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Motivic integration in non-holonomic geometry

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From April 2003 to November 2003, I organized a seminar so called "E-mail seminar 2003" for the preparation of our RIMS workshop:

"New methods and subjects in singularity theory" (25th – 28th November 2003).

There I selected two topics of the seminar:

non-holonomic geometry and **motivic integration**.

The details on the E-mail seminar 2003 can be seen in the web page

`http://www.math.sci.hokudai.ac.jp/~ishikawa/benkyo.html`

which has been presented only in Japanese so far.

In this note, as an informal and private report on E-mail seminar, I am going to show an attempt to apply the idea of motivic integrations to non-holonomic geometry.

In §1, after a rough review on the idea of motivic integration, we introduce the non-holonomic arc spaces and motivic integrations on them.

In §2, we review the results on Goursat systems due to Montgomery and Zhitomirskii [10], and we introduce a new invariant, in this note, on germs of Goursat systems using the space of singular Legendre curves in the contact three space. The new results stated in this section will be completed by the forthcoming joint paper with Piotr Mormul and Misha Zhitomirskii. Moreover we introduce stringy invariants on Goursat systems by motivic integration on Legendre arc spaces.

In §3, as an appendix, we introduce the notion of natural liftings of forms to tangent bundle, and using it, define the prolongation of differential systems to arc spaces.

In §4, as an addendum, we consider the fundamental philosophy concerning "equations and solutions", so called "the dearest dream of middle aged Mr. Ishikawa".

For simplicity we work with complex case in this note.

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1 Motivic integration and non-holonomic geometry.

1.1 Motivic integrations for beginners.

For details, consult [3][13][4].

Let X be an algebraic variety over $\mathbf{C}$. We consider the space $\mathcal{L}^k(X)$ of k-jets of curves on X: If A is the function ring on X, then $\mathcal{L}^k(X)$ is the set of ring homomorphisms $A \to \mathbf{C}[t]/\langle t^{k+1}\rangle$.

Example 1.1 Let

$$X = \mathbf{C} \setminus \{0\} = \{x \neq 0\} = \{(x_1, x_2) \in \mathbf{C}^2 \mid x_1 x_2 = 1\}.$$

Then $A = \mathbf{C}[x_1, x_2]/\langle x_1x_2 - 1\rangle$. For a ring homomorphism $\varphi : A \to \mathbf{C}[t]/\langle t^{k+1}\rangle$ we set $\varphi(x_1) = \alpha(t), \varphi(x_2) = \beta(t)$. Then $0 = \varphi(x_1x_2 - 1) = \alpha(t)\beta(t) - 1$ in $\mathbf{C}[t]/\langle t^{k+1}\rangle$. This means that $\beta(t) = 1/\alpha(t)$ as k-jet.

We have $\mathcal{L}^0(X) = X$ and $\mathcal{L}^1(X) = TX$. If X is non-singular, then $\mathcal{L}^k(X) \to X$ is a fibration with fiber $\mathbf{C}^{dk}$, where $d = \dim(X)$.

For $k \geq \ell$, there is the canonical projection $\pi_{k\ell} : \mathcal{L}^k(X) \to \mathcal{L}^\ell(X)$. Then we consider the inverse limit $\mathcal{L}^\infty(X) := \varprojlim \mathcal{L}^k(X)$, *the space of formal arcs.* Denote the canonical projection by $\pi_k : \mathcal{L}^\infty(X) \to \mathcal{L}^k(X)$.

Let $K_0(Var_{\mathbf{C}})$ denotes the Grothendieck ring of complex algebraic varieties: To a complex algebraic variety V, there corresponds an element $[V] \in K_0(Var_{\mathbf{C}})$. If $V \cong W$, then $[V] = [W]$. We understand $K_0(Var_{\mathbf{C}})$ as the universal ring generated by $[V]$'s enjoying the relations:

(1) $[V] = [Z] + [V \setminus Z]$, $(Z \subseteq V :$ Zariski closed in $V)$,

(2) $[V][W] = [V \times W]$.

We have $0 = [\emptyset]$ and $1 = [\mathrm{pt}]$.

Example 1.2 By the relation (1), we have $[\mathbf{C}] = [\mathrm{pt}] + [\mathbf{C} \setminus \mathrm{pt}]$. Setting $\mathbf{L} := [\mathbf{C}]$, we have

$$[\mathbf{C} \setminus \mathrm{pt}] = \mathbf{L} - 1.$$

We set $\mathcal{M}_{\mathbf{C}} = K_0(Var_{\mathbf{C}})[\mathbf{L}^{-1}]$, the Laurent polynomial ring. Then we have

Proposition 1.3 (Denef-Loeser [4])

$$J_X(T) := \sum_{k=0}^{\infty}[\mathcal{L}^k(X)]T^k, \quad P_X(T) := \sum_{k=0}^{\infty}[\pi_k(\mathcal{L}(X))]T^k$$

are rational functions.

