in N\}$ be a perfect matching of the bipartite graph $G = (\mathcal{C}_N, \mathcal{I}_N, E)$ defined in Lemma 5.4. Then we can allocate $A_i = I_{\pi(i)} \subseteq C_i$ of the cake $X = (0, 1]$ to player $i \in N$ with $\sum_{i \in N} A_i = X$. Since a perfect matching can be obtained in polynomial time of n, we call this Procedure AllocateInterval($N(X), X, \mathcal{C}_{N(X)}$) and will use it in Procedure CutMinInterval($N(X), X, \mathcal{C}_{N(X)}$).

Next we consider the remaining case. Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ with a separable interval. Let $\mathcal{Y}$ be the set of all separable intervals in X and let

$$y^* = \max_{Y=(y', y'']\in\mathcal{Y}} y'. \tag{24}$$

That is, y^* is the largest left endpoint of the separable intervals in X. Let $\mathcal{Y}_{y^*}$ be the set of all separable intervals in X whose left endpoints are y^* (Figure 10), i.e.,

$$\mathcal{Y}_{y^*} = \{Y = (y', y''] \in \mathcal{Y} \mid y' = y^*\}. \tag{25}$$

For each interval $Y = (y', y'']$ of X, let

$$\gamma(Y) = \text{csize}(Y) - n_Y\, \rho_{\min}. \tag{26}$$

Let $Y = (y', y'']$ be a separable interval of the minimal interval X of minimum density $\rho_{\min}$. Then Y is a minimal interval with respect to density and

$$n_Y\, \rho_{\min} < \text{csize}(Y) < (n_Y + 1)\, \rho_{\min} \tag{27}$$

and we have

$$0 < \gamma(Y) < \rho_{\min}. \tag{28}$$

Actually, $\rho(Y) = \frac{\text{csize}(Y)}{n_Y} > \rho_{\min}$ and $\gamma(Y) = \text{csize}(Y) - n_Y\, \rho_{\min} > 0$ for each $Y \subset X$ since X is a minimal interval of minimum density $\rho_{\min}$. Furthermore, $\text{csize}(Y) < (n_Y + 1)\, \rho_{\min}$ for a separable interval Y of the minimal interval X of minimum density $\rho_{\min}$.

Let γ^* be the minimum $\gamma(Y)$ among the separable intervals $Y = (y^*, y'']$ with the largest left endpoint y^*, i.e.,

$$\gamma^* = \min_{Y\in\mathcal{Y}_{y^*}} \gamma(Y). \tag{29}$$

Clearly, by Eqs. (28), (29),

$$0 < \gamma^* < \rho_{\min}. \tag{30}$$

Let Z_{y^*} be the set of right endpoints of the separable intervals whose left endpoints are y^*, i.e.,

$$Z_{y^*} = \{y'' \mid Y = (y^*, y''] \in \mathcal{Y}_{y^*}\}. \tag{31}$$

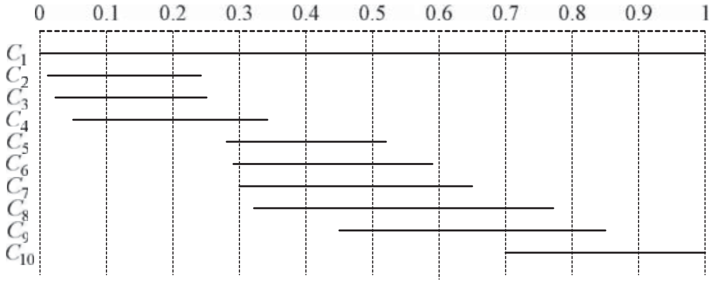

Figure 10: Players $N = \{1, 2, \ldots, 10\}$ and their valuation intervals $C_1 = (0, 1]$, $C_2 = (0.01, 0.24]$, $C_3 = (0.02, 0.25]$, $C_4 = (0.05, 0.34]$, $C_5 = (0.28, 0.52]$, $C_6 = (0.29, 0.59]$, $C_7 = (0.3, 0.65]$, $C_8 = (0.32, 0.77]$, $C_9 = (0.45, 0.85]$, $C_{10} = (0.7, 1]$. In this case, $X = (0, 1]$ is a minimal interval of minimum density $\rho_{\min} = 0.1$, and there are several separable intervals of $X = (0, 1]$ such as $(0.01, 0.25]$, $(0.01, 0.59]$, $(0.01, 1]$, $(0.28, 0.65]$, $(0.28, 0.77]$, $(0.28, 0.85]$. The largest left endpoint y^* of the separable intervals in X is 0.28 and the set of separable intervals with the largest left endpoint $y^* = 0.28$ is $\{(0.28, 0.65],\ (0.28, 0.77],\ (0.28, 0.85]\}$.

Let $\mathcal{Y}_{y^*}^{\gamma^*}$ be the set of separable intervals $Y = (y^*,\, y'']$ in $\mathcal{Y}_{y^*}$ with $\gamma(Y) = \gamma^*$, i.e.,

$$\mathcal{Y}_{y^*}^{\gamma^*} = \{Y = (y^*,\, y''] \in \mathcal{Y}_{y^*} \mid \gamma(Y) = \gamma^*\}. \tag{32}$$

Let $Z_{y^*}^{\gamma^*}$ be the set of right endpoints of the separable intervals in $\mathcal{Y}_{y^*}^{\gamma^*}$ and J be the cardinality of $Z_{y^*}^{\gamma^*}$, i.e.,

$$Z_{y^*}^{\gamma^*} = \{y'' \mid Y = (y^*,\, y''] \in \mathcal{Y}_{y^*}^{\gamma^*}\}, \quad J = |Z_{y^*}^{\gamma^*}|. \tag{33}$$

Let

$$Z_{y^*}^{\gamma^*} = \{z_1^*, z_2^*, \ldots, z_J^*\}, \quad z_1^* < z_2^* < \cdots < z_J^*. \tag{34}$$

For each $j = 1, 2, \ldots, J$, let

$$Y_j = (y^*, z_j^*]. \tag{35}$$

For simplicity, we also consider

$$z_0^* = y^* + \gamma^*, \quad Y_0 = (y^*,\, z_0^*]. \tag{36}$$

In Figure 10, $\rho_{\min} = 0.1$, $y^* = 0.28$, $\mathcal{Y}_{y^*} = \{(0.28, 0.65],\ (0.28, 0.77],\ (0.28, 0.85]\}$, $Z_{y^*} = \{0.65, 0.77, 0.85\}$, $\gamma((0.28, 0.65]) = 0.65 - 0.28 - 0.3 = 0.07$, $\gamma((0.28, 0.77]) = 0.77 - 0.28 - 0.4 = 0.09$, $\gamma((0.28, 0.85]) = 0.85 - 0.28 - 0.5 = 0.07$ and $\gamma^* = 0.07$. Thus, $\mathcal{Y}_{y^*}^{\gamma^*} = \{(0.28, 0.65],\ (0.28, 0.85]\}$, $Z_{y^*}^{\gamma^*} = \{0.65, 0.85\}$, $J = 2$, $z_1^* = 0.65 < z_2^* = 0.85$.

Then we have the following lemma and corollary.

Lemma 5.5 Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ in the cake C. Let $Y = (y^*,\, z] \subset X$ be an interval such that there exists $C_i = (\alpha_i,\ \beta_i] \in \mathcal{C}_{N(X)}$ with $y^* \le \alpha_i$ and $z = \beta_i$. Then $\gamma(Y) = \gamma^*$ for $z \in Z_{y^*}^{\gamma^*}$ and $\gamma(Y) > \gamma^*$ for $z \notin Z_{y^*}^{\gamma^*}$, i.e.,

$$\gamma(Y) \begin{cases} = & \gamma^* \quad (z \in Z_{y^*}^{\gamma^*}) \\ > & \gamma^* \quad (z \notin Z_{y^*}^{\gamma^*}). \end{cases} \tag{37}$$

Proof: By the definition of $\gamma(Y)$ of $Y = (y^*,\ z] \subset X$ in Eq.(26),

$$\gamma(Y) = \text{csize}(Y) - n_Y \rho_{\min} = z - y^* - n_Y \rho_{\min}.$$

It is clear that if $z \in Z_{y^*}^{\gamma^*}$ then $\gamma(Y) = \gamma^*$ by the definitions of $\mathcal{Y}_{y^*}^{\gamma^*}$ and $Z_{y^*}^{\gamma^*}$. Therefore, we can assume $z \notin Z_{y^*}^{\gamma^*}$ below.

