数理解析研究所講究録 1372 巻 2004 年 183-189

Boundary control problems for viscoelastic systems with long memory

Jin-Soo Hwang (神戸大学大学院・自然科学研究科 黄 ジン守)
Division of Mathematical and Material Science,
The Graduate School of Science and Technology, Kobe University

Shin-ichi Nakagiri (神戸大学工学部 中桐 信一)
Department of Applied Mathematics, Faculty of Engineering, Kobe University, JAPAN.

1 Introduction

In the study of viscoelastic materials, the state of stresses at the instant t defends on the strain at the instant t, but also on the strains at the instants previous to the present instant t. From this standpoint of view, the viscoelastic equations with *long memory* are introduced. The qualification of long memory is given by the Volterra integrals on the effects of memory of materials. We shall give the description of the linear viscoelastic systems with long memory in the three dimensional Euclidean space $\mathbf{R}^3$.

Let Ω be an open and bounded set in $\mathbf{R}^3$ with sufficiently smooth boundary $\partial\Omega$. Let $T > 0$ be fixed and let $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\partial\Omega$. We denote by $y = (y_1, y_2, y_3)$ a displacement field in $\mathbf{R}^3$ and a_{ijkh} are the coefficients of instantaneous elasticity. The system of linear viscoelastic equations with long memory is described by

$$\frac{\partial^2 y_i}{\partial t^2} - \frac{\partial}{\partial x_j} a_{ijkh}\epsilon_{kh}(y) - \int_0^t \frac{\partial}{\partial x_j} b_{ijkh}(t-s,x)\epsilon_{kh}(y)ds = f_i, \quad i = 1,2,3, \tag{1.1}$$

where $\epsilon_{kh}(y) = \frac{1}{2}(\frac{\partial y_h}{\partial x_k} + \frac{\partial y_k}{\partial x_h})$ is a strain tensor element, b_{ijkh} are the coefficients of elasticity by taking into the memory effects of the material, $f = (f_1, f_2, f_3)$ is an external force. Throughout this paper we assume that the coefficients a_{ijkh} and b_{ijkh} satisfy

$$\begin{cases} a_{ijkh},\ b_{ijkh} \in L^\infty(Q) \text{ for all } i,j,k,h, \\ a_{ijkh} = a_{jikh} = a_{khij}, \\ a_{ijkh}\xi_{ij}\xi_{kh} \geq \alpha\xi_{ij}\xi_{ij}, \quad \exists\alpha > 0, \quad \forall\xi_{ij} \in \mathbf{R}, \quad \xi_{ij} = \xi_{ji}; \end{cases} \tag{1.2}$$

and that a_{ijkh} and b_{ijkh} have the following time regularities:

$$\begin{cases} t \to a_{ijkh}(t,\cdot) \ \in L^\infty(\Omega) \ \text{ is continuousely differentiable and} \\ \frac{\partial a_{ijkh}}{\partial t} \ \in L^\infty(Q) \text{ for all } \ i,j,k,h; \\ t \to b_{ijkh}(t,\cdot) \ \in L^\infty(\Omega) \ \text{ is continuousely differentiable and} \\ \frac{\partial b_{ijkh}}{\partial t} \ \in L^\infty(Q) \text{ for all } \ i,j,k,h. \end{cases} \tag{1.3}$$

The purpose of this paper is to study the boundary control problems for the viscoelastic system (1.1). First we consider the following Neumann boundary control system

$$\begin{cases} \dfrac{\partial^2 y_i(v)}{\partial t^2} - \dfrac{\partial}{\partial x_j} a_{ijkh}\epsilon_{kh}(y(v)) - \displaystyle\int_0^t \dfrac{\partial}{\partial x_j} b_{ijkh}(t-s,x)\epsilon_{kh}(y(v))ds = f_i \text{ in } Q, \\ \Big[a_{ijkh}\epsilon_{kh}(y(v)) + \displaystyle\int_0^t b_{ijkh}(t-s,x)\epsilon_{kh}(y(v))ds\Big]\mathbf{n}_j = v_i \text{ on } \Sigma, \\ y_i(v;0,x) = y_i^0, \quad \dfrac{\partial y_i}{\partial t}(v;0,x) = y_i^1 \text{ in } \Omega, \quad i=1,2,3, \end{cases} \tag{1.4}$$