Example 1.4 Let $X = PT^*\mathbf{C}^2 \to \mathbf{C}^2$ be the projective cotangent bundle over $\mathbf{C}^2$. Note that $X \cong \mathbf{C}^2 \times CP^1$ and $\dim X = 3$. Then we see

$$\begin{aligned} [\mathcal{L}^k(X)] &= [\mathbf{C}^2 \times \mathbf{C} \times \mathbf{C}^{3k}] + [\mathbf{C}^2 \times \text{pt} \times \mathbf{C}^{3k}] \\ &= \mathbf{L}^{3k+3} + \mathbf{L}^{3k+2} = \mathbf{L}^{3k+2}(\mathbf{L}+1). \end{aligned}$$

Thus we have

$$\begin{aligned} J_X(T) &= \textstyle\sum_{k=0}^{\infty} \mathbf{L}^{3k+2}(\mathbf{L}+1)T^k \\ &= \mathbf{L}^2(\mathbf{L}+1)\textstyle\sum_{k=0}^{\infty}(\mathbf{L}^3T)^k = \dfrac{\mathbf{L}^2(\mathbf{L}+1)}{1-\mathbf{L}^3T}. \end{aligned}$$

1.2 Motivic measure.

On an algebraic variety Y, we consider subsets in Y generated from algebraic subvarieties by the operations of taking finite union, finite intersection and complement. We call them constructible subsets in Y.

A subset $A \subseteq \mathcal{L}^\infty(X)$ in the formal arc space is called a *constructible subset* if $A = \pi_k^{-1}(C)$ for a constructible subset $C \subseteq \mathcal{L}^k(X)$ of the algebraic variety $\mathcal{L}^k(X)$. Then we define the *motivic measure* of A by

$$\mu(A) := \lim_{k\to\infty} \frac{[\pi_k(A)]}{\mathbf{L}^{dk}} \in \widehat{\mathcal{M}}_{\mathbf{C}},$$

where $d = \dim(X)$ and $\widehat{\mathcal{M}}_{\mathbf{C}}$ is the completion of $\mathcal{M}_{\mathbf{C}}$ relatively to the weight

$$\text{weight}([X]\mathbf{L}^{-n}) = \dim(X) - n.$$

Thus a sequence with maximal weights tending to $-\infty$ is regarded to converge to 0. For example we have $\mathbf{L}^{-n} \to 0 (n \to \infty)$.

Example 1.5 Let $A = X = PT^*\mathbf{C}^2$. Then

$$\mu(\mathcal{L}(X)) = \lim_{k\to\infty} \frac{\mathbf{L}^{3k+2}(\mathbf{L}+1)}{\mathbf{L}^{3k}} = \mathbf{L}^2(\mathbf{L}+1) = [X].$$

Let $A \subseteq \mathcal{L}^\infty(X)$ be a constructible subset. Let $\alpha : A \to \mathbf{Z} \cup \{\infty\}$ be a function. Suppose, for any $m \in \mathbf{Z} \cup \{\infty\}$, $\alpha^{-1}(m)$ is a constructible subset. Then we define the *motivic integration* of α by

$$\int_A \mathbf{L}^{-\alpha} d\mu := \sum_{m\in\mathbf{Z}} \mu(A \cap \alpha^{-1}(m))\mathbf{L}^{-m}.$$

Example 1.6 Let X be a non-singular variety, and D a non-singular hypersurface in X. Consider the function $\text{ord}(D) : \mathcal{L}^\infty(X) \to \mathbf{N} \cap \{\infty\}$ by $\text{ord}(D)(\gamma) := \text{ord}_0(f \circ \gamma)$, for a local generator f of D. Then $\mu((\text{ord}(D)^{-1}(m))$ is equal to $[X \setminus D]$ if $m = 0$, $[D](1 - \frac{1}{\mathbf{L}})$ if $m = 1$, $[D](\frac{1}{\mathbf{L}} - \frac{1}{\mathbf{L}^2})$ if $m = 2$, and so on. Thus we have

$$\begin{aligned} \textstyle\int_{\mathcal{L}^\infty(X)} \mathbf{L}^{-\text{ord}(D)} d\mu &= [X \setminus D] + [D](1 - \frac{1}{\mathbf{L}})\frac{1}{\mathbf{L}} + [D](\frac{1}{\mathbf{L}} - \frac{1}{\mathbf{L}^2})\frac{1}{\mathbf{L}^2} + \cdots \\ &= [X] - [D]\frac{\mathbf{L}}{1+\mathbf{L}}. \end{aligned}$$

1.3 Non-holonomic jets and arcs.

Let M be a complex manifold. We define

$$C^{\mathrm{hol}}((\mathbf{C},0),M)=\mathcal{L}^{\mathrm{hol}}(M):=\{c\mid c:(\mathbf{C},0)\to M \text{ is a holomorphic germ}\},$$

the space of holomorphic arcs on M, and

$$\mathcal{L}^k(M):=\{j^kc(0)\mid c\in C^{\mathrm{hol}}((\mathbf{C},0),M)\},(k=0,1,2,\dots).$$

If M is a non-singular algebraic variety, then $\mathcal{L}^k(M)$ coincides with the one defined in the previous subsections.

We have $\mathcal{L}^0(M)=M$ and $\mathcal{L}^1(M)=TM$. In general, $\mathcal{L}^k(M)$ is a complex manifold with $\dim(\mathcal{L}^k(M))=(k+1)\dim(M)$. If $k\geq\ell$, then there is the canonical projection $\pi_{k,\ell}:\mathcal{L}^k(M)\to\mathcal{L}^\ell(M)$. We set $\mathcal{L}^\infty(M)=\varprojlim\mathcal{L}^k(M)$, *the space of formal arcs*. Notice that $\mathcal{L}^{\mathrm{hol}}(M)\subset\mathcal{L}^\infty(M)$.