We first consider the case when $Y = (y^*,\ z]$ is not a separable interval. In this case, if $Y = (y^*,\ z]$ is a minimal interval with respect to density then $\text{csize}(Y) \geq (n_Y + 1)\ \rho_{\min}$ and by Eq. (30),

$$\gamma(Y) = \text{csize}(Y) - n_Y \rho_{\min} \geq \rho_{\min} > \gamma^*.$$

Otherwise (i.e., if $Y = (y^*,\ z]$ is not a minimal interval with respect to density), let

$$y' = \min_{C_j = (\alpha_j, \beta_j] \in \mathcal{C}_{N(X)} : C_j \subset Y = (y^*\ z]} \alpha_j.$$

Let $C_j = (\alpha_j, \beta_j] \in \mathcal{C}_{N(X)}$ satisfy $\alpha_j = y'$ and $\beta_j \leq z$. Then $y^* < y' \leq \alpha_i$ since the valuation interval $C_i = (\alpha_i,\ \beta_i] \in \mathcal{C}_{N(X)}$ satisfies $\alpha_i \geq y^*$ and $z = \beta_i$ (and $C_i \subset Y = (y^*\ z]$) and $Y = (y^*,\ z]$ is not a minimal interval with respect to density. Let $Y' = (y', z] \subset Y = (y^*,\ z]$. Then, both valuation intervals $C_i = (\alpha_i, \beta_i], C_j = (\alpha_j, \beta_j] \in \mathcal{C}_{N(X)}$ with $y' = \alpha_j$ and $z = \beta_i$ are contained in $Y' = (y', z]$. Thus, $Y' = (y', z]$ is a minimal interval with respect to density and $n_{Y'} = n_Y$. Note that, Y' is not a separable interval since y^* is the largest left endpoint of the separable intervals of X. Thus, $\text{csize}(Y') \geq (n_{Y'} + 1)\ \rho_{\min}$,

$$\begin{aligned} \text{csize}(Y) &> \text{csize}(Y') \geq (n_{Y'} + 1)\ \rho_{\min} = (n_Y + 1)\ \rho_{\min}, \text{ and} \\ \gamma(Y) &= \text{csize}(Y) - n_Y \rho_{\min} \geq \rho_{\min} > \gamma^*, \end{aligned}$$

since $\text{csize}(Y) = z - y^* > z - y' = \text{csize}(Y')$.

We next consider the case when $Y = (y^*,\ z]$ is a separable interval. Thus, $n_Y \rho_{\min} < \text{csize}(Y) < (n_Y + 1)\rho_{\min}$. By the definition of $Z_{y^*}^{\gamma^*} = \{y'' \mid Y = (y^*,\ y''] \in \mathcal{Y}_{y^*}^{\gamma^*}\}$ and Eq. (29), we have

$$\gamma(Y) = \text{csize}(Y) - n_Y \rho_{\min} > \gamma^*$$

since $z \notin Z_{y^*}^{\gamma^*}$. □

Collorary 5.3 Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ in the cake C. Let $Y = (y^*,\ z] \subset X$ be an interval such that there is a valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ with $\alpha_i \geq y^*$ and $z = \beta_i$. If $z \notin Z_{y^*}^{\gamma^*}$ and $z > z_j^*$ for some $Y_j = (y^*,\ z_j^*]$ $(j = 0, 1, \ldots, J)$, then

$$z - z_j^* > \rho_{\min}(n_Y - n_{Y_j}). \tag{38}$$

Proof: Since $z - z_j^* = \text{csize}(Y) - \text{csize}(Y_j)$, we have

$$\begin{aligned} z - z_j^* &= \text{csize}(Y) - \text{csize}(Y_j) = \rho_{\min} n_Y + \gamma(Y) - (\rho_{\min} n_{Y_j} + \gamma(Y_j)) \\ &= (\gamma(Y) - \gamma(Y_j)) + \rho_{\min}(n_Y - n_{Y_j}) \\ &> \rho_{\min}(n_Y - n_{Y_j}) \end{aligned}$$

by Lemma 5.5 and $\gamma(Y_j) = \gamma^* < \gamma(Y)$. □

The following lemma can be obtained using almost the same argument and almost the same notation used in Proof of Lemma 3.1.

Lemma 5.6 Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ in the cake C. Let $Y_j = (y^*, z_j^*]$ $(j = 1, 2, \ldots, J)$ be a minimal interval of X with respect to density defined in Eq.(35). Then the valuation intervals $\mathcal{C}_{N(Y_j)}$ is solid (i.e., $\bigcup_{C_i \in \mathcal{C}_{N(Y_j)}} C_i = Y_j$).

Let $X = (x', x'']$ be a minimal interval of minimum density $\rho_{\min}$ in the cake C and let $S = N(X)$ and $\mathcal{D}_S = \{D_i = C_i \mid C_i \in \mathcal{C}_N, C_i \subseteq X\} = \mathcal{C}_{N(X)}$. For each $j = 1, 2, \ldots, J$, let

$$\begin{aligned} Z_j &= (z_{j-1}^*, z_j^*], & \mathcal{D}_{S(Z_j)} &= \{D_i \in \mathcal{D}_S \mid D_i \subseteq Y_j, D_i \not\subseteq Y_{j-1}\}, \quad (39) \\ S(Z_j) &= \{i \in S \mid D_i \in \mathcal{D}_{S(Z_j)}\}, & n'_{Z_j} &= |S(Z_j)|. \quad (40) \end{aligned}$$

Note that $\mathcal{D}_{S(Z_j)} = \{D_i \in \mathcal{D}_S \mid D_i \subseteq Y_j\} \setminus \{D_i \in \mathcal{D}_S \mid D_i \subseteq Y_{j-1}\}$. Furthermore, for each $j = 1, 2, \ldots, J$, let

$$\mathcal{D}'_{S(Z_j)} = \{D'_i = D_i \setminus Y_{j-1} \mid D_i \in \mathcal{D}_{S(Z_j)}\}. \quad (41)$$

We consider the cake-cutting problem for the cake Z_j, players $S(Z_j)$, solid valuation intervals $\mathcal{D}'_{S(Z_j)}$ for each $j = 1, 2, \ldots, J$. Note that, $D'_i = D_i \setminus Y_{j-1} \in \mathcal{D}'_{S(Z_j)}$ is always contained in $Z_j = (z_{j-1}^*, z_j^*]$, although valuation interval $D_i = (d'_i, d''_i] \in \mathcal{D}_{S(Z_j)}$ may not be in $Z_j = (z_{j-1}^*, z_j^*]$ (i.e., $d'_i < z_{j-1}^*$ may happen). Of course, $y^* \le d'_i$ and $z_{j-1}^* < d''_i \le z_j^*$ hold. Note also that, for each $j = 1, 2, \ldots, J$, the valuation intervals $\mathcal{D}'_{S(Z_j)}$ is solid, i.e.,

$$\bigcup_{D'_i \in \mathcal{D}'_{S(Z_j)}} D'_i = Z_j. \quad (42)$$

This can be obtained as follows: By Lemma 5.6, for each $Y_j = (y^*, z_j^*]$ $(j = 1, 2, \ldots, J)$, $\mathcal{D}_{N(Y_j)}$ is solid (i.e., $\bigcup_{D_i \in \mathcal{D}_{N(Y_j)}} D_i = Y_j$), and thus, for each point $z \in Z_j = (z_{j-1}^*, z_j^*] = Y_j \setminus Y_{j-1}$, there is a valuation interval $D_i \in \mathcal{D}_{N(Y_j)}$ containing z. The interval D_i is not in $\mathcal{D}_{N(Y_{j-1})}$, since $z \notin Y_{j-1}$. Thus, z is in $D'_i = D_i \setminus Y_{j-1} \in \mathcal{D}'_{S(Z_j)}$. This implies that the valuation intervals $\mathcal{D}'_{S(Z_j)}$ is solid and Eq.(42) holds.

Note that there is no valuation interval of $\mathcal{D}_S = \mathcal{C}_{N(X)}$ contained in $Y_0 = (y^*, z_0^*] \subset X$ $(n_{Y_0} = 0)$, since if there were a valuation interval $D_i \in \mathcal{D}_S$ contained in Y_0, then D_i would be a minimal interval with respect to density and $n_{D_i} \ge 1$ and $\rho(D_i) \le \mathrm{csize}(Y_0) = z_0^* - y^* = \gamma^* < \rho_{\min}$, a contradiction that X is a minimal interval of minimum density $\rho_{\min}$. Thus, we have the following lemma.