where $f = (f_1, f_2, f_3) \in [L^2(0,T;(H^1(\Omega))')]^3$, $y_0 = (y_1^0, y_2^0, y_3^0) \in [L^2(\Omega)]^3$, $y_1 = (y_1^1, y_2^1, y_3^1) \in [(H^1(\Omega))']^3$, $\mathbf{n}_j$ is the j-th outward normal to $\partial\Omega$, and the boundary control variables $v = (v_1, v_2, v_3)$ are assumed to satisfy the condition

$$v_i \in L^2(\Sigma), \quad i = 1,2,3. \tag{1.5}$$

It is verified by the method of transposition (cf. Lions [2], Lions and Magenes [3], that there is a unique *transposed* solution $y(v) \in [L^2(Q)]^3$ of (1.4) for each v satisfying (1.5). Therefore, for the controlled system (1.4) we can attach the following quadratic cost functional given by

$$J(v) = \sum_{i=1}^{3} \int_Q (y_i(v) - z_{di})^2 dxdt + \sum_{i=1}^{3} \nu_i \int_\Sigma v_i{}^2 dxdt, \tag{1.6}$$

where z_{di} are desired values in $L^2(Q)$ and $\nu_i > 0, i = 1,2,3$.

Next we consider the following Dirichlet boundary control system

$$\begin{cases} \dfrac{\partial^2 y_i(v)}{\partial t^2} - \dfrac{\partial}{\partial x_j} a_{ijkh}\epsilon_{kh}(y(v)) - \displaystyle\int_0^t \dfrac{\partial}{\partial x_j} b_{ijkh}(t-s,x)\epsilon_{kh}(y(v))ds = f_i \text{ in } Q, \\ y_i(v) = v_i \text{ on } \Sigma, \\ y_i(v;0,x) = y_i^0, \quad \dfrac{\partial y_i}{\partial t}(v;0,x) = y_i^1 \text{ in } \Omega, \quad i=1,2,3, \end{cases} \tag{1.7}$$

where $f = (f_1, f_2, f_3) \in [L^2(Q)]^3$, $y_0 = (y_1^0, y_2^0, y_3^0) \in [H^1(\Omega)]^3$ and $y_1 = (y_1^1, y_2^1, y_3^1) \in [L^2(\Omega)]^3$. Further in (1.7) we assume the stronger regularity condition on $v = (v_1, v_2, v_3)$ such that

$$v_i \in H_0^2(\Sigma), \quad i = 1,2,3. \tag{1.8}$$

We can verify that there exists a unique *weak* solution $y(v)$ of (1.7) in the sense of Dautray and Lions [1] for each v satisfying (1.8). The solution $y(v)$ has the regularity $y(v) \in [L^2(Q)]^3$, $y(v) \in [C([0,T];L^2(Q))]^3$. Hence, for the control system (1.7) we can attach the following two types of quadratic cost functionals:

$$J(v) = \sum_{i=1}^{3} \int_Q (y_i(v) - z_{di})^2 dxdt + \sum_{i=1}^{3} \nu_i \|v_i\|^2_{H_0^2(\Sigma)}; \tag{1.9}$$

$$J(v) = \sum_{i=1}^{3} \int_\Omega (y_i(v;T) - z_{di})^2 dx + \sum_{i=1}^{3} \nu_i \|v_i\|^2_{H_0^2(\Sigma)}. \tag{1.10}$$

In (1.9) and (1.10) z_{di} are desired values in $L^2(Q)$ and $L^2(\Omega)$, respectively, and $\nu_i > 0, i = 1,2,3$.

In this paper we establish the necessary conditions of optimality both for the Neumann boundary control system (1.4) with the cost (1.6) and the Dirichlet boundary control system (1.7) with the cost (1.9) or (1.10) by introducing proper adjoint systems.

2 Neumann boundary control problems

In this section we study the Neumann boundary control problems of (1.4). To formulate the problems, we need to introduce the *transposed* solution of (1.4) by the transposition method.