In some sense, $\mathcal{L}^{\mathrm{hol}}(M)$ and $\mathcal{L}^\infty(M)$ are "infinite dimensional complex manifolds". In fact we can define "holomorphic structures" on any quotient space of any space of holomorphic mappings.

Now, let $E\subset TM$ be a holomorphic subbundle. We assume that E is *non-holonomic*, i.e., that E satisfies Hörmander condition: Any tangent vector to M is obtained by finite iterations and summations of Lie brackets of local sections to E.

We set

$$\mathcal{L}^{\mathrm{hol}}_E(M)=\{c\in\mathcal{L}^{\mathrm{hol}}(M)\mid \dot{c}(t)\in E,t\in(\mathbf{C},0)\},$$

where $\dot{c}(t)$ denotes the differential of $c(t)$ by t. Moreover we set

$$\mathcal{L}^k_E(M)=\{j^kc(0)\mid c\in\mathcal{L}^{\mathrm{hol}}_E(M)\},\ \mathcal{L}^\infty_E(M)=\varprojlim\mathcal{L}^k_E(M).$$

Note that $\mathcal{L}^0_E(M)=M,\mathcal{L}^1_E(M)=E$. Thus $\mathcal{L}^1_E(M)=\dim(M)+\mathrm{rank}(E)$. In general, $\mathcal{L}^k_E(M)$ is an analytic subset of $\mathcal{L}^k(M)$.

Example 1.7 Let $X=M=PT^*\mathbf{C}^2$. Let $E\subset TX$ be the canonical contact structure on X. If $\theta=\xi dx+\eta dy$ is the canonical 1-form on $T^*\mathbf{C}^2$, then $E:=\{\xi dx+\eta dy=0\}$. If we set $p=-\xi/\eta$ on the open subset $\{\eta\neq0\}$, we have $E=\{dy-pdx=0\}$. Then $\mathcal{L}^k_E(X)$ is in fact an algebraic subset in $\mathcal{L}^k(X)$

$$[\mathcal{L}^k_E(X)]=[\mathbf{C}^2\times\mathbf{C}\times\mathbf{C}^{2k}]+[\mathbf{C}^2\times\mathrm{pt}\times\mathbf{C}^{2k}]=\mathbf{L}^{2k+2}(\mathbf{L}+1).$$

Now we define

$$\begin{aligned}J_E(T) &:= \textstyle\sum_{k=0}^\infty[\mathcal{L}^k_E(X)]T^k=\sum_{k=0}^\infty\mathbf{L}^{2k+2}(\mathbf{L}+1)T^k\\ &= \frac{\mathbf{L}^2(\mathbf{L}+1)}{1-\mathbf{L}^2T}.\end{aligned}$$

For the motivic measure, we have

$$\mu(\mathcal{L}_E(X)) = \lim_{k\to\infty} \frac{\mathbf{L}^{2k+2}(\mathbf{L}+1)}{\mathbf{L}^{3k}} = \lim_{k\to\infty} \frac{\mathbf{L}^2(\mathbf{L}+1)}{\mathbf{L}^k} = 0.$$

It seems natural to define another measure $\mu_E(A)$ for $A \subseteq \mathcal{L}_E(X)$ by

$$\mu_E(A) := \lim_{k\to\infty} \frac{[\pi_k(A)]}{\mathbf{L}^{rk}},$$

where $r = \mathrm{rank}E$.

Example 1.8 Let $X = PT^*\mathbf{C}^2$ and E the contact distribution on $PT^*\mathbf{C}^2$. Then we have

$$\mu_E(\mathcal{L}_E^\infty(X)) = \lim_{k\to\infty} \frac{\mathbf{L}^{2k+2}(\mathbf{L}+1)}{\mathbf{L}^{2k}} = \mathbf{L}^2(\mathbf{L}+1) = [X].$$

2 Singular Legendre curves and Goursat systems.

This section is written based on my talk given at the famous Singularity Seminar, Hokkaido University, Japan, and given at the workshop in Matsumoto, Japan, on January 2004.

2.1 Singular Legendre curves.

Let us consider the manifold $PT^*\mathbf{C}^2$ of contact elements on the plane $\mathbf{C}^2$ with the canonical contact structure E.

A holomorphic curve-germ $c : (\mathbf{C}, 0) \to PT^*\mathbf{C}^2$ is called a *Legendre curve* if the velocity vector $\dot{c}(t) \in E(\subset T(PT^*\mathbf{C}^2))$, for $t \in (\mathbf{C}, 0)$. Here we do not assume c is an immersion.

In term of local coordinates x, y, p with $E = \{\alpha = dy - pdx = 0\}$, the Legendre condition means that $c^*\alpha = 0$. If we set $c(t) = (x(t), y(t), p(t))$, then the y-component is given by $y(t) = \int p(t)dx(t) = \int p(t)\frac{dx(t)}{dt}dt$.

