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Combinatorial aspects of the mixed Hodge structure

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ABSTRACT. *This is a review article on the combinatorial aspects of the mixed Hodge structure of the cohomology group of 1) an affine hypersurface in a torus and of 2) a Milnor fibre of the isolated hypersurface singularity. In the first part, we calculate the fibre integrals of the affine hypersurface in a torus in the form of their Mellin transforms. The relations between poles of Mellin transforms of fibre integrals, the mixed Hodge structure of the cohomology of the hypersruface, the hypergeometric differential equation, and the Euler characteristic of fibres are clarified. In the second part we give a purely combinatorial method to compute spectral pairs of the singularity.*

0 Introduction

This note consists of two part. In the first part (§1- §5), we review first the mixed Hodge structure (MHS) of the cohomology group of a hypersurface in a torus and then propose to calculate concretely fibre integrals associated to it. We establish an expression of the position of poles of the Mellin transform with the aid of the mixed Hodge structure of an hypersurface Z_f define by a $\Delta-$regular polynomial explained by V. Batyrev [2]. The trial to relate the asymptotic behaviour of a fibre integral with the Hodge structure of the fibre variety goes back to [22] where Varchenko established the equivalence of the asymptotic Hodge structure and the mixed Hodge structure in the sense of Deligne-Steenbrink for the case of plane curves and (semi-)quasihomogneous singularities.

The relation between the poles of the Mellin transform and the mixed Hodge structure has been explained for examples of isolated complete intersections of space curve type in [19].

In this note, we illustrate the clarity of this approach in taking the example of a hypersurface in a torus defined by so called simpliciable polynomial (see Definition 2).

The aim of the second part of the article (§6 -§7) is to give a survey on the combinatorial aspects of the MHS of the cohomology of the Milnor fibre defined by a single function germ with isolated singularity (hypersurface singularity).

In the case of a convenient germ f, A.G.Kouchnirenko [11] established a formula of Milnor number $\mu(f) = dim H^{n-1}(X_t)$ for the Milnor fibre $X_t = \{x \in \mathbf{C}^n; |x| \leq \epsilon, f(x) = t\}$ for small enough ϵ and generic $t \neq 0$. Based on a fundamental theory by J.H.M.Steenbrink [16], V.I. Danilov [3] (almost simultaneously Anatoly N.Kirillov [10] also) has calculated the MHS $H^{p,q}(H^{n-1}(X_t))$ under the assumption that f is non-degenerate and simplicial (see Definition 5).

Despite these remarkable results, their description of $H^{n-1}(X_t)$ is not refined enough to study more advanced question on the topology and the analysis on the Milnor fibre X_t. For example to calculate the Gauss-Manin system of the fibre integrals $\int_{\gamma_j(t)} \omega_i$, $\gamma_j(t) \in H_{n-1}(X_t)$, $\omega_i \in H^{n-1}(X_t)$ we must know the precise disposition of representatives $\omega_i \in H^{n-1}(X_t)$ with respect to the Newton diagram $\Gamma(f)$. Or, at least, to describe the basis $\{\omega_1, \cdots, \omega_\mu\}$ in terms of integer points on $\mathbf{R}^n_+$ by means of combinatorics associated to $\Gamma(f)$. This task has been carried by A.Douai [5] for the case $n = 2$ and non-degenerate f to obtain a concrete expression of the the Gauss-Manin system on $H^1(X_t)$. So far as it is known to me, the question of combinatorial description of the $H^{p,q}(H^{n-1}(X_t))$ is still open. Here we try to give an answer to this question (§6, **Algorithm**).

Quite recently, an algorithm to compute $H^{p,q}(H^{n-1}(X_t))$ together with the monodromy action on it has appeared (see [14]). It is implemented in the computer algebra system SINGULAR in the library gaussman.lib. Everybody who wants to verify combinatorial statements on $H^{p,q}(H^{n-1}(X_t))$ can achieve it in computing non-trivial examples by means of this tool.

1 Hypersurface in a torus

Let Δ be a convex $n-$dimensional convex polyhedron in $\mathbf{R}^n$ with all vertices in $\mathbf{Z}^n$. Let us define a ring $S_\Delta \subset \mathbf{C}[x_1^{\pm}, \cdots, x_n^{\pm}]$ of the Laurent polynomial ring as follows:

$$S_\Delta := \mathbf{C} \oplus \bigoplus_{\frac{\vec{\alpha}}{k} \in \Delta, \exists k \geq 1} \mathbf{C} \cdot x^{\vec{\alpha}}. \tag{2.1}$$

We denote by $\Delta(f)$ the convex hull of the set $\vec{\alpha} \in supp(f)$ and call it the Newton polyhedron of $f(x)$. We introduce the following Jacobi ideal:

$$J_{f,\Delta} = \langle x_1 \frac{\partial f}{\partial x_1}, \cdots, x_n \frac{\partial f}{\partial x_n} \rangle \cdot S_{\Delta(f)}. \tag{2.2}$$

Let τ be a $\ell-$dimensional face of $\Delta(f)$ and define

$$f^\tau(x) = \sum_{\vec{\alpha} \in \tau \cap supp(f)} a_{\vec{\alpha}} x^{\vec{\alpha}}, \tag{2.3}$$

where $f(x) = \sum_{\vec{\alpha} \in supp(f)} a_{\vec{\alpha}} x^{\vec{\alpha}}$. The Laurent polynomial $f(x)$ is called Δ regular, if $\Delta(f) = \Delta$ and for every $\ell-$dimensional face $\tau \subset \Delta(f)$ $(\ell \geq 0)$ the polynomial equations:

$$f^\tau(x) = x_1 \frac{\partial f^\tau}{\partial x_1} = \cdots = x_n \frac{\partial f^\tau}{\partial x_n} = 0,$$

have no common solutions in $\mathbf{T}^n = (\mathbf{C}^\times)^n$.

Proposition 1.1 *Let f be a Laurent polynomial such that $\Delta(f) = \Delta$. Then the following conditions are equivalent.*

(i) The elements $x_1 \frac{\partial f}{\partial x_1}, \cdots, x_n \frac{\partial f}{\partial x_n}$ gives rise to a regular sequence in $S_{\Delta(f)}$

(ii)

$$dim(\frac{S_\Delta}{J_{f,\Delta}}) = n!vol(\Delta).$$

(iii) f is $\Delta-$regular.

It is possible to introduce a filtration on S_Δ, namely $\vec{\alpha} \in S_k$ if and only if $\frac{\vec{\alpha}}{k} \in \Delta$. Consequently we have an increasing filtration;

$$\mathbf{C} \cong \{0\} = S_0 \subset S_1 \subset \cdots \subset S_n \subset \cdots,$$

that induces a decreasing filtration on $\frac{S_\Delta}{J_{f,\Delta}}$:

$$F^n(\frac{S_\Delta}{J_{f,\Delta}}) \subset F^{n-1}(\frac{S_\Delta}{J_{f,\Delta}}) \subset \cdots \subset F^0(\frac{S_\Delta}{J_{f,\Delta}}).$$

This is called the Hodge filtation of $\frac{S_\Delta}{J_{f,\Delta}}$. It is worthy to remark here that the Hodge filtration ends up with $n-$th term.

Let us remind us of the notion of Ehrhart polynomial:

Definition 1 *Let Δ be an $n-$dimensional convex polytope. Denote the Poincaré series of graded algebra S_Δ by*

$$P_\Delta(t) = \sum_{k\geq 0} \ell(k\Delta)t^k,$$

$$Q_\Delta(t) = \sum_{k\geq 0} \ell^*(k\Delta)t^k,$$

where $\ell(k\Delta)$ (resp.$\ell^*(k\Delta)$) represents the number of integer points in $k\Delta$. (resp. interior integer points in $k\Delta$.) Then

$$\Psi_\Delta(t) = \sum_{k=0}^{n} \psi_k(\Delta)t^k = (1-t)^{n+1}P_\Delta(t),$$

$$\Phi_\Delta(t) = \sum_{k=0}^{n} \varphi_k(\Delta)t^k = (1-t)^{n+1}Q_\Delta(t),$$

are called Ehrhart polynomials which satisfy

$$t^{n+1}\Psi_\Delta(t^{-1}) = \Phi_\Delta(t).$$

Further, the main object of our study will be the cohomology group of the hypersurface $Z_f := \{x \in \mathbf{T}^n; f(x) = 0\}$. We have an important isomorphism on the Hodge filtration of $PH^{n-1}(Z_f)$.

Theorem 1.2 *([2]) For the primitive part $PH^{n-1}(Z_f)$ of $H^{n-1}(Z_f)$, the following isomorphism holds;*

$$\frac{F^i PH^{n-1}(Z_f)}{F^{i+1}PH^{n-1}(Z_f)} \cong Gr_F^{n-i}\left(\frac{S_\Delta}{J_{f,\Delta}}\right) = \frac{F^i(\frac{S_\Delta}{J_{f,\Delta}})}{F^{i+1}(\frac{S_\Delta}{J_{f,\Delta}})}. \tag{2.4}$$

Furthermore

$$dim\ Gr_F^{n-i}\left(\frac{S_\Delta}{J_{f,\Delta}}\right) = \sum_{q\geq 0} h^{i,q}(PH^{n-1}(Z_f)) = \psi_{n-i}(\Delta),$$

for $i \leq n-1$.

As for the weight filtration, we have the following characterization. We understand the notion of the stratum of the support of the algebra $\frac{S_\Delta}{J_{f,\Delta}}$ in identifying a polynomial $x^{\vec{\alpha}} \in S_\Delta$ with $\vec{\alpha} \in \mathbf{Z}^n$. We call $(n-j)-$dimensional stratum of $supp(S_\Delta)$ the set of those points $\vec{i}$ from $k\Delta$, $k = 1, 2, \cdots$ such that $\frac{\vec{i}}{k}$ is located on the $(n-j)-$dimensional face of Δ and not on any $(n-j-1)-$dimensional face $\Delta' \subset \Delta$.