Lemma 2.1 Assume that $f \in [L^2(0,T;(H^1(\Omega))')]^3$, $y_0 \in [L^2(\Omega)]^3$, $y_1 \in [(H^1(\Omega))']^3$ and (1.5) in the system (1.4). Then there exists a unique element $y = (y_1, y_2, y_3)$ in $[L^2(Q)]^3$ such that

$$\begin{cases} \sum_{i=1}^{3} \int_Q y_i\Big(\frac{\partial^2 \phi_i}{\partial t^2} - \frac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(\phi) - \int_t^T \frac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(\phi)ds\Big)dxdt \\ = \sum_{i=1}^{3} \int_Q f_i\phi_i dxdt - \sum_{i=1}^{3}\int_\Omega y_i^0 \frac{\partial \phi_i}{\partial t}(0,x)dx + \sum_{i=1}^{3}\int_\Omega y_i^1 \phi_i(0,x)dx + \sum_{i=1}^{3}\int_\Sigma v_i\phi_i d\Sigma \end{cases}$$

for all function ϕ such that $\phi \in X$, where

$$\begin{aligned} X = \Big\{ \ & \phi = (\phi_1, \phi_2, \phi_3) \ \mid \ \phi_i \in L^2(0,T;H^1(\Omega)), \\ & \frac{\partial^2 \phi_i}{\partial t^2} - \frac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(\phi) - \int_t^T \frac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(\phi)ds \ \in L^2(Q), \\ & \Big[a_{ijkh}\epsilon_{ij}(\phi) + \int_t^T b_{ijkh}(s-t,x)\epsilon_{ij}(\phi)ds\Big]\mathbf{n}_h = 0 \ \text{ on } \Sigma, \\ & \phi_i(T,x) = \frac{\partial \phi_i}{\partial t}(T,x) = 0, \ \ i = 1,2,3\Big\}. \end{aligned}$$

Here we note that

$$\phi_i \in C([0,T];H^1(\Omega)), \quad \frac{\partial \phi_i}{\partial t} \in C([0,T];L^2(\Omega)), \quad \phi_i|_\Sigma \in H^{\frac{1}{2}}(\Sigma) \subset L^2(\Sigma)$$

for all $\phi = (\phi_1, \phi_2, \phi_3) \in X$.

By Lemma 2.1, for the system (1.4) we can consider the cost given by

$$J(v) = \sum_{i=1}^{3} \int_Q (y_i(v) - z_{di})^2 dxdt + \sum_{i=1}^{3} \nu_i \int_\Sigma v_i^2 d\Sigma, \quad v \in [L^2(\Sigma)]^3, \tag{2.1}$$

where $\nu_i > 0$ and $z_{di} \in L^2(Q)$, $i = 1,2,3$. Let $\mathcal{U}_{ad} \subset [L^2(\Sigma)]^3$ be a closed and convex set of admissible controls. The element $u \in \mathcal{U}_{ad}$ such that

$$\inf_{v \in \mathcal{U}_{ad}} J(v) = J(u) \tag{2.2}$$

is called the optimal control. It is easily verified that the optimal control u for the cost (2.1) exists uniquely by the positivity $\nu_i > 0$ for $i = 1,2,3$. Then the optimality condition is given by

$$\sum_{i=1}^{3} \int_Q (y_i(u) - z_{di})(y_i(v) - y_i(u))dxdt + \sum_{i=1}^{3} \nu_i \int_Q (u_i)(v_i - u_i)d\Sigma \geq 0, \ \ \forall v \in \mathcal{U}_{ad}, \tag{2.3}$$

where u is the optimal control for (2.1). We want to write down the condition (2.3) in terms of adjoint state equation. For this, we introduce the adjoint system by

$$\begin{cases} \frac{\partial^2 p_i(u)}{\partial t^2} - \frac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(p(u)) - \int_t^T \frac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds = y_i(u) - z_{di} \ \text{ in } \ Q, \\ \Big[a_{ijkh}\epsilon_{ij}(p(u)) + \int_t^T b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds\Big]\mathbf{n}_h = 0 \ \text{ on } \Sigma, \\ p_i(u;T,x) = 0, \quad \frac{\partial p_i}{\partial t}(u;T,x) = 0 \ \text{ in } \Omega, \ i = 1,2,3. \end{cases} \tag{2.4}$$

Since $y_i(u) - z_{di} \in L^2(Q)$, $i = 1,2,3$ by assumption, we can verify that the weak solution $p = (p_1, p_2, p_3) \in X$ of (2.4) exists uniquely. The optimality condition can be obtained by the following theorem.

