Alternative Randomization for Valuing American Options

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1 Introduction

European-style options, which can only be exercised at its maturity, have closed-form formulas for their values in the standard model pioneered by Black and Scholes [1] and Merton [8]. Although a vast majority of traded options are of American-style optimally exercised before the maturity, there are no closed-form formulas for their values even in the standard model. The principal difficulty in analyzing American options may be the absence of an explicit expression for the early exercise boundary, which is an optimal level of critical asset value where early exercise occurs.

Due to the lack of closed-form formulas for American option values, many approximate and/or numerical solutions have been developed so far. Broadie and Detemple [2] numerically evaluated recent methods for computing American option values. From those numerical experiments, it comes out that a numerical procedure developed by Carr [3] is fast and accurate among existing methods. Carr's procedure is based on valuing an American option with a randomized maturity, so that it is called the randomization approach. The purpose of this paper is to improve Carr's randomization approach by introducing alternative randomization based on an order statistic from an exponential population.

2 Free Boundary Problem

Let \((S_t)_{t\geq 0}\) be the stock price governed by the risk-neutralized diffusion process

\[
\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t, \quad t \geq 0,
\]

(2.1)

where \(r > 0\) is the risk-free interest rate, \(\delta \geq 0\) is a continuous dividend rate, \(\sigma > 0\) is a volatility of the asset returns, and \((W_t)_{t\geq 0}\) is a standard Wiener process on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})\). We consider an American put option written on \((S_t)_{t\geq 0}\), which has maturity date \(T\) and strike price \(K\). Let

\[
P \equiv P(t, S_t) = P(t, S_t; K, r, \delta), \quad 0 \leq t \leq T,
\]

denote the value of the American put option at time \(t\). See Remark 1 below for the call value.

*This paper is an abbreviated version of Kimura [5].
McKean [7] showed that the alive American put value $P$ and an early exercise boundary $(B_t)_{t\in[0,T]}$ can be jointly obtained by solving a free boundary problem, which is specified by the Black-Scholes-Merton PDE
\[
\frac{1}{2}\sigma^2 S^2 P_{SS} + (r - \delta)SP_S - rP + P_t = 0, \quad S > B_t, \tag{2.2}
\]

Together with the boundary conditions
\[
\lim_{S\uparrow\infty} P(t, S) = 0, \tag{2.3}
\]
\[
\lim_{S \downarrow B_t} P(t, S) = K - B_t, \tag{2.4}
\]
\[
\lim_{S \downarrow B_t} P_S(t, S) = -1, \tag{2.5}
\]

And the terminal condition
\[
P(T, S) = (K - S)^+. \tag{2.6}
\]

The condition (2.4) is often called the value-matching condition and (2.5) is called the smooth-pasting or high-contact condition.

It is sometimes convenient to work with the equations where the current time $t$ is replaced by the remaining time until maturity $s \equiv T - t$. From (2.2)–(2.6), the put price for the reversed process $\hat{P}(s, S) \equiv P(T - s, S_{T-s})$ satisfies the PDE
\[
\frac{1}{2}\sigma^2 S^2 \hat{P}_{SS} + (r - \delta)S\hat{P}_S - r\hat{P} - \hat{P}_s = 0, \quad S > \hat{B}_s, \tag{2.7}
\]

With the boundary conditions
\[
\lim_{S\uparrow\infty} \hat{P}(s, S) = 0, \tag{2.8}
\]
\[
\lim_{S \downarrow \hat{B}_s} \hat{P}(s, S) = K - \hat{B}_s, \tag{2.9}
\]
\[
\lim_{S \downarrow \hat{B}_s} \hat{P}_S(s, S) = -1, \tag{2.10}
\]

And the initial condition
\[
\hat{P}(0, S) = (K - S)^+. \tag{2.11}
\]

**Remark 1** Let $C(0, S; K, r, \delta)$ denote the initial value of the associated American call option with the same parameters as those in the put option. McDonald and Schroder [6] proved the parity relation
\[
C(0, S; K, r, \delta) = P(0, K; S, \delta, r). \tag{2.12}
\]