Two Legendre curves $c, c' : (\mathbf{C}, 0) \to PT^*\mathbf{C}^2$ are called *contactomorphic* (resp. *diffeomorphic*) if there exist a diffeomorphism $\sigma : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ and a contactomorphism (resp. a diffeomorphism) $\tau : (PT^*\mathbf{C}^2, c(0)) \to (PT^*\mathbf{C}^2, c'(0))$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbf{C}, 0) & \xrightarrow{c} & (PT^*\mathbf{C}^2, c(0)) \\ \sigma \downarrow & & \downarrow \tau \\ (\mathbf{C}, 0) & \xrightarrow{c'} & (PT^*\mathbf{C}^2, c'(0)), \end{array}$$

where a diffeomorphism τ is called a contactomorphism if $\tau_* E = E$.

Then we have

Theorem 2.1 ([17][5]) *Let $c, c' : (\mathbf{C}, 0) \to PT^*\mathbf{C}^2$ be Legendre curves of finite type. Then c and c' are contactomorphic if and only if they are diffeomorphic.*

A Legendre curve is called *of finite type* if it is of finite type as a space curve-germ, i.e. if it is determined by its finite jet up to diffeomorphisms. If a Legendre curve is of finite type, then it is of finite order (non-flat), and it has unique integral lifting to any prolongation.

In [5], we have constructed Mather's type theory of Legendre curves.

The following is one of important open problem:

Problem(Zhitomirskii[17]): *Characterize diffeomorphism classes of space curve-germs which are realized by Legendre curves.*

2.2 Goursat systems.

Let N be a complex manifold of dimension n ($n \geq 2$). Let E be a germ of differential system of rank 2 on N at x_0. This means that $E \subseteq TN$ is a subbundle of rank 2. In other words, $E = \langle X, Y \rangle_{\mathbf{C}}$ for a local holomorphic vector fields X, Y over N at x_0 which are pointwise linearly independent. As a dual expression we have $E = \{\omega_1 = 0, \ldots, \omega_{n-2} = 0\}$ for a local independent holomorphic 1-forms $\omega_1, \ldots, \omega_{n-2}$.

For example, the contact structure $E \subset T\mathbf{C}^3$ is expressed by

$$E = \left\langle \frac{\partial}{\partial p}, \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} \right\rangle = \{dy - pdx = 0\}.$$

For a differential system E, we set $E^2 = E + [E, E]$.

We call E a *Goursat system* if E^2 is of rank 3, namely, of corank $n-3$, and, setting $E = E_{n-2}, E^2 = E_{n-3}$, if $E_{i-1} = (E_i)^2$ is of corank $i-1$, for $i = n-2, n-3, \ldots, 2, 1$ ([1][10][2]).

Then we have

Theorem 2.2 ([10]) *For any $n \geq 3$, there exist a complex n-manifold $\mathbf{M}^n$ and a Goursat system $\mathbf{E}_{n-2}$ ("Monster Goursat") on $\mathbf{M}$ such that any Goursat germ (N^n, x_0, E) is isomorphic to $(\mathbf{M}, y_0, \mathbf{E})$ for some $y_0 \in \mathbf{M}$.*

That (N^n, x_0, E) is isomorphic to $(\mathbf{M}, y_0, \mathbf{E})$ means that there exists a holomorphic diffeomorphism $\Phi : (N, x_0) \to (\mathbf{M}, y_0)$ satisfying $\Phi_* E = \mathbf{E}$.

Now we describe the Monster Goursat systems in short.

For $n = 2$, $\mathbf{M}^2 := \mathbf{C}^2$ and $\mathbf{E}_0 = T\mathbf{C}^2$.

For $n = 3$, we set $\mathbf{M}^3 := P\mathbf{E}_1 = PT\mathbf{C}^2 = PT^*\mathbf{C}^2$, the manifold of tangent lines of $\mathbf{C}^2$. We set

$$\mathbf{E}_1 = \{v \in T\mathbf{M}_3 \mid T_\ell \mathbf{M}_3, \pi_* v \in \ell\},$$

for the canonical projection $\pi : \mathbf{M}^3 \to \mathbf{M}_2$. Then $\mathbf{E}_1$ is the canonical contact structure on $\mathbf{M}^3$, and $\mathbf{E}_1$ is a Goursat system. By Darboux's theorem, $\mathbf{E}_1$ is *transitive*, namely, for any points $y_0, y_0' \in \mathbf{M}_3$, there exists an isomorphism $(\mathbf{M}_3, y_0, \mathbf{E}_1) \cong (\mathbf{M}_3, y_0', \mathbf{E}_1)$.

For $n = 4$, we set $\mathbf{M}_4 = P\mathbf{E}_1$, the fiberwise projectivization of $\mathbf{E}_1$. For the projection $\pi : \mathbf{M}_4 \to \mathbf{M}_3$ we set

$$\mathbf{E}_2 = \{v \in T\mathbf{M}^4 \mid v \in T_\ell \mathbf{M}^4, \pi_* v \in \ell\}.$$

The differential system $\mathbf{E}_2$ is the prolongation of $\mathbf{E}_1$. We see $(\mathbf{M}^4, \mathbf{E}_2)$ is Goursat and transitive. However note that $(\mathbf{M}^4, \mathbf{E}_2)$ is *directionally non-transitive*. Namely, the self-isomorphisms on $(\mathbf{M}^4, y_0, \mathbf{E}_2)$ does not act on the projective line $P((\mathbf{E}_2)_{y_0})$ transitively. In fact the fibers of $\pi : \mathbf{M}_4 \to \mathbf{M}_3$ give special directions (Cauchy characteristics of the system $(\mathbf{E}_2)^2$ of corank one).