Theorem 1.3 *The weight filtration on $PH^{n-1}(Z_f)$ is defined as a decreasing filtration*

$$0 = W_{n-2} \subset W_{n-1} \subset \cdots \subset W_{2n-2} = PH^{n-1}(Z_f),$$

such that $W_{n+i-1} \cong \{$ the integer points located on the strata with dimension $\geq (n-i)$ of $supp(\frac{S_\Delta}{J_{f,\Delta}})$ but not on the $(n-i-1)-$ dimensional stratum.$\}$ for $0 \leq i \leq n-2$.

This theorem is an easy consequence of the Theorem 8.2 [2]. First we notice that the following exact sequence takes place,

$$0 \to H^n(\mathbf{T}) \to H^n(\mathbf{T} \setminus Z_f) \overset{Res}{\to} H^{n-1}(Z_f) \to 0.$$

The Poincaré residue mapping Res gives a morphism of mixed Hodge structure of the Hodge type $(-1,-1)$,

$$Res(F^j\, H^n(\mathbf{T}\setminus Z_f)) = F^{j-1}\, H^{n-1}(Z_f),\;\; Res(W_j\, H^n(\mathbf{T}\setminus Z_f)) = W_{j-2}\, H^{n-1}(Z_f).$$

Thus we have,

$$0 \to W_{n+i}\, H^n(\mathbf{T}) \to W_{n+i}H^n(\mathbf{T}\setminus Z_f) \overset{Res}{\to} W_{n+i-2}H^{n-1}(Z_f) \to 0,$$

for $i = 2, \cdots, n-1$ where

$$W_{2n-1}\, H^n(\mathbf{T}) = \cdots = W_{n-1}\, H^n(\mathbf{T}) = 0, \tag{2.5}$$

and $dim\ W_{2n}\ H^n(\mathbf{T}) = 1$.

This filtration induces a natural graduation $Gr_i^W\ PH^{n-1}(Z_f) := W_i/W_{i-1}$. In view of the equality (2.5) the Poincaré residue mapping Res gives an isomorphism

$$Res: W_{n+i}H^n(\mathbf{T}\setminus Z_f) \overset{Res}{\to} W_{n+i-2}H^{n-1}(Z_f),$$

for $i = 1, \cdots, n-1$. The algebraic structure of the space $W_{n+i}H^n(\mathbf{T}\setminus Z_f)$, $i = 1, \cdots, n-1$ has already been established by Theorem 8.2 [2].

2 Preliminary combinatorics

Let us consider a polynomial

$$f(x) = \sum_{1\le i\le M} x^{\vec{\alpha}(i)} \tag{2.1}$$

with $M \ge N+1$. Here $\vec{\alpha}(i)$ denotes the multi-index

$$\vec{\alpha}(i) = (\alpha_1^i, \cdots, \alpha_N^i) \in \mathbf{Z}^N.$$

In the case when $M > N$ we associate to $f(x)$ another polynomial in $M-1$ variables $f^\sigma(x,x')$

$$f^\sigma(x,x') = \sum_{i=1}^{M-N-1} x'_i x_1^{\alpha(\sigma(i))} + \sum_{j=M-N}^{M} x_i^{\alpha(\sigma(j))} \tag{2.2}$$

with $\sigma \in \mathrm{S}_M$, the permutation group of M elements. Here we used the notation of the multi-index:

$$\vec{\alpha}(\sigma(i)) = (\alpha_1^{\sigma(i)}, \cdots, \alpha_N^{\sigma(i)}) \in \mathbf{Z}^N.$$

In this situation, the expression $u(f^\sigma(x,x') + s)$ is a polynomial depending on $(M+1)$ variables $(x_1, \cdots, x_N, x'_1, \cdots, x'_{M-N-1}, s, u)$. Further we shall assume

$$supp(f^\sigma) \cap int(\Delta(f^\sigma)) = \emptyset. \tag{2.3}$$

Here $\Delta(f^\sigma)$ denotes the Newton polyhedron of $f^\sigma(x,x')$.

Remark 1 *A polynomial that depends on $(M+1)$−variables and contains $(M+1)$ monomials is called of Delsarte type. Jean Delsarte proposed to study algebraic cycles on the hypersurface defined by a polynomial of this class.*

Let us introduce new variables $T_1, \cdots T_{M+1}$:

$$T_1 = ux'_1 x_1^{\vec{\alpha}(\sigma(1))}, T_2 = ux'_2 x_2^{\vec{\alpha}(\sigma(2))}, \cdots \tag{2.4}$$

$$T_{M-N-1} = ux'_{M-N-2} x^{\vec{\alpha}(\sigma(M-N-1))}, T_{M-N} = ux^{\vec{\alpha}(\sigma(M-N))}, \cdots, T_{M+1} = us.$$

To express the situation in a compact form, we use the following notations:

$$\Xi :=^t (x_1, \cdots, x_N, x'_1, \cdots, x'_{M-N-1}, u, s), \tag{2.5}$$

$$Log\, T :=^t (log\, T_1, \cdots, log\, T_{M+1}) =^t (\tau_1, \cdots, \tau_{M+1}), \tag{2.6}$$

$$Log\, \Xi :=^t (\log\, x_1, \cdots, \log\, x_N, \log\, x'_1, \cdots, \log\, x'_{M-N-1}, \log\, u, , \log\, s). \tag{2.7}$$

In making use of these notations , we have the relation

$$\tau_1 = log\, u + log\, x'_1 + < \vec{\alpha}(\sigma(1)), log\, x >, \cdots, \tag{2.8}$$

$$\tau_{M-N-1} = log\, u + log\, x'_{M-N-1} + < \vec{\alpha}(\sigma(M-N-1)), log\, x >,$$

$$\tau_{M-N} = log\, u + < \vec{\alpha}(\sigma(M-N)), log\, x >, \cdots, \tau_{M+1} = log\, u + log\, s.$$

We can rewrite the relation (2.8) with the aid of a matrix $\mathsf{L}^\sigma \in End(\mathbf{Z}^{M+1})$, as follows:

$$Log\, T = \mathsf{L}^\sigma \cdot Log\, X. \tag{2.9}$$

where

$$\mathsf{L}^\sigma = \begin{bmatrix} \alpha_1^{\sigma(1)} & \cdots & \alpha_N^{\sigma(1)} & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \alpha_1^{\sigma(2)} & \cdots & \alpha_N^{\sigma(2)} & 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ \vdots & \cdots & \vdots & 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\ \alpha_1^{\sigma(M-N-1)} & \cdots & \alpha_N^{\sigma(M-N-1)} & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ \alpha_1^{\sigma(M-N)} & \cdots & \alpha_N^{\sigma(M-N)} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \alpha_1^{\sigma(M)} & \cdots & \alpha_N^{\sigma(M)} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}, \tag{2.10}$$

Further we shall assume that the determinant of the matrix L^σ is positive. This assumption is always satisfied without loss of generality, if we permute certain column vectors of the matrix, which evidently corresponds to the change of positions of variables x. We denote the determinant by $\gamma^\sigma = det(L^\sigma)$. The row vectors of L^σ will be denoted by $\vec{e}_1^\sigma, \cdots, \vec{e}_{M+1}^\sigma$. Later we will make use of the notation of variables $X := (X_1, \cdots X_{M-1}) := (x_1, \cdots, x_N, x'_1, \cdots, x'_{M-N-1})$ and that of the polynomial $f^\sigma(x, x') = f^\sigma(X)$.

Definition 2 *We call that polynomial $f(x)$ is simpliciable if for every $\sigma \in \mathsf{S}_M$, $det(L^\sigma) = \gamma^\sigma \neq 0$.*

For $\tau \subset \Delta(f^\sigma)$ we denote by $\Sigma(\tau)$ a $(dim\ \tau + 1)-$ dimensional simplex consisting all segments connecting $\{0\}$ and a point of τ. Let us define a graded algebra

$$S_\tau := \bigcup_{\frac{\alpha}{k} \in \Sigma(\tau), \exists k \geq 1} \mathbf{C}X^\alpha. \tag{2.11}$$

and a polynomial

$$f^{\sigma,\tau}(X) := \sum_{\alpha \in supp(f^\sigma) \cap \tau} X^\alpha \tag{2.12}$$

Lemma 2.1 *If $f(x)$ is a simpliciable polynomial, then $f^\sigma(X)$ is $\Delta(f^\sigma)-$ regular.*

Proof *The condition $det(L^\sigma) = \gamma^\sigma \neq 0$ yields that $X_1 \frac{\partial f^{\sigma,\tau}}{\partial X_1}$, $X_2 \frac{\partial f^{\sigma,\tau}}{\partial X_2}$, $\cdots X_{M-1} \frac{\partial f^{\sigma,\tau}}{\partial X_{M-1}}$, form a regular sequence in S_τ for any face $\tau \subset \Delta(f^\sigma)$.* **Q.E.D.**

3 Mellin transforms

In this section we proceed to the calculation of the Mellin transform of the fibre integrals associated to the hypersurface $Z_{f^\sigma+s} = \{X \in \mathbf{T}^{M-1}; f^\sigma(X)+s = 0\}$ defined by a simpliciable polynomial. First of all we consider the fibre integral taken along the fibre $\gamma(s) \in H_{M-2}(Z_{f^\sigma+s})$ as follows,

$$I^\sigma_{X^{\mathbf{J}},\partial\gamma}(s) := \int_{\gamma(s)} \frac{X^{\mathbf{J}-\mathbf{1}}dX}{df^\sigma(X)} = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\gamma(s)} \frac{X^{\mathbf{J}}dX}{(f^\sigma(X)+s)X^{\mathbf{1}}} \tag{3.1}$$

where $\partial\gamma(s) \in H_{M-1}(\mathbf{T}^{M-1} \setminus Z_{f^\sigma+s})$ is a cycle obtained after the application of ∂, Leray's coboundary operator. Here $X^{\mathbf{1}} = X_1 \cdots X_{M-1}$, $X^{\mathbf{J}} = X_1^{i_1} \cdots, X_{M-1}^{i_{M-1}}$. See the works by F.Pham and V.A.Vassiliev ([23]) for the Leray's coboundary operator.