Let $B^P_t \equiv B^P_t(K, r, \delta)$ and $B^C_t \equiv B^C_t(K, r, \delta)$ denote the early exercise boundaries of the American put and call options, respectively. Carr and Chesney [4] showed symmetry relation
\[
B^C_t(K, r, \delta) = \frac{K^2}{B^P_t(K, \delta, r)}. \tag{2.13}
\]

Due to the parity/symmetry relations, the results for the American call option can be derived from the associated put option.
3 Randomization Approach

Carr [3] developed a valuing method for the American put. Carr's randomization approach consists of the following steps:

1. Randomize the maturity date by an *exponentially* distributed random variable $\tilde{T}$ with mean $E[\tilde{T}] = \lambda^{-1} = T$ in order to value the so-called Canadian option.

2. Extend the result to the case that $\tilde{T}$ is distributed as the $n$-stage Erlangian distribution with the same mean $E[\tilde{T}] = T$.

3. Take the limit of the randomized option value by letting $n \to \infty$ to obtain the underlying American option value.

To understand the meaning of the step 3 above, Figure 1 illustrates the convergence of the $n$-stage Erlangian distribution to Dirac's delta function concentrated at the mean $\lambda^{-1} = 1$.

![Figure 1: n-stage Erlangian pdf ($\lambda^{-1} = 1, n = 1, 2, 4, 8, 16, 32$)](image)

Actually, the idea of Carr's randomization is not new. In the theory of integral transforms, this idea goes by the name of the Post-Widder inversion formula [9]: For a continuous function $g(t)$ ($t \geq 0$), define

$$g_n^*(T) = \int_0^\infty g(t) \frac{(nt/T)^{n-1}}{(n-1)!} \frac{n}{T} e^{-nt/T} dt.$$ \hspace{1cm} (3.1)

Then, we have

$$\lim_{n \to \infty} g_n^*(T) = g(T),$$ \hspace{1cm} (3.2)

which is the essential point of Carr's randomization method.

For $\lambda > 0$, let

$$P^* \equiv P^*(\lambda, S) = \int_0^\infty \lambda e^{-\lambda s} \hat{P}(s, S) ds$$ \hspace{1cm} (3.3)
be the Laplace-Carson transform (LCT) of \( \hat{P}(s, S) \). Then, from (2.7)-(2.11), \( P^*(\lambda, S) \) satisfies the ODE

\[
\frac{1}{2}\sigma^2 S^2 P_{SS}^* + (r - \delta)SP_S^* - (\lambda + r)P^* + \lambda(K - S)^+ = 0, \quad S > L^*,
\]

(3.4)
together with the boundary conditions

\[
\begin{align*}
\lim_{S\uparrow \infty} P^*(\lambda, S) &= 0, \\
\lim_{S\downarrow L^*} P^*(\lambda, S) &= K - L^*, \\
\lim_{S\downarrow L^*} P_S^*(\lambda, S) &= -1.
\end{align*}
\]

(3.5)(3.6)(3.7)
The early exercise boundary \( L^* \equiv L^*(\lambda) \) is given by the LCT of \( \hat{B}_s = B_{T-s} \)

\[
L^*(\lambda) = \int_0^\infty \lambda e^{-\lambda s}\hat{B}_s ds,
\]

(3.8)
which is a constant due to the memoryless property of the exponential distribution.

**Theorem 1**

\[
P^*(\lambda, S) = \begin{cases} 
  K - S, & S \leq L^* \\
  \frac{\lambda}{\lambda + r} K - \frac{\lambda}{\lambda + \delta} S + c(S) + b(S) + d(S), & L^* < S < K \\
  p(S) + b(S) + d(S), & S \geq K
\end{cases}
\]