Lemma 5.7 Each interval $Z_j = (z_{j-1}^*, z_j^*]$ $(j = 1, 2, \ldots, J)$ is a minimal interval with minimum density $\rho'_{\min} = \rho_{\min}$ for the cake-cutting problem for the cake Z_j, players $S(Z_j) = \{i \in S \mid D_i \in \mathcal{D}_{S(Z_j)}\}$, valuation intervals $\mathcal{D}'_{S(Z_j)}$ in Eq.(39) and the density ρ'. Furthermore, the valuation intervals $\mathcal{D}'_{S(Z_j)}$ is solid and Eq.(42) holds.

Proof: As described above, the valuation intervals $\mathcal{D}'_{S(Z_j)}$ is solid and Eq.(42) holds for each $j = 1, 2, \ldots, J$.

Thus, we show below that each $Z_j = (z_{j-1}^*, z_j^*]$ $(j = 1, 2, \ldots, J)$ is a minimal interval with minimum density $\rho_{\min}$. It is clear that $\rho'(Z_j) = \frac{\mathrm{csize}(Z_j)}{n'_{Z_j}} = \rho_{\min}$, since

$$\begin{aligned} Y_j &= (y^*, z_j^*], \quad Y_{j-1} = (y^*, z_{j-1}^*], \quad Z_j = Y_j \setminus Y_{j-1}, \\ \mathrm{csize}(Y_j) &= \rho_{\min} n_{Y_j} + \gamma^*, \quad \mathrm{csize}(Y_{j-1}) = \rho_{\min} n_{Y_{j-1}} + \gamma^*, \quad n'_{Z_j} = n_{Y_j} - n_{Y_{j-1}}, \\ \mathrm{csize}(Z_j) &= \mathrm{csize}(Y_j) - \mathrm{csize}(Y_{j-1}) = \rho_{\min}(n_{Y_j} - n_{Y_{j-1}}) = \rho_{\min} n'_{Z_j}. \end{aligned}$$

Let $Z = (z', z'']$ be a proper subinterval of Z_j (i.e., $Z \subset Z_j$) such that z' is z^*_{j-1} or z' is a left endpoint of some valuation interval and that z'' is a right endpoint of some valuation interval in $\mathcal{D}'_{S(Z_j)}$.

If $z' \neq z^*_{j-1}$ then $\rho'(Z) = \rho(Z) > \rho_{\min}$, since $Z \subset X$ ($Z \neq X$) and X is a minimal interval with minimum density $\rho_{\min}$. Thus, we assume $z' = z^*_{j-1} < z'' < z^*_j$ ($Z = (z^*_{j-1}, z'']$). Now consider the intervals $Y'_j = (y^*, z'']$ and $Y_{j-1} = (y^*, z^*_{j-1}]$. Let $n'_Z = |\mathcal{C}_{N(Y'_j)} \setminus \mathcal{C}_{N(Y_{j-1})}|$. Then $n'_Z = n_{Y'_j} - n_{Y_{j-1}}$ by $Y_{j-1} \subset Y'_j$. By Corollary 5.3, we have

$$\text{csize}(Z) = z'' - z^*_{j-1} > \rho_{\min}(n_{Y'_j} - n_{Y_{j-1}}) = \rho_{\min}\, n'_Z$$

and $\rho'(Z) = \frac{\text{csize}(Z)}{n'_Z} > \rho_{\min}$.

Thus, $Z_j = (z^*_{j-1}, z^*_j]$ is a minimal interval with minimum density $\rho'_{\min} = \rho_{\min}$. □

We also consider the remaining cake-cutting problem of type (ii) after deletion of the interval $(z^*_0, z^*_J]$. Note that $(z^*_0, z^*_J] = Z_1 + Z_2 + \cdots + Z_J$. Let

$$S((z^*_0, z^*_J]) = S(Z_1) + S(Z_2) + \cdots + S(Z_J). \tag{43}$$

Thus, $S((z^*_0, z^*_J])$ is the set of players whose valuation intervals are contained in $Y^*_J = (y^*, z^*_J]$. Let

$$\begin{aligned} S' &= S \setminus S((z^*_0, z^*_J]), & (44)\\ X' &= X \setminus (z^*_0, z^*_J], & (45)\\ \mathcal{D}'_{S'} &= \{D'_i \mid D'_i = D_i \setminus (z^*_0, z^*_J], D_i \in \mathcal{D}_S, D_i \not\subseteq Y_J\}. & (46) \end{aligned}$$

Then, we reduce the remaining cake-cutting problem for the cake X', players S' and valuations $\mathcal{D}'_{S'}$ by shrinking of $(z^*_0, z^*_J]$ to the cake-cutting problem for the cake $X'^{(S)}$, players S' and solid valuation intervals $\mathcal{D}'^{(S)}_{S'}$, where $X'^{(S)}$, $D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}$ and $\mathcal{D}'^{(S)}_{S'}$ are obtained from X', $D'_i \in \mathcal{D}'_{S'}$ and $\mathcal{D}'_{S'}$ by shrinking of $(z^*_0, z^*_J]$, respectively.

Then the following lemmas holds.

Lemma 5.8 Let $X'^{(S)}$ be the interval obtained from $X' = X \setminus (z^*_0, z^*_J]$ in Eq.(45) by shrinking of $(z^*_0, z^*_J]$. Then $X'^{(S)}$ is a minimal interval with minimum density $\rho'_{\min} = \rho_{\min}$ in the cake-cutting problem for the cake $X'^{(S)}$, players S' in Eq.(44), solid valuation intervals $\mathcal{D}'^{(S)}_{S'}$ ($\bigcup_{D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}} D'^{(S)}_i = X'^{(S)}$) obtained from $\mathcal{D}'_{S'}$ in (46) and the density ρ'.

Proof: We will show that $X'^{(S)}$ is a minimal interval with minimum density $\rho_{\min}$.

We can obtain $\rho'(X'^{(S)}) = \rho_{\min}$ by the argument in Lemma 5.7. Actually, since $Y_J = (y^*, z^*_J]$, $z^*_J - z^*_0 = n_{Y_J}\rho_{\min}$, $\text{csize}(X) = \rho_{\min} n_X$, and $n_{X'^{(S)}} = n_X - n_{Y_J}$, we have

$$\begin{aligned} \text{csize}(X'^{(S)}) &= \text{csize}(X) - (z^*_J - z^*_0) = n_X\rho_{\min} - n_{Y_J}\rho_{\min}\\ &= (n_X - n_{Y_J})\rho_{\min} = n_{X'^{(S)}}\rho_{\min} \quad \text{and} \end{aligned}$$

$$\rho'(X'^{(S)}) = \frac{\text{csize}(X'^{(S)})}{n_{X'^{(S)}}} = \rho_{\min}.$$

Let $Z = (z', z''] \not\subseteq (y^*, z^*_J]$ be an interval in X such that $Z'^{(S)}$, obtained from $Z' = Z \setminus (z^*_0, z^*_J] \subset X'$ by shrinking of $(z^*_0, z^*_J]$, is a proper subinterval in $X'^{(S)}$ (i.e., $Z'^{(S)} \subset X'^{(S)}$). Thus, $z' < y^*$ or $z'' > z^*_J$. To prove that $X'^{(S)}$ is a minimal interval with minimum density

$\rho'_{\min} = \rho_{\min}$ we will show that $\rho'(Z'^{(S)}) > \rho_{\min}$ by dividing into two subcases: Case (i) the case of $z' < y^*$ and Case (ii) the case of $y^* \leq z'$ and $z'' > z^*_J$.

As noted before, there is no valuation interval of $\mathcal{D}_S = \mathcal{C}_{N(X)} = \{C_i \in \mathcal{C}_N \mid C_i \subseteq X\}$ contained in $Y_0 = (y^*, z^*_0]$. Similarly, there is no valuation interval of $\mathcal{D}'_{S'}$ (and of $\mathcal{D}'^{(S)}_{S'}$) contained in $Y_0 = (y^*, z^*_0]$, since each $D_i \in \mathcal{D}_S$ with $D'_i = D_i \setminus (z^*_0, z^*_J] \in \mathcal{D}'_{S'}$ is not contained $Y_J = (y^*, z^*_J]$ by Eq.(46) and $D_i \setminus Y_J \neq \emptyset$.