The Mellin transform of $I^\sigma_{X^J,\partial\gamma}(s)$ is defined by the following integral:

$$M^\sigma_{X^J}(z) := \int_\Pi (-s)^z I^\sigma_{X^J,\partial\gamma}(s) \frac{ds}{s}. \tag{3.2}$$

Here Π stands for a cycle in $\mathbf{C}$ that avoids the poles of $I_{X^J,\partial\gamma}(s)$. We assume that on the set $\partial\gamma^\Pi := \cup_{s\in\Pi}\partial(s,\gamma(s))$, $\Re(f^\sigma(X)+s) \to +\infty$. We denote by $\mathcal{L}_q(\mathbf{J}, z)$ the inner product of $(\mathbf{J}, z, 1)$ with the $q-$th column vector of $(\mathsf{L}^\sigma)^{-1}$. Let us deform the integral (6.2) in making use of the definition (3.1):

$$\begin{aligned} M^\sigma_{X^J}(z) &= \int_{\mathbf{R}_- \times \partial\gamma^\Pi} e^{u(f^\sigma(X)+s)} X^{\mathbf{J}} u(-s)^z \frac{du}{u} \wedge \frac{dX}{X^{\mathbf{1}}} \wedge \frac{ds}{s} \\ &= \frac{1}{\gamma^\sigma} \int_{(L^\sigma)_*(\mathbf{R}_- \times \partial\gamma^\Pi)} e^{\Psi(T)} \prod_{q=1}^{M+1} T_q^{\mathcal{L}_q(\mathbf{J},z)} \prod_{q=1}^{M+1} \bigwedge \frac{dT_q}{T_q}. \end{aligned} \tag{3.3}$$

where

$$\Psi(T) = T_1(X,s,u) + \cdots + T_M(X,s,u) = u(f^\sigma(X)+s) \tag{3.4}$$

where each term $T_i(X,s,u)$ represents a monomial term of variables X, s, u of the polynomial (3.4). By virtue of the simple structure of the matrix L^σ (2.10), we can consider the simplex polyhedron $\tau^\sigma_q \in \mathbf{R}^{M-1}$ defined as $\langle \vec{e}^\sigma_1, \overset{q}{\cdot\vee\cdot}, \vec{e}^\sigma_{M+1} \rangle$. where we identify $\vec{e}^\sigma_i \in \mathbf{Z}^{M+1}$ with that of $\mathbf{Z}^{M-1}$ after ignoring the last two entries. It means that we identify $\vec{e}_i^\sigma$ with the $i-$th row vector of the matrix L^σ of which one removes the last two columns ${}^t(0,0,\cdots,0,1),{}^t(1,1,\cdots,1) \in \mathbf{Z}^{M+1}$. The chain $\mathbf{R}_- \times \partial\gamma^{\Pi}$ can be deformed in $\mathbf{C}^{M+1}$ so far as it does not encounter the singularity of the integrand.

Proposition 3.1 *1) The Mellin transform $M^\sigma_{X^J}(z)$ of the fibre integral associated to the simpliciable polynomial $f^\sigma(X)$ has the following form.*

$$M^\sigma_{X^J}(z) = g(z)\prod_{q=1}^{M}\Gamma(\mathcal{L}_q(\mathbf{J},z)), 1 \le q \le M+1, \tag{3.5}$$

where $g(z)$ is a rational function in $e^{\frac{\pi i z}{\gamma^\sigma}}$ with $\gamma^\sigma = (M-1)!vol(\Delta(f^\sigma))$. The linear function

$$\mathcal{L}_q(\mathbf{J},z) = {}^t(\mathbf{J},z,1)\vec{w}_q^\sigma, \tag{3.6}$$

where $\vec{w}_q^\sigma$ is the $q-$th column vector of the matrix $(\mathsf{L}^\sigma)^{-1}$.

2) The $M+1$ linear functions $\mathcal{L}_q(\mathbf{J},z)$ are classified into the following three groups.

$$\mathcal{L}_{M+1}(\mathbf{J},z) = \frac{B^\sigma_{M+1}}{\gamma^\sigma}z = \frac{\gamma^\sigma}{\gamma^\sigma}z = z. \tag{3.7$_1$}$$

For q such that $\vec{w}_q^\sigma = B_q^\sigma(\vec{v}_q^\sigma, 1, -1)$ for some $\vec{v}_q^\sigma \in \mathbf{Q}^{M-1}$, and $B_q^\sigma \neq 0$,

$$\mathcal{L}_q(\mathbf{J},z) = \frac{B_q^\sigma(<\vec{v}_q^\sigma, \mathbf{J}> + z - 1)}{\gamma^\sigma}. \tag{3.7$_2$}$$

For q such that $\vec{w}_q^\sigma = (\vec{v}_q^\sigma, 0, 0)$ for some $\vec{v}_q^\sigma \in \mathbf{Q}^{M-1}$, and $B_q^\sigma = 0$,

$$\mathcal{L}_q(\mathbf{J},z) = \frac{(<\vec{v}_q^\sigma, \mathbf{J}>)}{\gamma^\sigma}. \tag{3.7$_3$}$$

Here the case $(3.7)_3$ corresponds to such q that $\dim \tau_q^\sigma < M-1$.

3)

$$|B_q^\sigma| = (M-1)!vol(\tau_q^\sigma). \tag{3.8}$$

4) For $\mathbf{J} \in \tau_q^\sigma \cap \Delta(f^\sigma)$, with $\dim \tau_q^\sigma = M-1$, $\tau_q^\sigma \neq \Delta(f^\sigma)$,

$$\langle \vec{v}_q^\sigma, \mathbf{J}\rangle = 1.$$

$$\langle \vec{v}_{M+1}^\sigma, \mathbf{J}\rangle = 0.$$

Proof 1) The definition of the $\Gamma-$ function sounds as follows;

$$\int_{\bar{\mathbf{R}}_-} e^T(-T)^\sigma \frac{dT}{T} = (1-e^{2\pi i\sigma})\int_{\mathbf{R}_-} e^T(-T)^\sigma\frac{dT}{T} = (1-e^{2\pi i\sigma})\Gamma(\sigma),$$

for the unique nontrivial cycle $\bar{\mathbf{R}}_-$ turning around $T=0$ that begins and returns to $\Re T \to -\infty$. We apply it to the integral (3.3) and get (3.5). We consider an action λ on the chain $C_a = \bar{\mathbf{R}}_-$ or $\mathbf{R}_-$ on the complex T_a plane, $\lambda : C_a \to \lambda(C_a)$ defined by the relation

$$\int_{\lambda(C_a)} e^{T_a}(-T_a)^{\sigma_a}\frac{dT_a}{T_a} = \int_{(C_a)} e^{-T_a}(-e^{2\pi\sqrt{-1}}T_a)^{\sigma_a}\frac{dT_a}{T_a}.$$

By means of this action the chain $L_*(\mathbf{R}_- \times \gamma^{\Pi})$ turns out to be homologous to

$$\sum_{(j_1^{(\rho)},\cdots,j_{M+1}^{(\rho)})\in[1,\gamma^\sigma]^{M+1}} m_{j_1^{(\rho)},\cdots,j_{M+1}^{(\rho)}} \lambda^{j_1^{(\rho)}}(\mathbf{R}_-) \prod_{a'=2}^{\gamma^\sigma} \lambda^{j_{a'}^{(\rho)}}(\bar{\mathbf{R}}_-),$$

with $m_{j_1^{(\rho)},\cdots,j_{M+1}^{(\rho)}} \in \mathbf{Z}$. This situation explains the presence of the factor $g(z) = \sum_{(j_1^{(\rho)},\cdots,j_{M+1}^{(\rho)})\in[1,\gamma^\sigma]^{M+1}}$ $m_{j_1^{(\rho)},\cdots,j_{M+1}^{(\rho)}}$ $e^{2\pi\sqrt{-1}j_1^{(\rho)}\mathcal{L}_1(\mathbf{J},\mathbf{z},\zeta)} \prod_{a'=2}^{M+1} e^{2\pi\sqrt{-1}j_{a'}^{(\rho)}\mathcal{L}_{a'}(\mathbf{J},\mathbf{z},\zeta)}(1 - e^{2\pi\sqrt{-1}\mathcal{L}_{a'}(\mathbf{J},\mathbf{z},\zeta)})$ except for the Γ−f unction factor.

The points 2)- 5) are reduced to the linear algebra. For example 3) can be shown, if one remembers the definition of M minors of the matrix $\mathbf{L}^\sigma$ calculated in removing the M−th column. 4) If $\mathbf{J} \in \tau_q^\sigma$, the vector $\vec{e}_i^\sigma$ is orthogonal to $(\vec{v}_{M+1}^\sigma, 1, -1)$ for $i \neq q$ and $\langle \vec{e}_q^\sigma, B_q^\sigma(\vec{v}_{M+1}^\sigma, 1, -1)\rangle = \gamma^\sigma$. The result on the M−th and $(M+1)$−st element is explained by the fact that $\vec{e}_{M+1}^\sigma = (0, \cdots, 0, 1, 1)$ is orthogonal to $(\vec{v}_{M+1}^\sigma, 1, -1)$ for $1 \leq q \leq M$.

Q.E.D.