For $n = 5$, we set $\mathbf{M}_5 = P\mathbf{E}_2$ and

$$\mathbf{E}_3 = \{v \in T\mathbf{M}^5 \mid v \in T_\ell \mathbf{M}^5, \pi_* v \in \ell\},$$

for the projection $\pi : \mathbf{M}^5 \to \mathbf{M}^4$. $\mathbf{E}_3$ is the prolongation of $\mathbf{E}_2$. It is Goursat and no longer transitive.

Also for any $n \geq 6$, we define in the same way Goursat system $(\mathbf{M}^n, \mathbf{E}_{n-2})$ inductively.

2.3 Legendre invariant on Goursat systems.

Let $n \geq 3$. Consider the canonical projection $\pi_{n,3} : \mathbf{M}^n \to \mathbf{M}^3$. Let $\gamma : (\mathbf{C}, 0) \to \mathbf{M}^n$ be an integral curve to $\mathbf{E}_{n-2}$, namely $\dot\gamma(t) \in \mathbf{E}_{n-2}, (t \in (\mathbf{C}, 0))$. Then $\pi \circ \gamma : (\mathbf{R}, 0) \to \mathbf{M}^3 = PT^*\mathbf{C}^2$ is a Legendre curve.

Conversely if $c : (\mathbf{C}, 0) \to \mathbf{M}^3$ is a Legendre curve of finite type, there exists a unique integral lifting $\tilde c : (\mathbf{C}, 0) \to \mathbf{M}^n$ to $\mathbf{E}_{n-2}$ with $\pi \circ \tilde c = c$ for all $n \geq 3$.

Now we introduce an important invariant of Goursat systems: For $y_0 \in \mathbf{M}^n$, we denote by C_{y_0} the set of contactomorphism classes of Legendre curves $\pi \circ \gamma : (\mathbf{C}, 0) \to \mathbf{M}^3$ where $\gamma : (\mathbf{C}, 0) \to \mathbf{M}^n$ runs over integral curves to $\mathbf{E}_{n-2}$ with $\gamma(0) = y_0$ and $\pi \circ \gamma$ of finite type.

$$C_{y_0} = \left\{ \pi \circ \gamma \;\middle|\; \begin{array}{l} \gamma : (\mathbf{C}, 0) \to \mathbf{M}^n \text{ integral to } \mathbf{E}_{n-2}, \\ \gamma(0) = y_0, \pi \circ \gamma \text{ is of finite type} \end{array} \right\} / \sim,$$

where $\sim$ means the contactomorphism equivalence.

Then we have

Theorem 2.3 *Let $y_0, y_0' \in \mathbf{M}^n$, $(n \geq 3)$. Then the following conditions are equivalent to each other:*

(i) $(\mathbf{M}, y_0, \mathbf{E}) \cong (\mathbf{M}, y_0', \mathbf{E})$.

(ii) $C_{y_0} = C_{y_0'}$.

(iii) $C_{y_0} \cap C_{y_0'} \neq \emptyset$.

Let (N^n, x_0, E) be a germ of Goursat system of rank 2. Then there exists intrinsically the projection $\pi : (N^n, x_0) \to \mathbf{M}^3$ making E is the prolongation of the contact $\mathbf{E}_1$. Then we define intrinsically

$$C_E = \left\{ \pi \circ \gamma \;\middle|\; \begin{array}{l} \gamma : (\mathbf{C}, 0) \to \mathbf{M}^n \text{ integral to } \mathbf{E}_{n-2}, \\ \gamma(0) = x_0, \pi \circ \gamma \text{ is of finite type} \end{array} \right\} / \sim,$$

where $\sim$ means the contactomorphism equivalence.

Then we rewrite Theorem 2.3

Theorem 2.4 *Let (N^n, x_0, E) and (N'^m, x'_0, E') be germs of Goursat systems of rank two. Then the following conditions are equivalent to each other:*

(i) $(N, x_0, E) \cong (N', x'_0, E')$.

(ii) $C_E = C_{E'}$.

(iii) $C_E \cap C_{E'} \neq \emptyset$.

Thus the classification problem of Goursat systems is reduced to the classification problem of Legendre curves ([17][5]). On the other hand, the classification of Goursat systems([11][12]) gives an insight to the classification of each individual Legendre curve.

2.4 Legendre-Goursat duality.

Let $c : (\mathbf{C}, 0) \to PT^*\mathbf{C}^2$ be a Legendre curve of finite type. Then we define G_c to be the set of isomorphism classes of Goursat systems E satisfying $[c] \in C_E$. Actually G_c is realized, up to isomorphism, by nothing but the prolongation-deprolongation infinite sequence of Goursat systems which is obtained by lifting of c.