Theorem 2.1 The optimal control $u \in \mathcal{U}_{ad} \subset [L^2(\Sigma)]^3$ for the cost (2.1) is characterized by the following system of equations and inequality:

$$\begin{cases} \dfrac{\partial^2 y_i(v)}{\partial t^2} - \dfrac{\partial}{\partial x_j} a_{ijkh}\epsilon_{kh}(y(v)) - \displaystyle\int_0^t \dfrac{\partial}{\partial x_j} b_{ijkh}(t-s,x)\epsilon_{kh}(y(v))ds = f_i \text{ in } Q, \\ \Big[a_{ijkh}\epsilon_{kh}(y(v)) + \displaystyle\int_0^t b_{ijkh}(t-s,x)\epsilon_{kh}(y(v))ds\Big]\mathbf{n}_j = v_i \text{ on } \Sigma, \\ y_i(v;0,x) = y_i^0(x), \quad \dfrac{\partial y_i}{\partial t}(v;0,x) = y_i^1(x) \text{ in } \Omega, \quad i = 1,2,3, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2} - \dfrac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(p(u)) - \displaystyle\int_t^T \dfrac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds = y_i(u) - z_{di} \text{ in } Q, \\ \Big[a_{ijkh}\epsilon_{ij}(p(u)) + \displaystyle\int_t^T b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds\Big]\mathbf{n}_h = 0 \text{ on } \Sigma, \\ p_i(u;T,x) = 0, \quad \dfrac{\partial p_i}{\partial t}(u;T,x) = 0 \text{ in } \Omega, \quad i = 1,2,3, \end{cases}$$

$$\sum_{i=1}^{3} \int_\Sigma (p_i(u) + \nu_i u_i)(v_i - u_i)d\Sigma \geq 0, \quad \forall v = (v_1, v_2, v_3) \in \mathcal{U}_{ad} \subset [L^2(\Sigma)]^3.$$

Example 2.1 Assume that the admissible set $\mathcal{U}_{ad}$ is given by

$$\mathcal{U}_{ad} = \Big\{v = (v_1, v_2, v_3) \mid v_i \geq 0 \text{ on } \Sigma, \ i = 1,2,3\Big\}.$$

Then by Theorem 2.1 the optimal control $u = (u_1, u_2, u_3)$ is given by

$$u_i = \Big[a_{ijkh}\epsilon_{kh}(y) + \int_0^t b_{ijkh}(t-s,x)\epsilon_{kh}(y)ds\Big]\mathbf{n}_j, \quad i = 1,2,3,$$

where y is the solution of the following unilateral problem on y and p:

$$\begin{cases} \dfrac{\partial^2 y_i}{\partial t^2} - \dfrac{\partial}{\partial x_j} a_{ijkh}\epsilon_{kh}(y) - \displaystyle\int_0^t \dfrac{\partial}{\partial x_j} b_{ijkh}(t-s,x)\epsilon_{kh}(y)ds = f_i \text{ in } Q, \\ \dfrac{\partial^2 p_i}{\partial t^2} - \dfrac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(p) - \displaystyle\int_t^T \dfrac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(p)ds = y_i - z_{di} \text{ in } Q, \ i = 1,2,3, \end{cases}$$