(3.9)
where

\[
\begin{align*}
c(S) &= \frac{1}{\theta_+ - \theta_-} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r} \theta_- \right) K \left(\frac{S}{K}\right)^{\theta_+} \\
p(S) &= \frac{1}{\theta_+ - \theta_-} \frac{\lambda}{\lambda + \delta} \left(1 - \frac{r - \delta}{\lambda + r} \theta_+ \right) K \left(\frac{S}{K}\right)^{\theta_-} \\
b(S) &= -\frac{\theta_+}{\theta_-} c(L^*) \left(\frac{S}{L^*}\right)^{\theta_-} \\
d(S) &= -\frac{1}{\theta_-} \frac{\delta}{\lambda + \delta} L^* \left(\frac{S}{L^*}\right)^{\theta_-}
\end{align*}
\]

and the parameters \( \theta_{\pm} \) are two roots of the quadratic equation \( \frac{1}{2}\sigma^2 \theta^2 + (r - \delta - \frac{1}{2}\sigma^2)\theta - (\lambda + r) = 0 \), i.e.,

\[
\theta_{\pm} = \frac{1}{\sigma^2} \left\{ -(r - \delta - \frac{1}{2}\sigma^2) \pm \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)} \right\}.
\]

(3.14)

**Proof.** See Kimura [5].
Remark 2 The function $c(S) (p(S))$ appeared in (3.9) can be interpreted as the randomized value of a European call (put) paying $(S - K)^+ ((K - S)^+)$. Also, the function $b(S) (d(S))$ can be interpreted as the present value of interest (dividends) received below the early exercise boundary $L^*$. 

Remark 3 Carr’s result for $b(S)$ (when $\delta = 0$) is invalid. The correct one is

$$b^{(1)}(S) = \left(\frac{S}{S_1}\right)^{\gamma-\epsilon} qK \left(RrT + \frac{1}{2\epsilon p}\right) \left(\frac{S_1}{K}\right)^{\gamma+\epsilon},$$

(3.15)
in terms of his notation; cf. [3, Equation (15)].

Theorem 2

(i) The early exercise boundary $L^*$ of the Canadian-American put option satisfies the equation

$$\lambda \left(\frac{L^*}{K}\right)^{\theta_+} = r(\theta_+ - 1) - \delta \theta_+ \frac{L^*}{K}.$$  

(3.16)

(ii) For the limiting case $\lambda \to 0$, we have

$$L^*(0) = \lim_{s \to \infty} \hat{B}_s = \frac{r(\theta_+ - 1)}{\delta \theta_+} K = \frac{\theta^o_+}{\theta_+^o - 1} K,$$

(3.17)

where $\theta_+^o = \lim_{\lambda \to 0^+} \theta_+^o$. In particular, if $\delta = 0$, then

$$L^*(0) = \lim_{s \to \infty} \hat{B}_s = \frac{K}{\sigma^2 \left(1 + \frac{r}{2\sigma^2}\right)}.$$  

(3.18)

(iii) For the limiting case $\lambda \to \infty$, we have

$$\lim_{\lambda \to \infty} L^*(\lambda) = \hat{B}_0 = B_T = \min \left(\frac{r}{\delta}, 1\right) K.$$  

(3.19)

Proof. See Kimura [5].

4 New Randomization Based on an Order Statistic

Let $X_1, \ldots, X_{n+m}$ be independent and exponentially distributed random variables with parameter $\alpha (> 0)$, and let $X(i)$ denote the $i$-th smallest of these random variables ($i = 1, \ldots, n + m$). Then, the probability density function (pdf) of $X_{(n+1)}$ is

$$f(t) = \frac{(n + m)!}{n!(m-1)!} (1 - e^{-\alpha t})^n \alpha e^{-\alpha t}, \quad t \geq 0.$$  

(4.1)

The mean and variance of $X_{(n+1)}$ are given by

$$E[X_{(n+1)}] = \frac{1}{\alpha} \sum_{i=0}^{n} \frac{1}{m + i} \approx \frac{1}{\alpha} \ln \frac{2n + 2m + 1}{2m - 1},$$

(4.2)

$$V[X_{(n+1)}] = \frac{1}{\alpha^2} \sum_{i=0}^{n} \frac{1}{(m + i)^2} \approx \frac{1}{\alpha^2} \ln \frac{2n + 2}{(2m - 1)(2n + 2m + 1)}.$$  