Then we have

Theorem 2.5 *Let $c : (\mathbf{C}, 0) \to PT^*\mathbf{C}^2$ be a Legendre curve of finite type. Then the contactomorphism class $[c]$ of c is recovered from the sequence G_C: We have*

$$\bigcap_{[E] \in G_c} C_E = \{[c]\}.$$

Conversely, let E be a Goursat system of rank two. Assume E is not the contact system. Then the isomorphism class $[E$ of E is recovered from C_E: We have

$$\sup\{n \mid \#\pi_n \left(\bigcup_{[c] \in C_E} G_c \right) = 1\} = d(E),$$

where $d(E)$ denote the dimension of the base manifold of E, and π_n means the operation of taking isomorphism classes of Goursat systems over n-dimensional manifold. Moreover we have

$$\pi_{d(E)} \left(\bigcup_{[c] \in C_E} G_c \right) = \{[E]\}.$$

The basic idea of the proof lies on the finite determinacy([14]). I have provided the theory of finite determinacy on Legendre curves via contactomorphisms in [5].

The theory of Goursat systems can be regarded as a generalisation of the theory of ordinary differential equations. Thus we have to seek corresponding result to the above classifying theory of Goursat systems. Then first we need the characterisation of systems for partial differential equations([15], [16]). Our final goal should be the construction of geometric theory of third order partial differential equations.

2.5 Stringy invariants on Goursat systems.

For the Engel structure E, we set

$$\tilde{C}_E^k := \{j^k(\pi \circ \gamma)(0) \mid \gamma : (\mathbf{C}, 0) \to M^4 \text{ integral through the base point}\}.$$

Then

$$[\tilde{C}_E^k] = \mathbf{L}^{2(k-1)}(\mathbf{L}-1) + \cdots + (\mathbf{L}-1) = \frac{\mathbf{L}^{2k}-1}{\mathbf{L}+1}.$$

Thus we have *Engel-Poincaré series*:

$$P_{\text{Engel}}(T) := \sum_{k=0}^{\infty} [\tilde{C}_E^k] T^k = \frac{(\mathbf{L}-1)T}{(1-\mathbf{L}^2T)(1-T)}.$$

Moreover we have *Engel-motivic volume*:

$$\mu_{\text{Engel}}(\tilde{C}_E) \quad := \quad \lim_{k\to\infty} \frac{[\tilde{C}_E^k]}{\mathbf{L}^{2k}} = \lim_{k\to\infty} \frac{\mathbf{L}^{2k}-1}{\mathbf{L}^{2k}(\mathbf{L}+1)} = \frac{1-\mathbf{L}^{-2k}}{\mathbf{L}+1} = \frac{1}{\mathbf{L}+1}.$$

For the regular Goursat system E of rank two on 5-space, we have

$$[\tilde{C}_E^k] = (\mathbf{L}-1)\mathbf{L}^{2(k-2)}.$$

Therefore

$$\mu_E(\tilde{C}_E^k) = \lim_{k\to\infty} \frac{(\mathbf{L}-1)\mathbf{L}^{2(k-2)}}{\mathbf{L}^{2k}} = \frac{\mathbf{L}-1}{\mathbf{L}^4}.$$

Moreover we have

$$P_{\text{regular}}(T) = 1 + T + \frac{(\mathbf{L}-1)T^2}{1-\mathbf{L}T}.$$

For the singular Goursat system E of rank two on 5-space, we have

$$\mu_E(\tilde{C}_E^k) = \frac{\mathbf{L}^{2(k-2)}}{\mathbf{L}^{2k}} = \frac{1}{\mathbf{L}^4}.$$

Thus simply the "volume" distinguishes regular and singular Goursat systems of corank two on five space. Moreover we have

$$P_{\text{singular}}(T) = 1 + T + \frac{T^2}{1-\mathbf{L}^2T}.$$

3 Natural liftings and prolongations.

We recall the notion of natural liftings [6]. It should be essential for general consideration of non-holonomic arc spaces.

Let α be a differential form on a manifold M. Then there exists a differential form $\tilde{\alpha}$ on TM such that, for any local section $v : M \to TM$, the pull-back $v^*\tilde{\alpha}$ is equal to Lie derivative $L_v\alpha$. We call $\tilde{\alpha}$ the *natural lifting* of α.

For example, if $f : M \to \mathbf{R}$ is a 0-form, namely, a function on M. Then its natural lifting $\tilde{f} : TM \to \mathbf{R}$ is defined by

$$\tilde{f} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i,$$

where $(x_1, \ldots, x_n)$ is any local system of coordinates of M and $v_1, \ldots, v_n$ be the corresponding fiber coordinates of TM.

In general we see

$$\widetilde{d\alpha} = d\tilde{\alpha}.$$

$$\widetilde{\alpha \wedge \beta} = \tilde{\alpha} \wedge \pi^*\beta + \pi^*\alpha \wedge \tilde{\beta}.$$

Moreover the natural lifting is locally defined: If $f : M \to W$ be a mapping and α is a form on W, then we have

$$\widetilde{f^*\alpha} = (f_*)^*\tilde{\alpha}.$$

In particular, taking f the inclusion $U \hookrightarrow W$ of an open set U of W, then we have

$$\widetilde{\alpha|U} = \tilde{\alpha}|U.$$

For example, we have

$$\widetilde{dx_i} = d\tilde{x}_i = dv_i,$$

$$\widetilde{adx_i} = \tilde{a}dx_i + a\widetilde{dx_i} = \sum_{j=1}^{n} \frac{\partial a}{\partial x_j} v_j dx_i + a dv_i,$$

for a function a, and

$$db = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial b}{\partial x_i \partial x_j} v_i dx_j + \sum_{i=1}^{n} \frac{\partial b}{\partial x_i} dv_i,$$

for a function b.