$$\begin{cases} \Big[a_{ijkh}\epsilon_{kh}(y) + \displaystyle\int_0^t b_{ijkh}(t-s,x)\epsilon_{kh}(y)ds\Big]\mathbf{n}_j \geq 0 \text{ on } \Sigma, \\ p_i + \nu_i\Big[a_{ijkh}\epsilon_{kh}(y) + \displaystyle\int_0^t b_{ijkh}(t-s,x)\epsilon_{kh}(y)ds\Big]\mathbf{n}_j \geq 0 \text{ on } \Sigma, \\ \Big[a_{ijkh}\epsilon_{ij}(p) + \displaystyle\int_t^T b_{ijkh}(s-t,x)\epsilon_{ij}(p)ds\Big]\mathbf{n}_h = 0 \text{ on } \Sigma, \\ \Big[a_{ijkh}\epsilon_{kh}(y) + \displaystyle\int_0^t b_{ijkh}(t-s,x)\epsilon_{kh}(y)ds\Big]\mathbf{n}_j \times \\ \Big(p_i + \nu_i \Big[a_{ijkh}\epsilon_{kh}(y) + \displaystyle\int_0^t b_{ijkh}(t-s,x)\epsilon_{kh}(y)ds\Big]\mathbf{n}_j\Big) = 0, \text{ on } \Sigma, \ i = 1,2,3. \end{cases}$$

$$\begin{cases} y_i(0,x) = y_i^0(x), \quad \dfrac{\partial y_i}{\partial t}(0,x) = y_i^1(x) \text{ in } \Omega, \\ p_i(T,x) = 0, \quad \dfrac{\partial p_i}{\partial t}(T,x) = 0 \text{ in } \Omega, \quad i = 1,2,3. \end{cases}$$

We note that the above equations and inequalities are given in the sense of distribution.

3 Dirichlet boundary control problems

In this section we consider the Dirichlet boundary control system (1.7). At first, we define an inner product of $[H_0^2(\Sigma)]^3$ by

$$(\phi,\psi)_{[H_0^2(\Sigma)]^3}=\sum_{i=1}^{3}\int_\Sigma \Delta_\Gamma\phi_i(t,x)\Delta_\Gamma\psi_i(t,x)d\Gamma dt+\sum_{i=1}^{3}\int_\Sigma \frac{\partial^2}{\partial t^2}\phi_i(t,x)\frac{\partial^2}{\partial t^2}\psi_i(t,x)d\Gamma dt,$$

where Δ_Γ is the Laplace-Beltrami operator on $\Gamma=\partial\Omega$.

For each $v=(v_1,v_2,v_3)$ satisfying (1.8) we can construct $\varphi=(\varphi_1,\varphi_2,\varphi_3)$ such that

$$\begin{cases} \varphi_i\in H^2(\bar{Q}),\ \ \varphi_i=v_i \ \text{ on } \ \Sigma,\\ \varphi_i(0,x)=\dfrac{\partial}{\partial t}\varphi_i(0,x)=0,\ \ i=1,2,3. \end{cases}$$

Let $z_i=y_i(v)-\varphi_i$. Then we have the homogeneous Dirichlet boundary problem

$$\begin{cases} \dfrac{\partial^2 z_i(v)}{\partial t^2}-\dfrac{\partial}{\partial x_j}a_{ijkh}\epsilon_{kh}(z(v))-\displaystyle\int_0^t\frac{\partial}{\partial x_j}b_{ijkh}(t-s,x)\epsilon_{kh}(z(v))ds=g_i \ \text{ in } \ Q,\\ z_i(v)=0 \ \text{ on } \Sigma,\\ z_i(v;0,x)=y_i^0(x),\ \ \dfrac{\partial z_i}{\partial t}(v;0,x)=y_i^1(x) \ \text{ in } \Omega,\ \ i=1,2,3, \end{cases}\tag{3.1}$$

where

$$g_i=f_i-[\frac{\partial^2\varphi_i(v)}{\partial t^2}-\frac{\partial}{\partial x_j}a_{ijkh}\epsilon_{kh}(\varphi(v))-\int_0^t\frac{\partial}{\partial x_j}b_{ijkh}(t-s,x)\epsilon_{kh}(\varphi(v))ds]\in L^2(Q),\ \ i=1,2,3.$$

The system (3.1) admit a unique weak solution $z=(z_1(v),z_2(v),z_3(v))$ under the conditions $f\in[L^2(Q)]^3$, $y_0\in[H^1(\Omega)]^3$ and $y_1\in[L^2(\Omega)]^3$ and (1.8) (cf. Dautray and Lions [1]). Thus we have the solutions $y_i=z_i(v)+\varphi_i$, $i=1,2,3$ of (1.7). Hence $y\in[L^2(Q)]^3$ and $y(T)\in[L^2(\Omega)]^3$ follow.