Case (i) $z' < y^*$: We only discuss the case of $z' < y^* < z^*_0 < z'' \leq z^*_J$ (the other cases, i.e., $z' < z'' \leq y^*$ or $z' < y^* < z'' \leq z^*_0$ or $z' < y^* < z^*_0 < z^*_J < z''$, can be discussed similarly). After shrinking of $(z^*_0,\, z^*_J]$, $Z' = Z \setminus (z^*_0,\, z^*_J]$ becomes $Z'^{(S)} = (z', z^*_0] = Z'$ and we can consider $z'' = z^*_J$ by almost the same argument in Lemma 5.1. Thus,

$$Y_J = (y^*, z^*_J] \subset Z = (z', z^*_J], \quad \mathrm{csize}(Z) = z^*_J - z', \quad \rho_{\min} n_{Y_J} = z^*_J - z^*_0.$$

$$n_{Z'^{(S)}} = n_Z - n_{Y_J}, \quad \mathrm{csize}(Z'^{(S)}) = z^*_0 - z',$$

and we have

$$\rho'(Z'^{(S)}) = \frac{\mathrm{csize}(Z'^{(S)})}{n_{Z'^{(S)}}} = \frac{z^*_0 - z'}{n_{Z'^{(S)}}} > \rho_{\min}$$

by $Z'^{(S)} = (z', z^*_0] \subset X'^{(S)}$ and

$$\rho(Z) = \frac{\mathrm{csize}(Z)}{n_Z} = \frac{z^*_J - z^*_0 + z^*_0 - z'}{n_{Z'^{(S)}} + n_{Y_J}} = \frac{\rho_{\min} n_{Y_J} + z^*_0 - z'}{n_{Z'^{(S)}} + n_{Y_J}} > \rho_{\min}$$

since $Z = (z', z^*_J] \subset X$ and X is a minimal interval of minimum density $\rho_{\min}$.

Case (ii) $y^* \leq z'$ and $z'' > z^*_J$: We only discuss the case of $z^*_0 \leq z' < z^*_J < z''$ (the other cases, i.e., $y^* \leq z' < z^*_0 < z^*_J < z''$ or $z^*_J \leq z' < z''$, can be discussed similarly). By Corollary 5.3 for $j = J$, $Y_J = (y^*,\, z^*_J]$ and $Y = (y^*,\, z'']$, we have $Z'^{(S)} = Z' = Z \setminus (z^*_0,\, z^*_J] = (z^*_J,\, z''] \subset Y \setminus (z^*_0,\, z^*_J] = (y^*,\, z^*_0] + (z^*_J,\, z'']$,

$$\mathrm{csize}(Z'^{(S)}) = z'' - z^*_J = \mathrm{csize}(Y) - \mathrm{csize}(Y_J) > \rho_{\min}(n_Y - n_{Y_J})$$

and $n_{Z'^{(S)}} \leq n_Y - n_{Y_J}$ (note that $D_i = (\alpha_i, \beta_i]$ with $y^* \leq \alpha_i < z^*_0$ and $z^*_J < \beta_i \leq z''$ is in $Y = (y^*,\, z'']$, but not in $Y_J = (y^*,\, z^*_J]$, and thus, $D'_i = D_i \setminus (z^*_0, z^*_J] = (\alpha_i, z^*_0] + (z^*_J, \beta_i]$ is not contained in $Z'^{(S)} = Z' = Z \setminus (z^*_0,\, z^*_J] = (z^*_J,\, z'']$). Thus,

$$\rho'(Z'^{(S)}) = \frac{\mathrm{csize}(Z'^{(S)})}{n_{Z'^{(S)}}} \geq \frac{\mathrm{csize}(Z'^{(S)})}{n_Y - n_{Y_J}} > \rho_{\min}.$$

Thus, we have shown that $X'^{(S)}$ is a minimal interval of minimum density $\rho'_{\min} = \rho_{\min}$.

Finally, we will show that the valuation intervals $\mathcal{D}'^{(S)}_{S'}$ is solid, i.e., $\bigcup_{D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}} D'^{(S)}_i = X'^{(S)}$. Note that $\bigcup_{D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}} D'^{(S)}_i = X'^{(S)}$ if and only if $\bigcup_{D'_i \in \mathcal{D}'_{S'}} D'_i = X'$. It is clear that $\bigcup_{D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}} D'^{(S)}_i \subseteq X'^{(S)}$ and $\bigcup_{D'_i \in \mathcal{D}'_{S'}} D'_i \subseteq X'$.

Let $x \in X' = X \setminus (z^*_0, z^*_J]$. Note that the cake X is a minimal interval of minimum density $\rho_{\min}$ in maximal interval $H = H_\ell$ of minimum density $\rho_{\min}$, the players are $S = N(X) = \{i \in N(H) \mid C_i \subseteq X\}$ and valuation intervals $\mathcal{D}_S = \mathcal{C}_{N(X)} = \{C_i \in \mathcal{C}_{N(H)} \mid i \in S\}$ is solid, i.e., $\bigcup_{C_i \in \mathcal{D}_S} C_i = X$ by Lemma 3.2. Thus, there is a valuation interval $C_i \in \mathcal{D}_S$ that contains x.

If $x \in X' \setminus (y^*, z_0^*] = X \setminus (y^*, z_J^*]$, then C_i is not contained in $(y^*, z_J^*]$ since C_i contains x, and thus, both $D_i' = C_i \setminus (z_0^*, z_J^*] \in \mathcal{D}'_{S'}$ and $D_i'^{(S)} \in \mathcal{D}'^{(S)}_{S'}$ contain x. This implies

$$X' \setminus (y^*, z_0^*] \subseteq \bigcup_{D_i' \in \mathcal{D}'_{S'}} D_i' \quad \text{and} \quad X'^{(S)} \setminus (y^*, z_0^*] \subseteq \bigcup_{D_i'^{(S)} \in \mathcal{D}'^{(S)}_{S'}} D_i'^{(S)}.$$

Therefore, we can assume $x \in (y^*, z_0^*] \subset X' = X \setminus (z_0^*, z_J^*]$. The valuation interval C_i above containing x, however, may happen to be entirely contained in $Y_J = (y^*, z_J^*]$. In this case, C_i is in some $\mathcal{D}_{S(Z_j)} = \{D_i \in \mathcal{D}_S \mid D_i \subseteq Y_j, D_i \not\subseteq Y_{j-1}\}$ in Eq.(39) and player i is in $S(Z_j) = \{i \in S \mid D_i \in \mathcal{D}_{S(Z_j)}\}$, which implies $i \notin S' = S \setminus S((z_0^*, z_J^*])$. Of course, if C_i is not contained in $Y_J = (y^*, z_J^*]$, then both $D_i' = C_i \setminus (z_0^*, z_J^*]$ and $D_i'^{(S)}$ contain x.

We will show that, for $x \in (y^*, z_0^*]$, there is always such a valuation interval C_i containing x, but is not contained in $Y_J = (y^*, z_J^*]$. This will imply that both $D_i' = C_i \setminus (z_0^*, z_J^*] \in \mathcal{D}'_{S'}$ and $D_i'^{(S)} \in \mathcal{D}'^{(S)}_{S'}$ contain x. We divide the case of $x'' \geq z_J^*$ into two subcases: Case (i) $x'' = z_J^*$ and Case (ii) $x'' > z_J^*$.

Case (i) $x'' = z_J^*$: In this case, $X = (x', z_J^*]$ and each valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ satisfies $\beta_i \leq z_J^*$. Note that $Y_J = (y^*, z_J^*] \subset X = (x', z_J^*]$ and $x' < y^*$, since Y_J is a separable interval in X. Let $y_{\max}$ be the largest right endpoint among the valuation intervals $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ such that $\alpha_i < y^*$, i.e.,

$$y_{\max} = \max_{C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}:\ \alpha_i < y^*} \beta_i. \tag{47}$$

Let $C_i = (\alpha_i, \beta_i]$ be a valuation interval in $\mathcal{C}_{N(X)}$ with $\alpha_i < y^*$ and $\beta_i = y_{\max}$. Thus, $C_i = (\alpha_i, \beta_i]$ is not in $Y_J = (y^*, z_J^*]$.

If $y_{\max} \geq z_0^*$, then both $C_i = (\alpha_i, \beta_i]$ and $D_i' = C_i \setminus (z_0^*, z_J^*] = (\alpha_i, z_0^*] \in \mathcal{D}'_{S'}$ contain $x \in (y^*, z_0^*]$, and we have $\bigcup_{D_i'^{(S)} \in \mathcal{D}'^{(S)}_{S'}} D_i'^{(S)} = X'^{(S)}$ and $\bigcup_{D_i' \in \mathcal{D}'_{S'}} D_i' = X'$.