Set $T^2(M) := T(TM)$, and $T^k(M) := T(T^{k-1}M)$ inductively. For a function $f : M \to \mathbf{R}$, we set $f_1 := \tilde{f} : TM \to \mathbf{R}$, and $f_k : T^k(M) \to \mathbf{R}$ by $f_k := \widetilde{f_{k-1}}$ inductively. Then

$$f_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f}{\partial x_i \partial x_j} v_i \xi_j + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} w_i,$$

where $(x, v; \xi, w)$ is a system of coordinates of T^2M.

Let α be a differential form on M. Then we set $\alpha_1 := \tilde{\alpha}$, the natural lifting of α to TM. Inductively we set $\alpha_k := \widetilde{\alpha_{k-1}}$, the natural lifting to T^kM.

There exists a natural embedding $i_k : \mathcal{L}^k(M) \to T^k(M)$. Note that $\dim \mathcal{L}^k(M) = (k+1) \dim M$, while $\dim T^k(M) = 2^k \dim M$.

Let α be a differential form on M. Then we define $\pi^!\alpha := i^*\alpha_k$, for the projection $\pi = \pi_{k,0} : \mathcal{L}^k(M) \to M$. We call $\pi^!\alpha$ the *prolongation* of α to $\mathcal{L}^k(M)$.

Now we can prolong an differential ideal $\mathcal{I}$ of the de Rham algebra $\Omega(M)$ on M. to an differential ideal $\mathcal{I}_k$ to $\mathcal{L}^k(M)$ generated by prolongations α_k of all generators α of $\mathcal{I}$. The construction is applied also to sheaves of differential ideals in easy manner.

If $E \subset TM$ be a subbundle, then we consider the sheaf of differential ideals $\mathcal{E}^{\perp}$ generated by local sections to the subbundle $E^{\perp} \subset T^*E$ defined by

$$E^{\perp} := \{\alpha \in T^*M \mid \alpha|E = 0\}.$$

Then we have the sheaf of differential ideals $(\mathcal{E}^{\perp})_k$ on $\mathcal{L}^k(M)$.

4 The dearest dream of middle aged Mr. Ishikawa.

"Equation"should be one of the most popular notions in mathematics. If an equation is given, then students and researchers will be forced to find solutions to the given equation. What are solutions? The question causes of course non-trivial problems: Before trying to solve the equation, you have to make clear the space of possible solutions for the equations. For example, for the equation $x^2 + 1 = 0$, the finding solutions depending on whether you seek them in the real numbers or complex numbers. Even though we are aware of that very much, we pose the principle that the solutions determine the equation as well as the equation do the solutions. Then we can recall the basic framework of algebraic geometry. The principle is in fact realized by the theory of scheme.

The principle can be applied also to differential equations. In non-holonomic geometry, equations are, for instance, given by subbundles of a tangent bundle. Then solutions are integral submanifolds, or, in particular, integral curves. If fact studying of the space of integral curves is one of main problems of non-holonomic geometry as well as control theory ([9][8]).

Here is an easy statement:

Lemma 4.1 *Let E, E' be subbundles of TM. Then the following conditions are equivalent to each other:*

(1) $\mathcal{L}^{\mathrm{hol}}_E(M) = \mathcal{L}^{\mathrm{hol}}_{E'}(M)$,

(2) $\mathcal{L}^{\infty}_E(M) = \mathcal{L}^{\infty}_{E'}(M)$,

$(3)_k$ $\mathcal{L}^{k}_E(M) = \mathcal{L}^{k}_{E'}(M)$, $(k = 1, 2, 3, \dots)$,

(4) $E = E'$.

Also the following facts should belong to the principle I am speaking.

Example 4.2 Let M be a manifold. Let $g : TM \to \mathbf{R}_{\geq 0}$ be a Riemannian metric on M. Then we set

$$\mathrm{Geod}(M, g) := \{\gamma : (\mathbf{R}, 0) \to M \mid \gamma \text{ is a germ of geodesic.}\}.$$

Then we observe that, for a Riemannian metrics g, g' on M, $\mathrm{Geod}(M, g) = \mathrm{Geod}(M, g')$ if and only if $g = g'$. Also for a sub-Riemannian metrics g, g' on $E, E' \subset TM$ respectively, we see that the spaces of the sub-Riemannian geodesics $\mathrm{Geod}(M, E, g) = \mathrm{Geod}(M, E', g')$ if and only if $g = g'$.

Inspired by the idea of motivic integrations, we have considered in §1 the integral jet spaces of non-holonomic structures.

We examine the idea in particular to Goursat systems in §2.

In [7], we provided a realization of the principle that solutions determine the equation. In fact types of singularities of solutions distinguish certain type of equations. In this note, however, we consider the space of solution itself. Then topology of solution spaces and singularities of solution spaces, not of individual solution come into our scope. Thus my dearest day dream may be summarised as *to construct invariants of an equation from the topology and singularities of solution spaces.*

Here, as a conclusion of my speculative story, we describe an elementary idea for the investigation of solution spaces to differential equations.

First we start with the linear algebra.