Let us denote the set of such indices q with strictly positive (resp. strictly negative) B_q^σ by $I^+ \subset \{1, \cdots, M+1\}$, (resp. by $I^- \subset \{1, \cdots, M+1\}$). The set of indices q for which $B_q^\sigma = 0$ will be denoted by I^0. With these notations, one can formulate the following,

Corollary 3.2 *1) The Newton polyhedron admits the following representation,* $\Delta(f^\sigma) = \{\vec{i} \in \mathbf{R}^M;$ $\langle \vec{v}_q^\sigma, \vec{i}\rangle \geq 1$ *for* $q \in I^+$, $\langle \vec{v}_{\bar{q}}^\sigma, \vec{i}\rangle \leq 1$ *for* $\bar{q} \in I^-$, $\langle \vec{v}_{q^0}^\sigma, \vec{i}\rangle \geq 0$ *for* $q^0 \in I^0$ $\}$.

2) We denote by $\chi(Z_{f^\sigma+1})$ *the Euler-Poincaré characteristic of the hypersurface* $Z_{f^\sigma+1} = \{X \in \mathbf{T}^{M-1}; f^\sigma(X)+1\}$ *here under the constant* 1 *we understand a generic value for* $f^\sigma(X)$. *The following equality holds,*

$$\sum_{q\in I^+} B_q^\sigma = (M-1)!vol_{M-1}(\Delta(f^\sigma(X)+1)) = (-1)^M\chi(Z_{f^\sigma+1}). \tag{3.9}$$

3) $\sum_{q=1}^{M+1} B_q^\sigma = 0$. *In other words,*

$$\sum_{\bar{q}\in I^-} B_{\bar{q}}^\sigma = -(\sum_{q\in I^+} B_q^\sigma). \tag{3.10}$$

Proof 1) After the definition of vectors $\vec{v}_1^\sigma, \cdots, \vec{v}_M^\sigma$ we can argue as follows. If $\vec{i}$ does not belong to the hyperplane $\langle \vec{e}_1^\sigma, \overset{q}{\check{\cdots}}, \vec{e}_M^\sigma\rangle$, then $\langle \vec{v}_q^\sigma, \vec{i}\rangle = 1 + \frac{\gamma^\sigma}{B_q^\sigma}$. In the case when $q \in I^+$ (resp. $\bar{q} \in I^-$) $\langle \vec{v}_q^\sigma, \vec{i}\rangle > 1$(resp. $\langle \vec{v}_q^\sigma, \vec{i}\rangle < 1$) that is equivalent to say that all the points $\vec{i}$ of the Newton polyhedron $\Delta(f^\sigma)$ satisfy $\langle \vec{v}_q^\sigma, \vec{i}\rangle \geq 1$ for $q \in I^+$ (resp.$\langle \vec{v}_q^\sigma, \vec{i}\rangle \leq 1$ for $q \in I^-$). If $\vec{i} \in \langle \vec{e}_1^\sigma, \overset{q}{\check{\cdots}}, \vec{e}_M^\sigma\rangle$, then $\langle \vec{v}_q^\sigma, \vec{i}\rangle = 1$. For $q^0 \in I^0$, $\Delta(f^\sigma) \subset \{\vec{i}; \langle \vec{v}_{q^0}^\sigma, \vec{i}\rangle \geq 0\}$, because $\langle \vec{v}_{q^0}^\sigma, \vec{i}\rangle = 1$ for $\vec{i} \notin \langle \vec{e}_1^\sigma, \overset{q^0}{\check{\cdots}}, \vec{e}_M^\sigma\rangle$. As all possible cases are exhausted by I^+, I^-, I^0, $|I^+| + |I^-| + |I^0| = M$. This yields the statement. 2) Apply the Theorem by [9],[13] on the Euler characteristic. 7) The $(M+1)$−st column vector of $\mathbf{L}^\sigma$ is orthogonal to the M−th row vector of $\mathbf{L}^{\sigma-1}$, $(B_1^\sigma, \cdots, B_{M+1}^\sigma)$.

Corollary 3.3 *Under the above situation, the Mellin inverse of* $M_{X^{\mathbf{J}},\gamma}^\sigma(s)$ *with properly chosen periodic function* $g(z)$ *with period* γ^σ :

$$I_{X^{\mathbf{J}},\gamma'}^\sigma(s) = \int_{\check{\Pi}} g(z)\frac{\prod_{a\in I^+}\Gamma(\mathcal{L}_a(\mathbf{J},z))}{\prod_{\bar{a}\in I^-}\Gamma(1-\mathcal{L}_{\bar{a}}(\mathbf{J},z))}s^{-z}dz, \tag{3.11}$$

defines a convergent analytic function in $-\pi < arg\ s < \pi$.

Proof In applying the Stirling's formula

$$\Gamma(z+1) \sim (2\pi z)^{\frac{1}{2}} z^z e^{-z}, \quad \Re z \to +\infty,$$

to the integrand of (3.11), we take into account the relation (3.10). Here we remind us of the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\ \pi z}$. As for the choice of the rational function $g(z)$ one makes use of Nörlund's technique. In this way we can choose such $g(z)$ that the integrand is of exponential decay on $\check{\Pi}$. **Q.E.D.**

Example Let us illustrate the above procedures by a simple example.

$$f(x) = x_1^5 + x_1^2 x_2 + x_1 x_2^2 + x_2^4. \tag{3.12}$$

We have 4 possibilities to add a new variable x_1' so that the polynomial (3.12) becomes a simplicial.

$$f^{\sigma_1}(x,x') = x_1' x_1^5 + x_1^2 x_2 + x_1 x_2^2 + x_2^4.$$
$$f^{\sigma_2}(x,x') = x_1^5 + x_1' x_1^2 x_2 + x_1 x_2^2 + x_2^4.$$
$$f^{\sigma_3}(x,x') = x_1^5 + x_1^2 x_2 + x_1' x_1 x_2^2 + x_2^4.$$
$$f^{\sigma_4}(x,x') = x_1^5 + x_1^2 x_2 + x_1 x_2^2 + x_1' x_2^4.$$

Let us calculate L^{σ_3} and $(\mathsf{L}^{\sigma_3})^{-1}$.

$$\mathsf{L}^{\sigma_3} = \begin{bmatrix} 5 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$(\mathsf{L}^{\sigma_3})^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -4 & 0 & 1 & 0 \\ 2 & -5 & 0 & 3 & 0 \\ 1 & -6 & 7 & -2 & 0 \\ 8 & -20 & 0 & 5 & 7 \\ -8 & 20 & 0 & -5 & 0 \end{bmatrix}.$$

Let us denote by $\vec{e}_1 = (5,0,0)$,$\vec{e}_2 = (2,1,0)$,$\vec{e}_3 = (1,2,1)$, $\vec{e}_4 = (0,4,0)$,$\vec{e}_5 = (0,0,0)$. Then we have

$$vol(\tau_5) = 3!vol(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) = 7.$$

Similarly $vol(\tau_4) = 5$, $vol(\tau_3) = 0$, $vol(\tau_2) = 20$, $vol(\tau_1) = 8$. Remark $\tau_1 + \tau_3 + \tau_4 + \tau_5 = \tau_2$ (a subdivision of simplex into three simplices) which yields $7 + 8 + 5 = 20$. The face not affected (see Definition 3 below) by σ_3 is that spanned by $\vec{e}_1, \vec{e}_2, \vec{e}_4$.

4 Hodge structure of the fibre integrals

Now we can state the relationship between the Hodge structure of the $PH^{M-2}(Z_{f^\sigma})$ and the poles of the Mellin transform after suitable period function multiplication $\frac{1}{g(z)} M^{\sigma}_{X^J,\gamma'}(z) = \frac{\prod_{a \in I^+} \Gamma(\mathcal{L}_a(J,z))}{\prod_{a \in I^-} \Gamma(1-\mathcal{L}_a(J,z))}$. Here we remind us of the relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin \pi z}$. We will misuse the expression "the poles of the Mellin transform" in meaning those of $\frac{\prod_{a \in I^+} \Gamma(\mathcal{L}_a(J,z))}{\prod_{a \in I^-} \Gamma(1-\mathcal{L}_a(J,z))}$.

Theorem 4.1 *1) For $X^J \in Gr_F^p Gr_{M-2}^w PH^{M-2}(Z_{f^\sigma})$, $0 \le p \le M-1$, the following properties hold*

a)

$$0 < \langle \vec{v}^{\sigma}_{q^0}, \mathbf{J} \rangle < M - 1 - p \ \ for \ q^0 \in I^0$$

$$M - 1 - p < \langle \vec{v}^{\sigma}_{q}, \mathbf{J} \rangle < (M - 1 - p)(1 + \frac{\gamma^{\sigma}}{B^{\sigma}_{q}}) \ \ q \in I^+$$

$$(M - 1 - p)(1 + \frac{\gamma^{\sigma}}{B^{\sigma}_{\bar{q}}}) < \langle \vec{v}^{\sigma}_{\bar{q}}, \mathbf{J} \rangle < M - 1 - p \ \ for \ \bar{q} \in I^-$$

if not $\langle \vec{v}^{\sigma}_{q}, \mathbf{J} \rangle = 0$.

b) *The maximal pole of the Mellin transform satisfies;*

$$1 - (M - 1 - p)(1 + max_{q \in I^+} \frac{\gamma^{\sigma}}{B^{\sigma}_{q}}) < z < 2 + p - M.$$

Here the pole is not necessarily a simple pole.

2) For $X^{\mathbf{J}} \in Gr^p_F Gr^w_{M-1} PH^{M-2}(Z_{f^{\sigma}})$, $0 \leq p \leq M - 1$, *the following properties hold*

a) *There exists unique index* $q \in I^+$ *such that:*

$$\langle \vec{v}^{\sigma}_{q}, \mathbf{J} \rangle = M - 1 - p$$

b) *The maximal pole of the Mellin transform is the simple pole*

$$z = 2 + p - M.$$

3) For $X^{\mathbf{J}} \in Gr^p_F Gr^w_{M-2+r} PH^{M-2}(Z_{f^{\sigma}})$, $1 \leq r \leq M-3$, $0 \leq p \leq M-1$, *the following properties hold.*

a) *There exist* r *indices* $q_1, \cdots q_r \in I^+$ *such that:*

$$\langle \vec{v}^{\sigma}_{q_1}, \mathbf{J} \rangle = \langle \vec{v}^{\sigma}_{q_2}, \mathbf{J} \rangle = \cdots = \langle \vec{v}^{\sigma}_{q_r}, \mathbf{J} \rangle = M - 1 - p,$$

but no such $r + 1$ *pair of indices* $q_1, \cdots, q_{r+1}$.

b) *The maximal pole of the Mellin transform satisfies;*

$$z = 2 + p - M,$$

which is of order $\leq r + 1$ *i.e. there can be cancellation of poles.*

The defect number $(r + 1)-$ {order of poles } will be described in §5.