3.1 Case of distributive value observations

In this case the cost functional is given by

$$J(v)=\sum_{i=1}^{3}\int_Q(y_i(v)-z_{di})^2dxdt+\sum_{i=1}^{3}\nu_i\|v_i\|^2_{H_0^2(\Sigma)},\ \ \nu_i>0,\ i=1,2,3,\tag{3.2}$$

where $z_{di}\in L^2(Q)$, $i=1,2,3$. Let $\mathcal{U}_{ad}$ be a closed and convex subset of $H_0^2(\Sigma)$. Then there exists a unique optimal control $u\in\mathcal{U}_{ad}$ for the cost (3.2). The optimal control $u=(u_1,u_2,u_3)$ is characterized by

$$\sum_{i=1}^{3}\int_Q(y_i(u)-z_{di})(y_i(v)-y_i(u))dxdt+\sum_{i=1}^{3}\nu_i(u_i,v_i-u_i)_{H_0^2(\Sigma)}\geq 0,\quad \forall v=(v_1,v_2,v_3)\in\mathcal{U}_{ad}.$$

We introduce the adjoint system by

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2}-\dfrac{\partial}{\partial x_h}a_{ijkh}\epsilon_{ij}(p(u))-\displaystyle\int_t^T\frac{\partial}{\partial x_h}b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds=y_i(u)-z_{di} \ \text{ in } \ Q,\\ p_i(u)=0 \ \text{ on } \Sigma,\\ p_i(u;T,x)=0,\ \ \dfrac{\partial p_i}{\partial t}(v;T,x)=0 \ \text{ in } \Omega,\ \ i=1,2,3, \end{cases}$$

where $y_i(u) - z_{di} \in L^2(Q)$, $i = 1,2,3$. There exists a unique weak solution $p = p(u)$ of this adjoint system. Hence we have the following optimality condition for the cost (3.2).

Theorem 3.1 The optimal control $u \in \mathcal{U}_{ad} \subset [H_0^2(\Sigma)]^3$ or the cost (3.2) is characterized by the following system of equations and inequality:

$$\begin{cases} \dfrac{\partial^2 y_i(u)}{\partial t^2} - \dfrac{\partial}{\partial x_j} a_{ijkh}\epsilon_{kh}(y(u)) - \displaystyle\int_0^t \dfrac{\partial}{\partial x_j} b_{ijkh}(t-s,x)\epsilon_{kh}(y(u))ds = f_i \text{ in } Q, \\ y_i(u) = u_i \text{ on } \Sigma, \\ y_i(u;0,x) = y_i^0, \quad \dfrac{\partial y_i}{\partial t}(u;0,x) = y_i^1 \text{ in } \Omega, \quad i = 1,2,3, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2} - \dfrac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(p(u)) - \displaystyle\int_t^T \dfrac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds = y_i(u) - z_{di} \text{ in } Q, \\ p_i(u) = 0 \text{ on } \Sigma, \\ p_i(u;T,x) = 0, \quad \dfrac{\partial p_i}{\partial t}(v;T,x) = 0 \text{ in } \Omega, \quad i = 1,2,3, \end{cases}$$

$$\sum_{i=1}^{3}\int_\Sigma \Big[-a_{ijkh}\epsilon_{ij}(p) - \int_t^T b_{ijkh}(s-t,x)\epsilon_{ij}(p)ds\Big]\mathbf{n}_h(v_i - u_i)d\Sigma$$
$$+\sum_{i=1}^{3}\int_\Sigma \nu_i((\Delta_\Gamma + \frac{\partial^2}{\partial t^2})u_i)((\Delta_\Gamma + \frac{\partial^2}{\partial t^2})(v_i - u_i))d\Sigma \geq 0, \quad \forall v \in \mathcal{U}_{ad}.$$