Thus, we can assume $y_{\max} < z_0^*$. Let

$$N((x', y_{\max}]) = \{i \in N(X) \mid C_i \in \mathcal{C}_{N(X)}, C_i \subseteq (x', y_{\max}]\},$$

$$\mathcal{C}_{N((x', y_{\max}])} = \{C_i \in \mathcal{C}_{N(X)} \mid i \in N((x', y_{\max}])\}, \quad n_{(x', y_{\max}]} = |N((x', y_{\max}])|.$$

Then

$$\mathcal{C}_{N(X)} = \mathcal{C}_{N((x', y_{\max}])} + \mathcal{C}_{N((y^*, z_J^*])},$$

since each $C_j = (\alpha_j, \beta_j] \in \mathcal{C}_{N(X)}$ satisfies either $\alpha_j < y^*$ (i.e., $C_j = (\alpha_j, \beta_j] \in \mathcal{C}_{N((x', y_{\max}])}$) or $\alpha_j \geq y^*$ (i.e., $C_j = (\alpha_j, \beta_j] \in \mathcal{C}_{N((y^*, z_J^*])}$) by the definition of $y_{\max}$ in Eq.(47). Thus, we have $n_{(x', y_{\max}]} + n_{(y^*, z_J^*]} = n_X$ and

$$\begin{aligned} \text{csize}((x', y_{\max}]) &= y_{\max} - x' > n_{(x', y_{\max}]}\rho_{\min}, \\ n_{(x', y_{\max}]}\rho_{\min} &= n_X\rho_{\min} - n_{(y^*, z_J^*]}\rho_{\min} = z_J^* - x' - (z_J^* - z_0^*) = z_0^* - x', \end{aligned}$$

since $(x', y_{\max}] \subset X$ and X is a minimal interval of minimum density $\rho_{\min}$. Thus, we have $y_{\max} > z_0^*$, a contradiction.

Case (ii) $x'' > z_J^*$: In this case, we will show there is a valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ such that $\alpha_i < y^*$ and $z_J^* < \beta_i$.

Suppose contrarily that there is no valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ such that $\alpha_i < y^*$ and $z_J^* < \beta_i$. Thus, for each valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$, either ($\alpha_i < y^*$ and $\beta_i \leq z_J^*$) or ($y^* \leq \alpha_i$ and $z_J^* < \beta_i$) holds. Let

$$\mathcal{C}_{N((x', z_J^*])} = \{C_i \in \mathcal{C}_{N(X)} \mid C_i \subseteq (x', z_J^*]\}, \quad \mathcal{C}_{N((y^*, x''])} = \{C_i \in \mathcal{C}_{N(X)} \mid C_i \subseteq (y^*, x'']\}.$$

Note that $(x', z^*_J]$, $Y_J = (y^*, z^*_J] = (x', z^*_J] \cap (y^*, x'']$ and $(y^*, x'']$ are all minimal intervals with respect to density, and $\mathcal{C}_{N(Y_J)} = \mathcal{C}_{N((x',z^*_J])} \cap \mathcal{C}_{N((y^*,x''])} \subset \mathcal{C}_{N((y^*,x''])}$. We have $n_{(x',z^*_J]} + n_{(y^*,x'']} - n_{(y^*,z^*_J]} = n_X$ since there is no valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ such that $\alpha_i < y^*$ and $z^*_J < \beta_i$. We also have $z^*_J - x' > n_{(x',z^*_J]}\rho_{\min}$ since X is a minimal interval of minimum density $\rho_{\min}$. Furthermore, $x'' - z^*_J > (n_{(y^*,x'']} - n_{(y^*,z^*_J]})\rho_{\min}$ by Corollary 5.3. Thus, we have

$$\begin{aligned}\text{csize}(X) &= x'' - x' = (z^*_J - x') + (x'' - z^*_J) \\ &> n_{(x',z^*_J]}\rho_{\min} + (n_{(y^*,x'']} - n_{(y^*,z^*_J]})\rho_{\min} \\ &= (n_{(x',z^*_J]} + n_{(y^*,x'']} - n_{(y^*,z^*_J]})\rho_{\min} = n_X\rho_{\min} = \text{csize}(X),\end{aligned}$$

a contradiction.

Thus, when $x'' > z^*_J$, there is a valuation interval $C_i = (\alpha_i, \beta_i] \in \mathcal{C}_{N(X)}$ such that $\alpha_i < y^*$ and $z^*_J < \beta_i$ and $C_i = (\alpha_i, \beta_i]$ contains x and not contained in $(y^*, z^*_J]$ and we have

$$\bigcup_{D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}} D'^{(S)}_i = X'^{(S)} \text{ and } \bigcup_{D'_i \in \mathcal{D}'_{S'}} D'_i = X'.$$

□

Based on Lemmas 5.4, 5.7 and 5.8, we can write Procedure CutMinInterval($S, X, \mathcal{D}_S$) as follows.

Procedure 5.3 CutMinInterval($S, X, \mathcal{D}_S$**)** {
if $X = (x', x'']$ is nonseparable **then** AllocateInterval($S, X, \mathcal{D}_S$);
  // this finds an allocation of X to players S by Lemma 5.4
else // there is a separable interval in X
  Find y^*, γ^*, $\mathcal{Y}^{\gamma^*}_{y^*}$, and $Z^{\gamma^*}_{y^*}$ defined by Eqs. (24), (29), (32), and (33), respectively;
  Let $Z^{\gamma^*}_{y^*} = \{z^*_1, z^*_2, \ldots, z^*_J\}$ and assume
    $z^*_0 = y^* + \gamma^* < z^*_1 < z^*_2 < \cdots < z^*_J \leq z_{J+1} = x''$;
  for $j = 1$ **to** J **do**
    $Z_j = (z^*_{j-1}, z^*_j]$;
    cut the cake at both endpoints z^*_{j-1}, z^*_j of $Z_j = (z^*_{j-1}, z^*_j]$;
    let $\mathcal{D}_{S(Z_j)}$ and $\mathcal{D}'_{S(Z_j)}$ be defined in Eqs. (39) and (41);
    $S(Z_j) = \{i \in S \mid D_i \in \mathcal{D}_{S(Z_j)}\}$;
    CutMinInterval($S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)}$);
  $S' = S \setminus S((z^*_0, z^*_J])$; $X' = X \setminus (z^*_0, z^*_J]$;
  if $S' \neq \emptyset$ **then**
    $\mathcal{D}'_{S'} = \emptyset$;
    for each $D_i \in \mathcal{D}_S$ with $i \in S'$ **do** $D'_i = D_i \setminus (z^*_0, z^*_J]$; $\mathcal{D}'_{S'} = \mathcal{D}'_{S'} + \{D'_i\}$;
    Perform shrinking of $(z^*_0, z^*_J]$;
    Let $X'^{(S)}$, $D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}$ and $\mathcal{D}'^{(S)}_{S'}$ be obtained from
      X', $D'_i \in \mathcal{D}'_{S'}$ and $\mathcal{D}'_{S'}$ by shrinking of $(z^*_0, z^*_J]$, respectively;
    CutMinInterval($S', D'^{(S)}, \mathcal{D}'^{(S)}_{S'}$);
    Perform inverse shrinking of $(z^*_0, z^*_J]$;
}

Based on Lemmas 5.2, 5.4 and 5.8, we can show that Mechanism 5.1 correctly finds, in $O(n^3)$ time, an envy-free allocation $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ of the cake C to a set of n

players N with $\mathcal{A}_i = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}\}$ such that $A_i = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}} \subseteq C_i$ for each player $i \in N$ and the number of cuts made is at most $2n-2$. Actually, envy-freeness and truthfulness of Mechanism 5.1 can be obtained by induction on the number of calls on Procedure CutCake$(P, D, \mathcal{D}_P)$ by Lemma 5.2. Truthfulness of Mechanism 5.1 can be also shown in a similar way as in papers [2], [6]. We can show that the number of cuts is at most $2(n-1)$ in a similar way as in paper [6].

We will give a little more details below. For simplicity, we use $A_N = (A_i : i \in N)$ and A_i in place of $\mathcal{A}_N = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ and $\mathcal{A}_i$, respectively. Then the following lemmas can be obtained.

Lemma 5.9 For a cake X which is a minimal interval of minimum density $\rho_{\min}$, players S, and solid valuation intervals $\mathcal{D}_S = \{D_i \mid i \in S\}$ with $\bigcup_{i \in S} D_i = X$, Procedure CutMinInterval$(S, X, \mathcal{D}_S)$ satisfies the following (a) – (c).