Proof of the theorem can be achieved by a combination of Theorems 1.2, 1.3 and the Proposition 3.1, Corollary 3.2. We remember here that the $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \cdots$.

The above theorem mentions about how the Hodge structure of $PH^{M-2}(Z_{f^{\sigma}})$ influences on the poles of the Mellin transform. How about the original Hodge structure $PH^{N-1}(Z_f)$? To state this relationship, we need to introduce the following notion.

Definition 3 *The face* $\tau \in \Delta(f)$ *is called "not affected by* σ*"* $\in \mathsf{S}_M$ *if* $\tau \in \Delta(f^{\sigma})$ *after the extension of* $(i_1, \cdots, i_N) \in \tau \subset \mathbf{R}^N$ *into* $\mathbf{R}^M$ *transforming it into the vector* $(\mathbf{i}, 0) = (i_1, \cdots, i_N, 0, \cdots, 0, 0) \in \mathbf{R}^M$.

The face not affected by σ for the polynomial (2.2) is a face (or its sub-face) spanned by the vertices

$$\sum_{j=M-N}^{M} x_i^{\alpha(\sigma(j))}$$

i.e. vertices free of x'_i.

Theorem 4.2 *1) For $x^{\mathbf{i}} \in Gr_F^p Gr_{N-1}^w PH^{N-1}(Z_f)$, $0 \leq p \leq N$, for which $(\mathbf{i}, 0)$ lies in $supp(\frac{S_{\Delta(f^\sigma)}}{J_{f^\sigma,\Delta(f^\sigma)}})$ not affected by σ, the following properties hold*

a)

$$0 < \langle \vec{v}^\sigma_{q^0}, (\mathbf{i}, 0)\rangle < N - p \;\; for\; q^0 \in I^0,$$

$$N - p < \langle \vec{v}^\sigma_q, (\mathbf{i}, 0)\rangle < (N - p)(1 + \frac{\gamma^\sigma}{B^\sigma_q}) \;\; for\; q \in I^+,$$

$$(N - p)(1 + \frac{\gamma^\sigma}{B^\sigma_{\bar{q}}}) < \langle \vec{v}^\sigma_{\bar{q}}, (\mathbf{i}, 0)\rangle < N - p \;\; for\; \bar{q} \in I^-,$$

if not $\langle \vec{v}^\sigma_q, (\mathbf{i}, 0)\rangle = 0$, *or* $\langle \vec{v}^\sigma_{\bar{q}}, (\mathbf{i}, 0)\rangle = 0$.

b) *The maximal pole of the Mellin transform satisfies;*

$$1 - (N - p)(1 + max_{q\in I^+}\frac{\gamma^\sigma}{B^\sigma_q}) < z < 1 - N + p.$$

Here the pole is not necessarily a simple pole.

The proof is straightforward if one applies Theorem 4.1 to $\Delta(f)$. We consider the $N-$dimensional face $\tau^\sigma_q \subset \mathbf{Z}^N$ that is a $N-$dimensional simplex contained in $\Delta(f)$. One can verify that there exist $(\mathbf{i}, 0) \in supp(\frac{S_{\Delta(f^\sigma)}}{J_{f^\sigma,\Delta(f^\sigma)}})$ such that $x^{\mathbf{i}} \in Gr_F^p Gr_{N-1}^w PH^{N-1}(Z_f)$, $0 \leq p \leq N - 1$ for the cases $N = 2, 3, 4$ by means of polyhedra realizing the formulae 5.11, [4].

We remark the following simple combinatorial fact.

Proposition 4.3 *For every $x^{\mathbf{i}} \in Gr_F^p Gr_{N-1}^w PH^{N-1}(Z_f)$, there exists an element $\sigma \in \mathsf{S}_M$ such that $x^{\mathbf{i}}$ is not affected by σ. That is to say there exists $\sigma \in \mathsf{S}_M$ such that $x^{\mathbf{i}} \in S_{\Delta(f)} \cap S_{\Delta(f^\sigma)}$.*

5 Hypergeometric group associated to the fibre integrals

Let us introduce two differential operators of order $\Delta^\sigma := (M-1)!vol_{M-1}(\Delta(f^\sigma(X) + 1)) = |\chi(Z_{f^\sigma+1})| = |I^+| = |I^-|$;

$$P^\sigma_{\mathbf{J}}(\vartheta_s) = \prod_{q\in I^+} \prod_{j=0}^{B^\sigma_q - 1} \mathcal{L}_q(\mathbf{J}, -\vartheta_s + \frac{\gamma^\sigma j}{B^\sigma_q}) \tag{5.1}$$

$$Q^\sigma_{\mathbf{J}}(\vartheta_s) = \prod_{\bar{q}\in I^-} \prod_{j=1}^{-B^\sigma_{\bar{q}}} (-\mathcal{L}_{\bar{q}}(\mathbf{J}, -\vartheta_s - \gamma^\sigma(1 + \frac{j}{B^\sigma_{\bar{q}}}))), \tag{5.2}$$

where I^+, I^- are those sets of indices introduced in §3.

Theorem 5.1 *The fibre integral $I^\sigma_{X^{\mathbf{J}},\gamma}(s)$ is annihilated by the operator*

$$R^\sigma_{\mathbf{J}}(\vartheta_s) = P^\sigma_{\mathbf{J}}(\vartheta_s) - s^{\gamma^\sigma} Q^\sigma_{\mathbf{J}}(\vartheta_s), \tag{5.3$_1$}$$

that is to say

$$[P^\sigma_{\mathbf{J}}(\vartheta_s) - s^{\gamma^\sigma} Q^\sigma_{\mathbf{J}}(\vartheta_s)] I^\sigma_{X^{\mathbf{J}},\gamma}(s) = 0. \tag{5.4}$$

It is worthy to remark that the operator $R^{\sigma}_{\mathbf{J}}(\vartheta_s)$ is a push-forward of the Pochhammer hypergeometric operator of order Δ^{σ},

$$(5.3)_2 \qquad P^{\sigma}_{\mathbf{J}}(\gamma^{\sigma}\vartheta_t) - tQ^{\sigma}_{\mathbf{J}}(\gamma^{\sigma}\vartheta_t),$$

by the Kummer covering $t = s^{\gamma^{\sigma}}$. In certain cases, the operator (5.3) turns out to be reducible. Let us introduce the following set of rational numbers.

$$C^{+}(\mathbf{J}) = \cup_{q\in I^{+}} \cup_{0\leq j\leq B^{\sigma}_q-1} \{\frac{j}{B^{\sigma}_q} - \frac{(<\vec{v}^{\sigma}_q, \mathbf{J}> -1)}{\gamma^{\sigma}}\}.$$

$$C^{-}(\mathbf{J}) = \cup_{\bar{q}\in I^{-}} \cup_{1\leq j\leq -B^{\sigma}_{\bar{q}}-1} \{\frac{j}{B^{\sigma}_{\bar{q}}} - \frac{(<\vec{v}^{\sigma}_{\bar{q}}, \mathbf{J}> -1)}{\gamma^{\sigma}}\}.$$

$$C^{0}(\mathbf{J}) = C^{+}(\mathbf{J}) \cap C^{-}(\mathbf{J}).$$

We define a positive integer $\bar{\Delta}^{\sigma} = \sharp|C^{+}(\mathbf{J})\setminus C^{0}(\mathbf{J})| = \sharp|C^{-}(\mathbf{J})\setminus C^{0}(\mathbf{J})|$. Then "the nontrivial part" of $(5.3)_2$ (i.e. after the division by operators with rational function solution of type s^{α^0}, $\alpha^0 \in C^0(\mathbf{J})$) can be defined as

$$\bar{R}^{\sigma}_{\mathbf{J}}(\vartheta_t) = \prod_{\alpha^{+}\in C^{+}(\mathbf{J})\setminus C^{0}(\mathbf{J})} (\vartheta_t + \alpha^{+}) - t \prod_{\alpha^{-}\in C^{-}(\mathbf{J})\setminus C^{0}(\mathbf{J})} (\vartheta_t + \alpha^{-} + 1),$$

as an operator of order $\bar{\Delta}^{\sigma}$ up to multiplication by a constant to the variable "t".

We consider solutions $u_{\ell,m}(t), 1 \leq \ell \leq \bar{\Delta}^{\sigma}$, to the equation

$$(5.5) \qquad \bar{R}^{\sigma}_{\mathbf{J}}(\vartheta_t)u_{\ell,m}(t) = 0,$$

with the asymptotic behaviour

$$(5.5)_1 \qquad u_{\ell,m}(t) \cong t^{\rho^{\ell}_{\mathbf{J}}} \sum_{\nu=0}^{m} (log\ t)^{\nu} A_{\ell,\nu}(t).$$

Here $0 \leq m \leq m_{\ell}$, $\sum_{\ell}(m_{\ell}+1) = \bar{\Delta}^{\sigma}$, $A_{\ell}(t)$ holomorphic in the neighbourhood of $t = 0$. Similarly, we consider the asymptotic behaviour at $t = \infty$ of the solutions to (5.5)

$$v_{\ell,k}(t) \cong (\frac{1}{t})^{\bar{\rho}^{\ell}_{\mathbf{J}}} \sum_{\mu=0}^{k} (log\ t)^{\mu} B_{\ell}(\frac{1}{t}).$$

Here $0 \leq k \leq k_{\ell}$, $\sum_{\ell}(k_{\ell}+1) = \bar{\Delta}^{\sigma}$, $B_{\ell}(\frac{1}{t})$ holomorphic in the neighbourhood of $\frac{1}{t} = 0$. Here $m_{\ell}+1$ (resp.$k_{\ell}+1$) denotes the multiplicity of $-\rho^{\ell}_{\mathbf{J}}$ (resp. $-\bar{\rho}^{\ell}_{\mathbf{J}}$) in the set $C^{+}(\mathbf{J})\setminus C^{0}(\mathbf{J})$ (resp. $C^{-}(\mathbf{J})\setminus C^{0}(\mathbf{J})$).