3.2 Case of terminal value observations

In this case the cost functional is given by

$$J(v) = \sum_{i=1}^{3}\int_\Omega (y_i(v;T) - z_{di})^2 dx + \sum_{i=1}^{3}\nu_i\|v_i\|^2_{H_0^2(\Sigma)}, \quad \forall v = (v_1,v_2,v_3) \in [H_0^2(\Sigma)]^3, \tag{3.3}$$

where $z_{di} \in L^2(\Omega)$, $\nu_i > 0$, $i = 1,2,3$. Let $\mathcal{U}_{ad}$ be a closed and convex subset of $H_0^2(\Sigma)$. Then the optimal control $u = (u_1,u_2,u_3)$ for the cost (3.3) exists uniquely and is characterized by

$$\sum_{i=1}^{3}\int_\Omega (y_i(u;T) - z_{di})(y_i(v;T) - y_i(u;T))dx + \sum_{i=1}^{3}\nu_i(u_i, v_i - u_i)_{H_0^2(\Sigma)} \geq 0, \quad \forall v = (v_1,v_2,v_3) \in \mathcal{U}_{ad}.$$

We introduce the adjoint system by

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2} - \dfrac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(p(u)) - \displaystyle\int_t^T \dfrac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds = 0 \text{ in } Q, \\ p_i(u) = 0 \text{ on } \Sigma, \\ p_i(u;T,x) = 0 \text{ in } \Omega, \\ \dfrac{\partial p_i}{\partial t}(u;T,x) = y_i(u;T,x) - z_{di}(x) \text{ in } \Omega, \quad i = 1,2,3. \end{cases} \tag{3.4}$$

Since $y_i(u;T) - z_{di} \in L^2(\Omega)$, $i = 1,2,3$ by assumption, we can obtain the unique weak solution $p = p(u)$ of (3.4). Hence the optimality condition can be obtained by the following theorem.

Theorem 3.2 The optimal control $u \in \mathcal{U}_{ad} \subset [H_0^2(\Sigma)]^3$ for the cost (3.3) is characterized by the following system of equations and inequality:

$$\begin{cases} \dfrac{\partial^2 y_i(u)}{\partial t^2} - \dfrac{\partial}{\partial x_j} a_{ijkh}\epsilon_{kh}(y(u)) - \displaystyle\int_0^t \dfrac{\partial}{\partial x_j} b_{ijkh}(t-s,x)\epsilon_{kh}(y(u))ds = f_i \text{ in } Q, \\ y_i(u) = u_i \text{ on } \Sigma, \\ y_i(u;0,x) = y_i^0(x), \quad \dfrac{\partial y_i}{\partial t}(u;0,x) = y_i^1(x) \text{ in } \Omega, \quad i = 1,2,3, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 p_i(u)}{\partial t^2} - \dfrac{\partial}{\partial x_h} a_{ijkh}\epsilon_{ij}(p(u)) - \displaystyle\int_t^T \dfrac{\partial}{\partial x_h} b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds = 0 \text{ in } Q, \\ p_i(u) = 0 \text{ on } \Sigma, \\ p_i(u;T,x) = 0 \text{ in } \Omega, \\ \dfrac{\partial p_i}{\partial t}(u;T,x) = y_i(u;T,x) - z_{di}(x) \text{ in } \Omega, \quad i = 1,2,3, \end{cases}$$

$$\sum_{i=1}^{3}\int_\Sigma \Big[a_{ijkh}\epsilon_{ij}(p(u)) + \int_t^T b_{ijkh}(s-t,x)\epsilon_{ij}(p(u))ds\Big]\mathbf{n}_h(v_i - u_i)d\Sigma$$
$$+\sum_{i=1}^{3}\int_\Sigma \nu_i((\Delta_\Gamma + \frac{\partial^2}{\partial t^2})u_i)((\Delta_\Gamma + \frac{\partial^2}{\partial t^2})(v_i - u_i))d\Sigma \geq 0, \quad \forall v \in \mathcal{U}_{ad} \subset [H_0^2(\Sigma)]^3.$$

References

[1] R. Dautray and J. L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, *Springer-Verlag,* Vol. 5, Evolution Problems, 1992.

[2] J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, *Springer-Verlag Berlin Heidelberg New York*, 1971.

[3] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications I, II, *Springer-Verlag Berlin Heidelberg New York*, 1972.