Let Ω be the Grassmann algebra over $\mathbf{C}^{n*}$ (the dual vector space to $\mathbf{C}^n$;

$$\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2 \oplus \cdots ,$$

$\Omega_0 = \mathbf{C}, \Omega_1 = \mathbf{C}^{n*}, \Omega_2 = \mathbf{C}^{n*} \wedge \mathbf{C}^{n*}$, and so on. Let $\mathcal{I} \subseteq \Omega$ be a subset of Ω. Then define $S(\mathcal{I}) \subseteq \cup_{k=0}^{n} \mathrm{Gr}(k, \mathbf{C}^n)$ by

$$S(\mathcal{I}) := \{W \mid \alpha|W = 0, \text{ for any } \alpha \in \mathcal{I}\}.$$

Conversely, for each subset $\mathcal{S} \subset \cup_{k=0}^{n} \mathrm{Gr}(k, \mathbf{C}^n)$, we define $I(\mathcal{S}) \subseteq \Omega$ by

$$I(\mathcal{S}) := \{\alpha \mid \alpha|W = 0, \text{ for any } W \in \mathcal{S}\}.$$

We call $\mathcal{I} \subseteq \Omega$ *reduced* if $\mathcal{I} = I(S(\mathcal{I}))$. We call $\mathcal{S} \subseteq \cup_{k=0}^{n} \mathrm{Gr}(k, \mathbf{C}^n)$ *reduced* if $\mathcal{S} = S(I(\mathcal{S}))$.

Lemma 4.3

(1) $\mathcal{I} \subseteq I(S(\mathcal{I})), \quad \mathcal{S} \subseteq S(I(\mathcal{S}))$.

(2) *If* $\mathcal{I} \subseteq \mathcal{I}'$, *then* $S(\mathcal{I}) \supseteq S(\mathcal{I}')$.

(3) *If* $\mathcal{S} \subseteq \mathcal{S}'$, *then* $I(\mathcal{S}) \supseteq I(\mathcal{S}')$.

(4) $S(\mathcal{I}) = SIS(\mathcal{I}), \quad I(\mathcal{S}) = ISI(\mathcal{S})$.

(5) $IS(\mathcal{I}) = ISIS(\mathcal{I}), \quad SI(\mathcal{S}) = SISI(\mathcal{S})$.

Corollary 4.4 $\tilde{\mathcal{I}} := IS(\mathcal{I})$ *is reduced.* $\tilde{\mathcal{S}} := SI(\mathcal{S})$ *is reduced.*

Let us extend the above story to de Rham algebra.

Let $\Omega = \Omega_n$ be the de Rham algebra, namely the differentiable algebra of germs of differential forms on $(\mathbf{C}^n, 0)$:

$$\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2 \oplus \cdots ,$$

$\Omega_0 = \mathcal{O}_n$, Ω_1 is the space of germs of 1-forms and so on.

We denote by $H(k, n)$ the space of holomorphic map-germs $(\mathbf{C}^k, 0) \to (\mathbf{C}^n, 0)$. Let $\mathcal{I} \subseteq \Omega$ be a subset of Ω. Then define $S(\mathcal{I}) \subseteq \cup_{k=0}^{n} H(k, n)$ by

$$S(\mathcal{I}) := \{f \in \cup_{k=0}^{n} H(k, n) \mid f^*\alpha = 0, \text{ for any } \alpha \in \mathcal{I}\}.$$

Conversely, for each subset $\mathcal{S} \subset \cup_{k=0}^{n} H(k,n)$, we define $I(\mathcal{S}) \subseteq \Omega$ by

$$I(\mathcal{S}) := \{\alpha \in \Omega \mid f^*\alpha = 0, \text{ for any } f \in \mathcal{S}\}.$$

We call $\mathcal{I} \subseteq \Omega$ *reduced* if $\mathcal{I} = I(S(\mathcal{I}))$. We call $\mathcal{S} \subseteq \cup_{k=0}^{n} \mathrm{Gr}(k, \mathbf{C}^n)$ *reduced* if $\mathcal{S} = S(I(\mathcal{S}))$. The we have at last:

Lemma 4.5

(1) $\mathcal{I} \subseteq I(S(\mathcal{I})), \quad \mathcal{S} \subseteq S(I(\mathcal{S}))$.

(2) *If* $\mathcal{I} \subseteq \mathcal{I}'$, *then* $S(\mathcal{I}) \supseteq S(\mathcal{I}')$.

(3) *If* $\mathcal{S} \subseteq \mathcal{S}'$, *then* $I(\mathcal{S}) \supseteq I(\mathcal{S}')$.

(4) $S(\mathcal{I}) = SIS(\mathcal{I}), \quad I(\mathcal{S}) = ISI(\mathcal{S})$.

(5) $IS(\mathcal{I}) = ISIS(\mathcal{I}), \quad SI(\mathcal{S}) = SISI(\mathcal{S})$.

Corollary 4.6 $\widetilde{\mathcal{I}} := IS(\mathcal{I})$ *is reduced.* $\widetilde{\mathcal{S}} := SI(\mathcal{S})$ *is reduced.*

The method of motivic integrations in algebraic geometry can be regarded as a sort of perturbation of the correspondence between equations and solutions. So, we are going to realise the idea of deforming the space of integral curves in the forthcoming papers.

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