Under this situation, we define characteristic polynomials of the exponents of solutions to (5.5) at $t = 0$

$$X_{0,\mathbf{J}}(t) = \prod_{\ell=1}^{\bar{\Delta}^{\sigma}} (t - e^{2\pi\rho^{\ell}_{\mathbf{J}}\sqrt{-1}}) = \prod_{\alpha^{+}\in C^{+}\setminus C^{0}} (t - e^{-2\pi\sqrt{-1}\alpha^{+}}),$$

and $t = \infty$

$$X_{\infty,\mathbf{J}}(t) = \prod_{\ell=1}^{\bar{\Delta}^{\sigma}} (t - e^{2\pi\bar{\rho}^{\ell}_{\mathbf{J}}\sqrt{-1}}) = \prod_{\alpha^{+}\in C^{-}\setminus C^{0}} (t - e^{-2\pi\sqrt{-1}\alpha^{-}}).$$

Especially in the case $C^0 = \emptyset$, we have the following simple formulae.

Corollary 5.2 *The characteristic polynomials defined above can be calculated in the following way.*

$$X_{0,\mathbf{J}}(t) = \prod_{q \in I^+} (t^{B_q^\sigma} - e^{-2\pi(1-\langle \vec{v}_q^\sigma, \mathbf{J}\rangle)\frac{B_q^\sigma}{\gamma^\sigma}\sqrt{-1}}), \tag{5.6$_1$}$$

$$X_{\infty,\mathbf{J}}(t) = \prod_{q \in I^-} (t^{-B_q^\sigma} - e^{-2\pi(1-\langle \vec{v}_q^\sigma, \mathbf{J}\rangle)\frac{B_q^\sigma}{\gamma^\sigma}\sqrt{-1}}). \tag{5.6$_2$}$$

For the polynomials introduced in $(5.6)_1, (5.6)_2$, we introduce two vectors $(A_1, A_2, \cdots, A_{\bar{\Delta}^\sigma})$, $(B_1, B_2, \cdots, B_{\bar{\Delta}^\sigma}) \in \mathbf{C}^{\bar{\Delta}^\sigma}$, after the following relation:

$$X_{0,\mathbf{J}}(t) = t^{\bar{\Delta}^\sigma} + A_1 t^{\bar{\Delta}^\sigma - 1} + A_2 t^{\bar{\Delta}^\sigma - 2} + \cdots + A_{\bar{\Delta}^\sigma},$$

$$X_{\infty,\mathbf{J}}(t) = t^{\bar{\Delta}^\sigma} + B_1 t^{\bar{\Delta}^\sigma - 1} + B_2 t^{\bar{\Delta}^\sigma - 2} + \cdots + B_{\bar{\Delta}^\sigma}.$$

Let us denote by $\omega^i, i = 0, 1, 2, \cdots, \gamma^\sigma - 1$ the non-zero singular points of the equation (5.4) i.e. $\{s \in \mathbf{C}; \prod_{q \in I^+} B_q^\sigma - (\prod_{\bar{q} \in I^-} B_{\bar{q}}^\sigma) s^{\gamma^\sigma} = 0\}$.

Proposition 5.3 *A representation of the hypergeometric group (global monodromy group) of the solutions to* (5.5) *is given by*

$$M_0 = h_0^{\gamma^\sigma}, M_{\omega^0} = h_1 = (h_0 h_\infty)^{-1}, M_\infty = h_\infty^{\gamma^\sigma}, M_{\omega^i} = h_\infty^{-i} h_1 h_\infty^i (i = 1, 2, \cdots, \gamma^\sigma - 1), \tag{5.7}$$

for the matrices

$$h_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_{\bar{\Delta}^\sigma} \\ 1 & 0 & \ddots & 0 & -A_{\bar{\Delta}^\sigma - 1} \\ 0 & 1 & \ddots & 0 & -A_{\bar{\Delta}^\sigma - 2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}, \tag{5.8}$$

$$(h_\infty)^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_{\bar{\Delta}^\sigma} \\ 1 & 0 & \ddots & 0 & -B_{\bar{\Delta}^\sigma - 1} \\ 0 & 1 & \ddots & 0 & -B_{\bar{\Delta}^\sigma - 2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix}.$$

where M_{ω^i} denotes the monodromy action around the point $\omega^i \in \mathbf{CP}^1_s$.

proof The monodromies of the solutions annihilated by $\bar{R}_{\mathbf{J}}^\sigma(\vartheta_t)$ are given by h_0, (resp. h_1, h_∞) after [12]. at $t = 0$, (resp.$t = 1, \infty$). Let us think of a γ^σ−leaf covering $\tilde{\mathbf{CP}}^1_t$ of $\mathbf{CP}^1_s$ that corresponds to the Kummer covering $s^{\gamma^\sigma} = t$. In lifting up the path around $t = 1$ the first leaf of $\tilde{\mathbf{CP}}^1_s$, the monodromy h_1 is sent to the conjugation with a path around $t = \infty$. That is to say we have $M_{\omega^1} = h_\infty^{-1} h_1 h_\infty$. For other leaves the argument is similar. **Q.E.D.**

In combining the above result with that of Theorem 4.1, 3), we get the following.

Corollary 5.4 *For $X^{\mathbf{J}} \in Gr_F^p Gr_{M-2+r}^w PH^{M-2}(Z_{f^\sigma})$, $1 \leq r \leq M-2$, $0 \leq p \leq M-1$ the size of a Jordan cell of the monodromies M_0 with unit eigenvalue arising from the term of the form* $(5.5)_1$ *with $\alpha^+ = \rho_{\mathbf{J}}^\ell$ is $r + 1 \setminus \sharp\{\alpha^+ \in C^+(\mathbf{J}) \setminus C^0(\mathbf{J}); \alpha^+ \in \mathbf{Z}\}$.*

proof It is enough to remember the following relation for a cycle C avoiding $z+\alpha=0$:

$$(r+1)!\int_C \frac{s^{-z}}{(z+\alpha)^{r+1}}dz = \int_C s^{-z}[(\frac{d}{dz})^r \frac{1}{(z+\alpha)}]dz$$

$$= \int_C \frac{1}{(z+\alpha)}[(-\frac{d}{dz})^r s^{-z}]dz = \int_C \frac{1}{(z+\alpha)} s^{-z}(log\ s)^r dz = 2\pi\sqrt{-1}s^{\alpha}(log\ s)^r.$$

If the set $C^0(\mathbf{J})$ is empty, the order of the poles of the Mellin transform for $X^{\mathbf{J}} \in Gr_F^p Gr_{M-2+r}^w PH^{M-2}(Z_{f^{\sigma}})$ is $r+1$ after Theorem 4.1, 3)a). If $C^0(\mathbf{J})$ is not empty, the order of poles is reduced by $\sharp\{\alpha \in C^+(\mathbf{J}) \setminus C^0(\mathbf{J}); \alpha \in \mathbf{Z}\}$. **Q.E.D.**

6 Local Milnor fibre

We describe here the mixed Hodge structure of the local (vanishing) cohomology of the Milnor fibre. From combinatorial point of view, the local structure is considered as a combination of combinatorics treated in the global case.

Let us consider a germ $f(x) \in \mathbf{C}[[x_1,\cdots,x_n]]$ that defines the isolated singularity at $x=0$. That is to say dimension $\mu(f)$ (Milnor number) of the Milnor ring $A(f)$ defined below is finite:

$$A(f) := \frac{\mathbf{C}[[x_1,\cdots,x_n]]}{\langle \frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\rangle \mathbf{C}[[x_1,\cdots,x_n]]}. \tag{6.1}$$

For a convex set

$$\Gamma_+(f) := \text{convex hull of}\{\vec{\alpha}+\mathbf{R}_+^n; \vec{\alpha} \in supp(f)\setminus\{0\}\}, \tag{6.2}$$

we define Newton boundary of the germ $f(x)$, $\Gamma(f) :=$ union of all closed compact faces of $\Gamma_+(f)$.

We call a germ $f(x)$ convenient if it allows a decomposition as follows,

$$f(x) = g(x) + R(x),$$

with $g(x) = \sum_{i=1}^n a_i x_i^{n_i}$, $\prod_{i=1}^n a_i \neq 0$, $n_i \geq 2$ for all $i \in [1,n]$ and $supp(R) \subset \Gamma_+(g)$.

Definition 4 *A germ $f(x)$ is called non-degenerate with respect to its Newton boundary $\Gamma(f)$ if for every closed face $\tau \in \Gamma(f)$ the system of equations*

$$f^{\tau}(x) = x_1\frac{\partial f^{\tau}}{\partial x_1} = \cdots = x_n\frac{\partial f^{\tau}}{\partial x_n} = 0,$$

has no common solutions in $\mathbf{T}^n = (\mathbf{C}^{\times})^n$.