(a) CutMinInterval$(S, X, \mathcal{D}_S)$ returns an envy-free allocation $(A_i : i \in S)$ of X to players S such that $A_i \subseteq D_i \in \mathcal{D}_S$, $\mathrm{csize}(A_i) = \rho_{\min}$ for each $i \in S$ and $\sum_{i \in S} A_i = X$.

(b) CutMinInterval$(S, X, \mathcal{D}_S)$ runs in $O(s^3)$ time where $s = |S|$.

(c) The number of cuts made over X by CutMinInterval$(S, X, \mathcal{D}_S)$ is at most $2s-2$.

If X is nonseparable, then, by Lemma 5.4, AllocateInterval$(S, X, \mathcal{D}_S)$ finds an allocation $(A_i : i \in S)$ of X to players S such that $A_i \subseteq D_i \in \mathcal{D}_S$, $\mathrm{csize}(A_i) = \rho_{\min}$ for each $i \in S$ and $\sum_{i \in S} A_i = X$. The number of cuts made by AllocateInterval$(S, X, \mathcal{D}_S)$ is $s-1$. A perfect matching of a bipartite graph in Lemma 5.4 with $2s$ vertices can be obtained in $O(s^3)$ time. Thus, Lemma 5.9 holds.

If X is separable, Lemma 5.9 can be shown by induction on the number of recursive calls of CutMinInterval$(\cdot, \cdot, \cdot)$ in CutMinInterval$(S, X, \mathcal{D}_S)$ in total.

Assume that the lemma holds if CutMinInterval$(S, X, \mathcal{D}_S)$ contains at most $k \geq 0$ recursive calls. Consider when CutMinInterval$(S, X, \mathcal{D}_S)$ contains $k+1$ recursive calls. Thus, X has a separable interval and CutMinInterval$(S, X, \mathcal{D}_S)$ contains J recursive calls CutMinInterval$(S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)})$ for the cake Z_j which is a minimal interval of minimum density $\rho_{\min}$ by Lemma 5.7 and a recursive call CutMinInterval$(S', X'^{(S)}, \mathcal{D}'^{(S)}_{S'})$ for the cake $X'^{(S)}$ which is a minimal interval of minimum density $\rho_{\min}$ by Lemma 5.8 if $S' \neq \emptyset$. By the induction hypothesis, CutMinInterval$(S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)})$ finds an allocation $(A_i : i \in S(Z_j))$ of Z_j to players $S(Z_j)$ such that $A_i \subseteq D'_i \in \mathcal{D}'_{S(Z_j)}$ (thus, $A_i \subseteq D_i \in \mathcal{D}_S$), $\mathrm{csize}(A_i) = \rho_{\min}$ for each $i \in S(Z_j)$ and $\sum_{i \in S(Z_j)} A_i = Z_j$ for each $j = 1, 2, \ldots, J$. Furthermore, CutMinInterval$(S', X'^{(S)}, \mathcal{D}'^{(S)}_{S'})$ finds an allocation $(A'_i : i \in S')$ of $X'^{(S)}$ to players S' such that $A'_i \subseteq D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}$, $\mathrm{csize}(A'_i) = \rho_{\min}$ for each $i \in S'$ and $\sum_{i \in S'} A'_i = X'^{(S)}$. By inverse shrinking of $(z^*_0, z^*_J]$, we have the allocation $(A_i : i \in S')$ of $X' = X \setminus (z^*_0, z^*_J]$ to players S' such that $A_i \subseteq D'_i \in \mathcal{D}'_{S'}$ (thus, $A_i \subseteq D_i \in \mathcal{D}_S$), $\mathrm{csize}(A_i) = \rho_{\min}$ for each $i \in S'$ and $\sum_{i \in S'} A_i = X'$.

Thus, we can obtain that CutMinInterval$(S, X, \mathcal{D}_S)$ returns an allocation $(A_i : i \in S)$ of X to players S such that $A_i \subseteq D_i \in \mathcal{D}_S$, $\mathrm{csize}(A_i) = \rho_{\min}$ for each $i \in S$ and $\sum_{i \in S} A_i = X$. Since $A_i \subseteq D_i$, $\mathrm{csize}(A_i) = \rho_{\min}$ and $\mathrm{ut}_i(A_i) = \mathrm{csize}(A_i \cap D_i) = \mathrm{csize}(A_i) = \rho_{\min} = \mathrm{csize}(A_j) \geq \mathrm{csize}(A_j \cap D_i) = \mathrm{ut}_i(A_j)$ for each $i, j \in S$, the allocation $(A_i : i \in S)$ is envy-free. Thus, (a) is obtained.

Similalry, (b) and (c) of the lemma can be shown by induction on the number of recursive calls of CutMinInterval$(\cdot,\cdot,\cdot)$ in CutMinInterval$(S, X, \mathcal{D}_S)$ in total. Note that, all the separable intervals can be found in (s^2) time.

The following two lemmas can be obtained similarly.

Lemma 5.10 For a cake H which is a maximal interval of minimum density $\rho_{\min}$, players R, and solid valuation intervals $\mathcal{D}_R = \{D_i \mid i \in R\}$ with $\bigcup_{i \in R} D_i = H$, Procedure CutMaxInterval$(R, H, \mathcal{D}_R)$ satisfies the following (a) – (c).

(a) CutMaxInterval$(R, H, \mathcal{D}_R)$ returns an envy-free allocation $(A_i : i \in R)$ of H to players R with $A_i \subseteq D_i \in \mathcal{D}_R$, csize$(A_i) = \rho_{\min}$ for each $i \in R$ and $\sum_{i \in R} A_i = H$.

(b) CutMaxInterval$(R, H, \mathcal{D}_R)$ runs in $O(r^3)$ time where $r = |R|$.

(c) The number of cuts made over H in CutMaxInterval$(R, H, \mathcal{D}_R)$ is at most $2r - 2$.

Lemma 5.11 For a cake D, players P, and solid valuation intervals $\mathcal{D}_P = \{D_i \mid i \in P\}$ with $\bigcup_{i \in P} D_i = D$, Procedure CutCake$(P, D, \mathcal{D}_P)$ satisfies the following (a) – (c).

(a) CutCake$(P, D, \mathcal{D}_P)$ returns an envy-free allocation $(A_i : i \in P)$ of D to players P such that $A_i \subseteq D_i \in \mathcal{D}_P$, csize$(A_i) \geq \rho_{\min}$ for each $i \in P$ and $\sum_{i \in P} A_i = D$.

(b) CutCake$(P, D, \mathcal{D}_P)$ runs in $O(p^3)$ time where $p = |P|$.

(c) The number of cuts made over D in CutCake$(P, D, \mathcal{D}_P)$ is at most $2p - 2$.

By Lemma 5.11, the Mechanism 5.1 (Procedure CutCake$(N, C, \mathcal{C}_N)$ is envy-free and runs in $O(n^3)$ time, and the number of cuts made by Mechanism 5.1 is at most $2(n-1)$.

Note that, in CutCake$(P, D, \mathcal{D}_P)$, if D is a maximal interval of minimum density $\rho_{\min}$, then CutMaxInterval$(P, D, \mathcal{D}_P)$ is called and, by Lemma 5.10, CutMaxInterval$(P, D, \mathcal{D}_P)$ finds an envy-free allocation $(A_i : i \in P)$ of D to players P such that $A_i \subseteq D_i \in \mathcal{D}_P$, csize$(A_i) = \rho_{\min}$ for each $i \in P$ and $\sum_{i \in P} A_i = D$. On the other hand, if D is not a maximal interval of minimum density $\rho_{\min}$, then $P' \neq \emptyset$ after the deletion of all maximal intervals of minimum density $\rho_{\min}$, and CutCake$(P', D^{(S)}, \mathcal{D}^{(S)}_{P'})$ is recursively called, and the minimum density $\rho'_{\min}$ in CutCake$(P', D^{(S)}, \mathcal{D}^{(S)}_{P'})$ satisfies $\rho'_{\min} > \rho_{\min}$. Thus, by induction on the number of recursive calls of CutCake$(\cdot,\cdot,\cdot)$ in CutCake$(P, D, \mathcal{D}_P)$, we can show that CutCake$(P, D, \mathcal{D}_P)$ returns an envy-free allocation $(A_i : i \in P)$ of D to players P such that $A_i \subseteq D_i \in \mathcal{D}_P$, csize$(A_i) \geq \rho_{\min}$ for each $i \in P$ and $\sum_{i \in P} A_i = D$.