(4.3)
In addition, the modal value of $X_{(n+1)}$ is

$$M[X_{(n+1)}] \equiv \arg\max_t f(t) = \frac{1}{\alpha} \ln \frac{n+m}{m}. \tag{4.4}$$

If we let either $E[X_{(n+1)}] = T$ or $M[X_{(n+1)}] = T$, then $X_{(n+1)}$ can be another candidate for the random maturity $\tilde{T}$, because $\lim_{n,m \to \infty} V[X_{(n+1)}] = 0$. For computational convenience, we adopt the mode matching $M[X_{(n+1)}] = T$, so that $\alpha$ can be determined as

$$\alpha = \frac{1}{T} \ln \frac{n+m}{m}. \tag{4.5}$$

Figure 2 shows the differences between the mean and mode matchings in the order-statistic-based randomization. From the figures (a) and (b), we find almost no differences between these matchings for large values of $n$.

Figure 2: The pdf of the order statistic $X_{(n+1)}$ ($n = m = 1, 2, 4, 8, 16, 32$)

For a continuous function $g(t)$ ($t \geq 0$), define

$$g_{n,m}^*(T) = \frac{(n+m)!}{n!(m-1)!} \int_0^\infty g(t)(1-e^{-\alpha t})^n \alpha e^{-\alpha t} dt. \tag{4.6}$$

Then, we have

$$\lim_{n,m \to \infty} g_{n,m}^*(T) = g(T). \tag{4.7}$$

**Theorem 3** The sequence $(g_{n,m}^*)_{n,m \geq 1}$ satisfies the recursion

$$g_{0,m}^*(T) = \int_0^\infty mae^{-\alpha t} g(t) dt$$

$$g_{n,m}(T) = \frac{n+m}{n} g_{n-1,m}^*(T) - \frac{m}{n} g_{n-1,m+1}^*(T), \quad n \geq 1. \tag{4.8}$$
Figure 3: American & European put values
\[(t = 0, K = 100, T = 1, r = 0.05, \delta = 0, \sigma = 0.2)\]

\textbf{Proof.} See Kimura [5]. \hfill \square

A simple and practical setting for the parameters \(n\) and \(m\) is \(n = m\). For a set of the parameters \(\{t, S, K, T, r, \delta, \sigma\}\), if we have a functional program for computing \(P^*(\lambda, S)\) for any \(\lambda \geq 0\), then the \(N\)-th randomized approximation \(\pi_N \equiv g_{N,N}^* \approx P(t, S) \ (N \geq 1)\) can be obtained by the following algorithm:
\[
\alpha = \frac{1}{T-t} \ln 2 \\
\text{for} \ m = N \text{ to } 2N \text{ do} \\
g_{0,m}^* = P^*(m\alpha, S) \\
\text{next } m \\
\text{for} \ n = 1 \text{ to } N \text{ do} \\
g_{n,m}^* = \frac{n+m}{n}g_{n-1,m}^* - \frac{m}{n}g_{n-1,m+1}^* \\
\text{next } m \\
\text{next } n \\
\pi_N = g_{N,N}^*
\]

In order to speed up the convergence of \(N\)-th randomized approximation \(\pi_N\), Carr [3] suggested using the Richardson extrapolation scheme. In this paper, however, we use another extrapolation scheme defined below, from the error analysis of the \(N\)-th approximation \(\pi_N\); see Kimura [5] for details.
\[
\left\{ \begin{array}{l}
\pi_N^{(0)} = \pi_N, \\
\pi_N^{(k)} = \frac{1}{2^k - 1} \left\{ 2^k \pi_N^{(k-1)} - \pi_N^{(k-1)} \right\}, \quad N = 2^0, 2^1, 2^2, \ldots \\
\end{array} \right. 
\quad (4.9)
\]

Figure 3 illustrates the curve of an American put and the associated European put values as a function of the present asset value \(S\).
References