This notion is similar to that of $\Delta-$regular polynomial defined in the global case, but it treats only $\tau \in \Gamma(f)$. Let us denote by $\hat{\tau}$ the convex hull of $\tau \cup \{0\}$. Then the non-degeneracy of $f(x)$ is known to be equivalent to the finite dimensionality of the ring

$$A_{\tau} := \frac{S_{\hat{\tau}}}{\langle x_1\frac{\partial f^{\tau}}{\partial x_1},\cdots,x_n\frac{\partial f^{\tau}}{\partial x_n}\rangle S_{\hat{\tau}}}. \tag{6.3}$$

Here we followed the notation of (2.1) for the algebra $S_{\hat{\tau}}$. Let us denote by $\Gamma_-(f)$ union of all segments connecting $\alpha \in \Gamma(f)$ and $\{0\}$ or equivalently $\Gamma_-(f) = \bigcup_{\tau\subset\Gamma(f)}\hat{\tau}$. Let us denote by V_k $k-$dimensional volume of disjoint sets (there are ${}_nC_k$ such sets in total) $\Gamma_-(f)\cap$ { $k-$dimensional coordinate planes with $(n-k)$ zero coordinates }.

In this situation, we have the following theorem on the Milnor number $\mu(f)$.

Theorem 6.1 *([11]) Let $f(x)$ be a germ convenient and non-degenerate with respect to $\Gamma(f)$, then we have*

(6.4) $$\mu(f) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^n.$$

Definition 5 *We introduce the notion of simplicial Newton boundary which means that for each $\tau \subset \Gamma(f)$ the following inequality holds*

$$\sharp\{\Gamma_i \text{ face of } \Gamma(f); \dim \Gamma_i = \dim \tau + 1, \tau \subset \Gamma_i\} \leq n - \dim \tau.$$

As a matter of fact, we can formulate the above theorem by Kouchnirenko in a more precise form. We introduce a new $\mathbf{C}-$ vector space V_τ associated to a face $\tau \in \Gamma(f)$ not contained in a coordinate plane.

$$V_\tau = A_\tau \setminus (\oplus_{\tau^{(1)}\in\tau} A_{\tau^{(1)}} \setminus (\oplus_{\tau^{(2)}\in\tau} A_{\tau^{(2)}} \setminus (\cdots \setminus \{0\})\cdots),$$

where $\tau^{(j)} \in \tau$ denotes a codimension j face of τ contained in a coordinate plane. Here we remark that though τ not contained in a coordinate plane $\tau^{(j)} \in \tau$, $j \in [1, dim\tau]$ may be contained in a coordinate plane. We introduce another $\mathbf{C}-$ vector space W_τ corresponding to the interior points of $supp(V_\tau)$,

$$W_\tau = A_\tau \setminus (\oplus_{\tau^{(1)}\in\tau} A_{\tau^{(1)}} \setminus (\oplus_{\tau^{(2)}\in\tau} A_{\tau^{(2)}} \setminus (\cdots \setminus \{0\})\cdots),$$

where $\tau^{(j)} \in \tau$ denotes a codimension j face of τ not necessarily contained in a coordinate plane.

We say that a set $c(\sigma)$ is a copy of set σ if the relation $c(\sigma) = \pm\sigma + \vec{w}$, for some $\vec{w} \in \mathbf{Z}^n$ holds. Further on we use the notation $c^j(\sigma)$, $j = 1, 2, \cdots$ to distinguish different copies of a set σ.

Proposition 2.6 of [11], (5.6), (5.7) of [16] entail the following.

Proposition 6.2 *1) For A_τ, we have the following relations,*

$$dim\ A_\tau = \sum_{i=1}^{dim\tau+1} \varphi_i(\hat{\tau}) = (dim\ \tau + 1)!vol(\hat{\tau}).$$

2)

$$\mu(f) = \sum_{\tau \subset \text{coordinate planes}} (-1)^{n-1-dim\tau} dim\ A_\tau$$

3)

(6.5) $$A(f) \simeq \oplus_{(n-1)\text{dimensional faces } \tau\subset\Gamma(f)} V_\tau.$$

In the case of repetitive appearances of A_γ's, for some face γ in different $V_{\tau_1}, \cdots, V_{\tau_k}$ ($\gamma \subset \tau_1 \cap \cdots \cap \tau_k$) these copies of A_γ (or rather $supp(A_\gamma)$) shall be shifted and located anew in a way that they form a symmetry with respect to the Hodge filtration of A_{τ_i} for some $i \subset [1, k]$.

4) Let us denote by $s^{(\ell)}(\sigma)$ the shift of a set $\sigma \in F^i/F^{i+1}$ to another properly chosen set $s^{(\ell)}(\sigma) \in F^{i-\ell}/F^{i-\ell+1}$. Then we have another representation as follows,

$$A(f) \simeq \bigoplus_{\sigma\subset\Gamma(f)} \bigoplus_{\tau\subset\sigma} \bigoplus_{\ell=0}^{n-dim\sigma-1} \bigoplus_{j=1}^{{}_{n-dim\sigma-1}C_\ell} (-1)^\ell c^j(s^{(\ell)}(W_\tau)).$$

Here different copies of $c^j(s^{(\ell)}(W_\tau))$ shall be distributed in $\oplus_r F^{i-\ell}/F^{i-\ell+1}(A_{\tau_r})$, in such a way that $c^j(s^{(\ell)}(W_\tau)) \cap c^{j'}(s^{(\ell)}(W_\tau)) = \emptyset$ for all pairs $j \neq j'$.

A precise way to arrange copies in accordance with the Hodge filtration shall be explained in the **Algorithm** below.

Further we shall establish a connexion between the volume of a polyhedron and a set of integer points. Let τ be a $(k-1)-$dimensional face of $\Gamma(f)$ and $\hat{\tau}$ be a $k-$dimensional simplex. Let us denote by $\vec{m}_1, \cdots, \vec{m}_k$ vertices of $\hat{\tau} \setminus \{0\}$. We consider the cone

$$cone(\tau) = \{\sum_{i=1}^{k} b_i \vec{m}_i; b_i \geq 0\}, \tag{6.6}$$

associated to τ. We introduce a grading on the algebra $S_{\hat{\tau}}$. First we consider a piecewise linear function $h : \mathbf{N}^n \rightarrow \mathbf{N}$ satisfying $h|_{\Gamma(f)} = 1$. Then there exists $M > 0$ such that $h(\alpha) \subset \frac{1}{M}\mathbf{N}$ for all $\alpha \in \mathbf{N}^n$. We define $\phi = M \cdot h|_{\mathbf{N}^n}$. Let us denote by $\mathcal{A}_q$ algebra of polynomials written as a linear combination of monomials $x^\alpha, \phi(\alpha) \geq 1$. Denote by $\mathcal{A}_q(\tau)$ subalgebra of polynomials of $\mathcal{A}_q$ whose supports are contained in $cone(\tau)$. Then we can consider the Poincaré polynomial of $S_{\hat{\tau}}$ defined by

$$P_{S_{\hat{\tau}}}(t) := \sum_{q=0}^{\infty} dim_{\mathbf{C}}(\mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau)).$$

Then we have the following relationship

$$k!vol_k(\hat{\tau}) = \sharp\{\mathbf{Z}^n \cap \{cone(\tau) \setminus \bigcup_{i=1}^{k}(\vec{m}_i + cone(\tau))\}\} \doteq P_{S_{\hat{\tau}}}(t)(1-t)^k|_{t=1}. \tag{6.7}$$

Here we recall the fundamental theorem from [16] (3.10). To formulate it, we need to introduce preparatory notions. Let us consider a resolution of singularity X_0, that is to say a proper mapping $\rho : Y \rightarrow \mathbf{C}^n$ from a smooth algebraic variety $Y \supset \mathbf{C}^n$ such that 1) ρ is an isomorphism on $\mathbf{C}^n \setminus \{0\}$ and 2)$E = \rho^{-1}(X_0)$ is a divisor on Y with transversal intersections. Let E_0 be the proper image of X_0 through ρ, i.e. the closure of $\rho^{-1}(X_0 \setminus \{0\})$ in Y. Let us denote by $E_1, \cdots E_N$ the remaining irreducible components of E. Assume that $E = E_0 + \sum_{i=1}^{N} m_i E_i$ with multiplicities m_i of the divisor E_i. Let M be the least common multiplier (l.c.m.) of $m_1, \cdots, m_N$. We consider a covering $\pi : \tilde{\mathbf{C}} \rightarrow \mathbf{C}$ that sends z to z^M. For the pair of mappings (f, π) we denote the fibre product $Y \times_{\mathbf{C}} \tilde{\mathbf{C}}$ by $\tilde{X}$. Let $D_i = \pi^{-1}(E_i)_{red}, i \in [1, N]$ be the reduced part of $\pi^{-1}(E_i)$. If we consider the morphism $\tilde{f} : \tilde{X} \rightarrow \tilde{\mathbf{C}}$, and its special fibre $D := \tilde{f}^{-1}(0)$, then we have $D = \sum_{i=1}^{m} D_i$. We will use the notations,

$$D^{(k)} = \coprod_{i_0 < \cdots < i_k} (D_{i_0} \cap \cdots \cap D_{i_k})_{red}, {}' D^{(r)} = \coprod_{0 < i_0 < \cdots < i_r} (D_{i_0} \cap \cdots \cap D_{i_r})_{red}.$$

Under these circumstances we have the following theorem ([3], [16]) on the vanishing cohomology $H^{r+k}(X_\infty)$.

Theorem 6.3 *There exists a spectral sequence $E_1^{r,k}$ converging to $H^{r+k}(X_\infty)$ satisfying the following properties.*

1) It converges to the weight filtration on $H^{r+k}(X_\infty)$, i.e. $E_\infty^{r,k} = Gr_k^W\, H^{r+k}(X_\infty)$,

2) It degenerates at the term E_2 and $E_2 = E_\infty$.