As mentioned before, truthfulness of Mechanism 5.1 can be shown in a similar way as in papers [2], [6]. Thus, we have the following theorem.

Theorem 5.1 Mechanism 5.1 is envy-free and truthful, and the number of cuts made by Mechanism 5.1 on the cake is at most $2(n-1)$. Mechanism 5.1 runs in $O(n^3)$ time.

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6 Appendix: Proof of Theorem 5.1

Although the truthfulness of Mechanism 5.1 can be obtained in a similar way as in papers [2], [6] as metioned before, we give a proof for completeness.

Lemma 6.1 Mechanism 5.1 (mechanism $\mathcal{M}$) is truthful.

Proof: Let $\mathcal{C}_N = \{C_1, C_2, \ldots, C_n\}$ be an arbitrary input to the mechanism $\mathcal{M}$ and $A_N = (A_j : j \in N)$ be an allocation of the cake C to n players N obtained by $\mathcal{M}$ with A_j for each $j \in N$. Let $\mathcal{C}'_N(i) = \{C_1, C_2, \ldots, C_{i-1}, C'_i, C_{i+1}, \ldots, C_n\}$ be an input to the mechanism $\mathcal{M}$ in which only player i gives a false valuation interval C'_i and let an allocation of the cake C to n players N obtained by $\mathcal{M}$ be $A'_N(i) = (A'_j : j \in N)$ with A'_j for each $j \in N$. Note that, by Lemma 5.11, we have

$$A_i \subseteq C_i, \quad A'_i \subseteq C'_i, \quad A_j, A'_j \subseteq C_j \text{ for all } j \in N \setminus \{i\}. \tag{48}$$

Thus, the value $\mathrm{ut}_i(A_i)$ of A_i for player i and the value $\mathrm{ut}_i(A'_i)$ of A'_i for player i are

$$\mathrm{ut}_i(A_i) = \mathrm{csize}(A_i \cap C_i) = \mathrm{csize}(A_i), \quad \mathrm{ut}_i(A'_i) = \mathrm{csize}(A'_i \cap C_i) \le \mathrm{csize}(A'_i).$$

We will show that $\mathrm{ut}_i(A_i) \ge \mathrm{ut}_i(A'_i)$.

Let $H_1 = (h'_1, h''_1]$, $H_2 = (h'_2, h''_2], \ldots,$ $H_L = (h'_L, h''_L]$ be all the maximal intervals of minimum density $\rho_{\min}$ of solid valuation intervals $\mathcal{C}_N = \{C_1, C_2, \ldots, C_n\}$ with density ρ. Let H'_1, $H'_2, \ldots,$ $H'_{L'}$ be all the maximal intervals of minimum density $\rho'_{\min}$ of solid valuation intervals $\mathcal{C}'_N(i) = \{C_1, C_2, \ldots, C_{i-1}, C'_i, C_{i+1}, \ldots, C_n\}$ with density ρ' (the argument below can be extended to the case when $\mathcal{C}'_N(i)$ is not solid valuation intervals). Thus, $\rho(H_\ell) = \rho_{\min}$ for each $\ell = 1, 2, \ldots, L$, and $\rho'(H'_{\ell'}) = \rho'_{\min}$ for each $\ell' = 1, 2, \ldots, L'$. We divide into three cases: (i) $\rho'_{\min} < \rho_{\min}$, (ii) $\rho'_{\min} > \rho_{\min}$, (iii) $\rho'_{\min} = \rho_{\min}$.

(i) $\rho'_{\min} < \rho_{\min}$. In this case, we can show that C'_i is contained in some $H'_{\ell'}$. By symmetry, we can assume $\ell' = 1$ and $C'_i \subseteq H'_1$.

Suppose contrarily that $C'_i \not\subseteq H'_1$. Then we have the following.

If $C_i \not\subseteq H'_1$ then $\rho'(H'_1) = \rho(H'_1)$ since $C'_i \not\subseteq H'_1$, and $\rho'_{\min} = \rho'(H'_1) = \rho(H'_1) \ge \rho_{\min}$. Otherwise (i.e., if $C_i \subseteq H'_1$), the number $n'_{N(H'_1)}$ of valuation intervals of $\mathcal{C}'_N$ in H'_1 is equal to the number of $n_{N(H'_1)}$ of valuation intervals of $\mathcal{C}_N$ in H'_1 minus 1 (i.e., $n'_{N(H'_1)} = n_{N(H'_1)} - 1$), since C_i is in H'_1, but C'_i is not in H'_1 and $\rho'_{\min} = \rho'(H'_1) > \rho(H'_1) \geq \rho_{\min}$. Thus, in either case, a contradiction that $\rho'_{\min} < \rho_{\min}$.

Thus, we have $C'_i \subseteq H'_1$. Since the allocation $A'_N(i) = (A'_j : j \in N)$ obtained by mechanism $\mathcal{M}$ satisfies $A'_i \subseteq C'_i \subseteq H'_1$ we have $\rho'_{\min} n_{H'_1} = \mathrm{csize}(H'_1)$ and $\mathrm{ut}_i(A'_i) = \mathrm{csize}(A'_i \cap C_i) \leq \mathrm{csize}(A'_i) = \rho'_{\min} < \rho_{\min} \leq \mathrm{ut}_i(A_i)$.

(ii) $\rho'_{\min} > \rho_{\min}$. In this case, we can show that C_i is contained in some H_ℓ by an argument similar to one in the case (i). By symmetry, we can assume $\ell = 1$ and $C_i \subseteq H_1$.

Since $\rho'(X_i) \geq \rho'_{\min}$ for each interval X_i with $C'_i \subseteq X_i$, we have $\mathrm{csize}(A'_i) \geq \rho'_{\min}$. Similarly, for each $j \in N(H_1)$ with valuation interval $C_j \subseteq H_1$ and for each interval X_j with $C_j \subseteq X_j$, we have $\rho'(X_j) \geq \rho'_{\min}$ and $\mathrm{csize}(A'_j) \geq \rho'_{\min}$. Since $\rho'_{\min} > \rho_{\min}$, $A'_j \subseteq C_j$ for all $j \in N(H_1) \setminus \{i\}$, and $A'_i \subseteq C'_i$ ($A'_i \cap C_i \subseteq C_i$, but $A'_i \subseteq C_i$ may not hold), we have $\mathrm{csize}(H_1) \geq \sum_{j \in N(H_1)} \mathrm{csize}(A'_j \cap C_j)$ and

$$
\begin{aligned}
\mathrm{ut}_i(A'_i) &= \mathrm{csize}(A'_i \cap C_i) \\
&\leq \mathrm{csize}(H_1) - \sum_{j \in N(H_1)\setminus\{i\}} \mathrm{csize}(A'_j \cap C_j) = \mathrm{csize}(H_1) - \sum_{j \in N(H_1)\setminus\{i\}} \mathrm{csize}(A'_j) \\
&\leq \mathrm{csize}(H_1) - \rho'_{\min}(n_{H_1} - 1) \\
&= n_{N(H_1)}\rho_{\min} - \rho'_{\min}(n_{H_1} - 1) = \rho_{\min} - (n_{N(H_1)} - 1)(\rho'_{\min} - \rho_{\min}) \\
&< \rho_{\min} = \mathrm{ut}_i(A_i).
\end{aligned}
$$

(iii) $\rho'_{\min} = \rho_{\min}$. By symmetry we can assume that this case can be divided into four subcases as follows:

(a) $C_i \subseteq H_1$ and $C'_i \subseteq H'_1$,

(b) $C_i \subseteq H_1$ and $C'_i \not\subseteq H'_{\ell'}$ for all $\ell' = 1, 2, \ldots, L'$,

(c) $C_i \not\subseteq H_\ell$ for all $\ell = 1, 2, \ldots, L$ and $C'_i \subseteq H'_1$,

(d) $C_i \not\subseteq H_\ell$ for all $\ell = 1, 2, \ldots, L$ and $C'_i \not\subseteq H'_{\ell'}$ for all $\ell' = 1, 2, \ldots, L'$.

In subcase (a), $\mathrm{ut}_i(A_i) = \mathrm{csize}(A_i) = \rho_{\min}$ and $\mathrm{csize}(A'_i) = \rho'_{\min} = \rho_{\min}$ and thus, we have $\mathrm{ut}_i(A'_i) = \mathrm{csize}(A'_i \cap C_i) \leq \mathrm{csize}(A'_i) = \rho'_{\min} = \rho_{\min} = \mathrm{ut}_i(A_i)$.

In subcase (c), $\mathrm{ut}_i(A_i) = \mathrm{csize}(A_i) \geq \rho_{\min}$ and $\mathrm{csize}(A'_i) = \rho'_{\min} = \rho_{\min}$ and thus, we have $\mathrm{ut}_i(A'_i) = \mathrm{csize}(A'_i \cap C_i) \leq \mathrm{csize}(A'_i) = \rho'_{\min} = \rho_{\min} \leq \mathrm{ut}_i(A_i)$.

In subcase (b), by the same argument for the case (ii), we can show $\mathrm{ut}_i(A'_i) = \mathrm{csize}(A'_i \cap C_i) \leq \mathrm{csize}(A'_i) \leq \mathrm{ut}_i(A_i)$ as follows.

Since $\rho'(X_i) \geq \rho'_{\min}$ for each interval X_i with $C'_i \subseteq X_i$, we have $\mathrm{csize}(A'_i) \geq \rho'_{\min}$. Similarly, for each $j \in N(H_1)$ with valuation interval $C_j \subseteq H_1$ and for each interval X_j with $C_j \subseteq X_j$, we have $\rho'(X_j) \geq \rho'_{\min}$ and $\mathrm{csize}(A'_j) \geq \rho'_{\min}$. Since $\rho'_{\min} = \rho_{\min}$ and $A'_j \subseteq C_j$ for all $j \in N(H_1) \setminus \{i\}$, $A'_i \subseteq C'_i$ ($A'_i \cap C_i \subseteq C_i$, but $A'_i \subseteq C_i$ may not hold), we have $\mathrm{csize}(H_1) \geq \sum_{j \in N(H_1)} \mathrm{csize}(A'_j \cap C_j)$ and

$$
\begin{aligned}
\mathrm{ut}_i(A'_i) &= \mathrm{csize}(A'_i \cap C_i) \\
&\leq \mathrm{csize}(H_1) - \sum_{j \in N(H_1)\setminus\{i\}} \mathrm{csize}(A'_j \cap C_j) = \mathrm{csize}(H_1) - \sum_{j \in N(H_1)\setminus\{i\}} \mathrm{csize}(A'_j) \\
&\leq \mathrm{csize}(H_1) - \rho'_{\min}(n_{H_1} - 1) = n_{N(H_1)}\rho_{\min} - \rho'_{\min}(n_{H_1} - 1) = \rho_{\min} = \mathrm{ut}_i(A_i).
\end{aligned}
$$

In subcase (d), since $C_i \not\subseteq H_\ell$ $(\ell = 1, 2, \ldots, L)$ and $C'_i \not\subseteq H'_{\ell'}$ $(\ell' = 1, 2, \ldots, L')$, we have $\rho(X_i) > \rho_{\min}$ for each X_i with $C_i \subseteq X_i$ and $\rho'(X'_i) > \rho'_{\min}$ for each X'_i with $C'_i \subseteq X'_i$. We show $\mathrm{ut}_i(A'_i) \leq \mathrm{ut}_i(A_i)$ by induction on the number of calls of Procedure CutCake$(P, D, \mathcal{D}_P)$.

Suppose that $C'_i \subseteq H_\ell$ for some $\ell = 1, 2, \ldots, L$. We can assume $C'_i \subseteq H_1$ by symmetry. Then, the number $n'_{N(H_1)}$ of valuation intervals of $\mathcal{C}'_N$ in H_1 is equal to the number of $n_{N(H_1)}$ of valuation intervals of $\mathcal{C}_N$ in H_1 plus 1 (i.e., $n'_{N(H_1)} = n_{N(H_1)} + 1$), since C'_i is in H_1, but C_i is not in H_1. This implies that $\rho'_{\min} \leq \rho'(H_1) < \rho(H_1) = \rho_{\min}$, a contradiction that $\rho'_{\min} = \rho_{\min}$.

Thus, we have that no H_ℓ ($\ell = 1, 2, \ldots, L$) contains C'_i. This implies that $\rho'(H_\ell) = \rho(H_\ell) = \rho_{\min} = \rho'_{\min}$ and H_ℓ is an interval of minimum density in valuation intervals $\mathcal{C}'_N$.

Similarly, we can show that no $H'_{\ell'}$ ($\ell' = 1, 2, \ldots, L'$) contains C_i, and each $H'_{\ell'}$ is an interval of minimum density in valuation intervals $\mathcal{C}_N$.

Thus, we can claim that each H_ℓ is a maximal interval of minimum density in valuation intervals $\mathcal{C}'_N$ and that each $H'_{\ell'}$ is a maximal interval of minimum density in valuation intervals $\mathcal{C}_N$.

Suppose contrarily that some H_ℓ were properly contained in a maximal interval $H'_{\ell'}$ of minimum density in valuation intervals $\mathcal{C}'_N$. Then H_ℓ would not be a maximal interval of minimum density in valuation intervals $\mathcal{C}_N$, since $H'_{\ell'}$ is an interval of minimum density in valuation intervals $\mathcal{C}_N$. This is a contradiction that H_ℓ is a maximal interval of minimum density in valuation intervals $\mathcal{C}_N$. Similarly, we can show that each $H'_{\ell'}$ is a maximal interval of minimum density in valuation intervals $\mathcal{C}_N$. Thus, we have $L' = L$ and, by symmetry, we can assume $H_\ell = H'_\ell$ for each $\ell = 1, 2, \ldots, L$.

When Procedure CutCake$(C, N, \mathcal{C}_N)$ is called, the cake-cutting problem of type (ii) for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{D}_P = \{D_k = C_k \setminus \sum_{\ell=1}^{L} H_\ell \mid C_k \in \mathcal{C}_N \setminus \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\}$ is obtained. Similarly, when Procedure CutCake$(C, N, \mathcal{C}'_N)$ is called, the cake-cutting problem of type (ii) for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{D}'_P = \{D'_k = C'_k \setminus \sum_{\ell=1}^{L} H_\ell \mid C'_k \in \mathcal{C}'_N \setminus \sum_{\ell=1}^{L} \mathcal{C}'_{N(H_\ell)}\}$ is obtained.

Note that $D_k = D'_k$ for all $k \in P \setminus \{i\}$ and that $D_i = C_i \setminus \sum_{\ell=1}^{L} H_\ell$ and $D'_i = C'_i \setminus \sum_{\ell=1}^{L} H_\ell$. Furthermore, note that, the allocated piece A_i to player i in the cake-cutting problem of type (ii) for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{D}_P = \{D_k = C_k \setminus \sum_{\ell=1}^{L} H_\ell \mid C_k \in \mathcal{C}_N \setminus \sum_{\ell=1}^{L} \mathcal{C}'_{N(H_\ell)}\}$ is the same as the allocated piece A_i to player i in the original cake-cutting problem. Similarly, the allocated piece A'_i to player i in the cake-cutting problem of type (ii) for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{D}'_P = \{D_k = C'_k \setminus \sum_{\ell=1}^{L} H_\ell \mid C'_k \in \mathcal{C}'_N \setminus \sum_{\ell=1}^{L} \mathcal{C}'_{N(H_\ell)}\}$ is the same as the allocated piece to player i in the original cake-cutting problem with valuation intervals $\mathcal{C}'_N(i) = \{C_1, C_2, \ldots, C_{i-1}, C'_i, C_{i+1}, \ldots, C_n\}$.

By induction hypothesis, $\mathrm{ut}_i(A_i) \geq \mathrm{ut}_i(A'_i)$ holds, where $\mathrm{ut}_i(A_i) = \mathrm{csize}(A_i \cap C_i) = \mathrm{csize}(A_i)$ is the value of the allocated piece A_i to player i in the cake-cutting problem of type (ii) for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{D}_P = \{D_k = C_k \setminus \sum_{\ell=1}^{L} H_\ell \mid C_k \in \mathcal{C}_N \setminus \sum_{\ell=1}^{L} \mathcal{C}_{N(H_\ell)}\}$ and $\mathrm{ut}_i(A'_i) = \mathrm{csize}(A'_i \cap C_i)$ is the value of the allocated piece A'_i to player i in the cake-cutting problem of type (ii) for the cake $D = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $P = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{D}'_P = \{D'_k = C'_k \setminus \sum_{\ell=1}^{L} H_\ell \mid C'_k \in \mathcal{C}'_N \setminus \sum_{\ell=1}^{L} \mathcal{C}'_{N(H_\ell)}\}$.

Thus, we have completed Proof of Theorem 5.1. □