3) The E_1 term is given by the formulae

$$\begin{aligned} E_1^{r,k} &= \oplus_{i \geq 0} H^{k+2r-2i}(D^{(2i-r)})(r-i) && for \quad r < 0, \\ &= H^k({}'D^{(r)}) \bigoplus \left(\oplus_{i > r} H^{k+2r-2i}(D^{(2i-r)})(r-i)\right) && for \quad r \geq 0 \end{aligned}.$$

We can classify the elements of A_τ after their eigenvalues under the action $x \to \zeta_*(x) = \zeta^{-h(\alpha)}x^\alpha$ with $\zeta = e^{\frac{2\pi\sqrt{-1}}{M}}$ that coincides with the action T_s of the semisimple part of the monodromy $T = T_s \cdot T_u$, where T_u denotes the unipotent part of T.

Let us introduce the Poincaré polynomial of $\mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau)$ in taking the monodromy action ζ_* into account,

$$P_{\mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau)}(t) := \sum_{0<\chi<1} h_\chi^{q,dim\tau-q} t^\chi. \tag{6.8}$$

$$\tilde{P}_{\mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau)}(t) := h_1^{q,q} t^q. \tag{6.9}$$

where

$$h_\chi^{q,dim\tau-q} := \sharp\{x^\alpha \in \mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau); h(\alpha) = \chi + q\},$$
$$h_1^{q,q} := \sharp\{x^\alpha \in \mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau); h(\alpha) = q\}.$$

The main theorem of [3] can be formulated as follows,

Theorem 6.4 *We suppose that $\Gamma(f)$ is a simplicial Newton boundary. Then Poincaré polynomials* (6.8), (6.9) *satisfy the following relations,*

(6.10)

$$P_{\mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau)}(t) = (-1)^{dim\tau-q} \sum_{\text{all faces}\gamma\subset\tau} \sum_{k\geq 0} (-1)^k_{dim\,\gamma+1} C_{p+k+1} \Big(\sum_{\alpha\in(k+1)\hat{\gamma}} t^{h(\alpha)} - \sum_{\alpha\in k\hat{\gamma}} t^{h(\alpha)} - \sum_{\alpha\in k\gamma} t^{h(\alpha)} \Big),$$

$$\sum_{q\geq 0} \tilde{P}_{\mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau)}(t) = \sum_{\text{all faces}\gamma\subset\tau} (t-1)^{dim\,\gamma}. \tag{6.11}$$

Let us recall fundamental notions around the spectral pairs of the singularity that reflect the interplay between the monodromy action T and the MHS of $H^{n-1}(X_\infty)$ [17]. The MHS on $H^{n-1}(X_\infty)$ consists of an increasing weight filtration $W_.$ and a decreasing Hodge filtration $F^{\cdot}$ ([16]). Let T_s be the semisimple part of T, and T_u unipotent, then T_s preserves the filtration $F^{\cdot}$ and $W_.$ whereas $N = log\,T_u$ satisfies $N(W_i) \subset W_{i-2}$ and $N(F^p) \subset F^{p-1}$. For eigenvalue χ of T, we define

$$H_\chi^{p,q} := Ker(T_s - \chi \cdot id_\mu; Gr^W_{p+q}\tilde{H}^{n-1}(X_\infty)),$$

$$dim\, H_\chi^{p,q} = h_\chi^{p,q},$$

where $\tilde{H}^{n-1}(X_\infty)$ denotes the reduced cohomology, $Gr_i^W = W_i/W_{i-1}$, and $Gr_F^p = F^p/F^{p+1}$. For $\alpha \in \mathbf{Q}$ and $w \in \mathbf{Z}$ we define integers $m_{\alpha,w}$ as follows. Write $\alpha = n-1-p-\beta$ with $0 \leq \beta < 1$ and let $\chi = e^{-2\pi\sqrt{-1}\alpha}$. If $\chi \neq 1$ then $m_{\alpha,w} = h_\chi^{p,w-p}$ while $m_{\alpha,w} = h_{\chi=1}^{p,w+1-p}$. The spectral pairs are collected in the invariant

$$Spp(f) = \sum m_{\alpha,w}(\alpha, w), \tag{6.12}$$

to be considered as an element of the free abelian group on $\mathbf{Q} \times \mathbf{Z}$. It is known that $Spp(f)$ is invariant under the symmetry $(\alpha, w) \to (n-2-\alpha, 2n-2-w)$ [17], Theorem 1.1, (ii).

Theorem 6.4 entails the relations

$$\sum_{dim\hat{\tau}=d} P_{\mathcal{A}_q(\tau)/\mathcal{A}_{q+1}(\tau)}(t) = \sum_{0<\chi<1} h_\chi^{q,d-q} t^\chi, \tag{6.13}$$

$$\sum_{q\geq 0}\sum_{\tau\subset\Gamma(f)} \tilde{P}_{A_q(\tau)/A_{q+1}(\tau)}(t) = \sum_{q\geq 0} h_1^{q,q} t^q. \tag{6.14}$$

As a corollary we have,

$$h_{\chi}^{n-1-p,n-1-q} = h_{\chi^{-1}}^{p,q},\ h_{\chi=1}^{p,p} = h_{\chi=1}^{n-p,n-p}. \tag{6.15}$$

We can write down the formula (6.10) in a more combinatorially clear way,
(6.16)

$$h_{\chi\neq 1}^{p,dim\ \tau-p}(D_{\hat{\tau}}) = (-1)^{dim\ \tau-p} \sum_{\text{all faces}\gamma\subset\tau}\ \sum_{k\geq 0}(-1)^{k}_{dim\ \gamma+1} C_{p+k+1}\big(\ell^*((k+1)\hat{\gamma}) - \ell^*(k\hat{\gamma}) - \ell^*(k\gamma)\big)\big),$$

where $D_{\hat{\tau}} = \mathbf{P}_{\tilde{\tau}} \cap \tilde{X}$ for $\tilde{\tau}$ suspension of $(\tau, 0) \subset \mathbf{R}^{n+1}$ with $(0, \cdots, 0, M) \in \mathbf{R}^{n+1}$.

Algorithm

Further we give an algorithm to get a basis of $A(f)$ in a purely combinatorial way. We shall achieve this task in making the decomposition of $A(f)$ in (6.5) more precise. This is the unique original part of this article.

Let $\vec{m}_1, \cdots, \vec{m}_k$ be vertices of a $k-$dimensional simplex face τ (if necessary we divide a non-simplex face into a sum of simplices). Here we remark the fact that for two simplices τ_1, τ_2 whose sum give a face $\Delta \subset \Gamma(f)$ i.e. $\Delta = \tau_1 \cup \tau_2$ and whose intersection is again a simplex γ ;$\gamma = \tau_1 \cap \tau_2$, we have

$$P_{S_\Delta}(t) = P_{S_{\tau_1}}(t) + P_{S_{\tau_1}}(t) - P_{S_\gamma}(t).$$

Thus the following procedure has meaning.

Definition 6 *Simplex subdivision $\delta_1, \cdots, \delta_m$ of faces of $\Gamma(f)$ means that for each $(n-1)$ dimensional compact face $\gamma \subset \Gamma(f)$, there exists an unique subdivision of it into a sum of $(n-1)-$ dimensional simplices,*

$$\gamma = \bigcup_{i\in I(\gamma)} \delta_i,$$

for a set of indices $I(\gamma) \subset [1, \cdots, m]$ associated to γ. Consequently,

$$\Gamma_-(f) = \bigcup_{i=1}^{m} \hat{\delta}_i,$$

is a subdivision into n dimensional simplices $\hat{\delta}_i$, $1 \leq i \leq m$.

We describe a combinatorial algorithm (not unique) to get a basis of $A(f)$ consisting of several steps.

1) *For a $(n-1)$dimensional simplex τ (whose vertices are $\vec{v}_1, \cdots, \vec{v}_n$) of a simplex subdivision, we construct the parallelepiped*

$$B_\tau := \{\mathbf{R}^n \cap \{cone(\tau) \setminus \bigcup_{i=1}^{n}(\vec{v}_i + cone(\tau))\}\}. \tag{6.17}$$

The inclusion relation $B_\tau \supset supp(A_\tau) \supset supp(V_\tau)$ can be easily seen from (6.6). For fixed subset of indices $\mathbf{J} \subset \{1, \cdots, n\}$ each vertex of the parallelepiped has the form

$$\vec{v}(\mathbf{J}) := \sum_{i\in\mathbf{J}} \vec{v}_i,$$

where no repetition of indices is allowed.

2) *To consider the set* $G_\tau = B_\tau \setminus \{$ *all open skeletons of dimension less than* $(n-1)$ *contained in* $F^0/F^1(A_\tau)$ $\}$ *In other words* $G_\tau = supp(W_\tau)$.

As a special case of copy, we introduce the notion of canonical copy $c_\tau(\alpha)$ of a point α with respect to a $(n-1)$dimensional simplex τ of the simplex subdivision (whose vertices are $\vec{v}_1, \cdots, \vec{v}_n$) that means the points α, $c_\tau(\alpha)$ are symmetrically located with respect to $\frac{1}{2}\sum_{i=1}^n \vec{v}_i$,

$$c_\tau(\alpha) + \alpha = \sum_{i=1}^n \vec{v}_i. \tag{6.18}$$

We shall choose basis of $A(f)$ in such a way that the symmetry property of Hodge numbers (6.15) can be realized. As for the integer points of A_τ on the intermediate Hodge filtration level $F^i/F^{i+1}(A_\tau)$, $1 \leq i \leq n-2$, the points of G_τ already realize this symmetry property. This can be seen from the arguments of [4], §5 where essentially $supp(A_\tau)$ is combinatorially described. Thus we shall further first care about the choice of $supp(A_\tau)$ on the extremal Hodge filtration levels $F^0/F^1(A_\tau)$ and $F^{n-1}/F^n(A_\tau)$.

3) *To count the number of interior points of each canonical copy* $c_\tau(\hat{\tau}^{int})$ *of* $\hat{\tau}^{int}$ *in* G_τ, *located on the Hodge filtration level* $F^0/F^1(A_\tau)$.

4) *For every* $(n-1)$ *simplex* τ *from simplex subdivision to exclude faces from* G_τ, *contained in* $F^{n-1}/F^n(A_\tau)$, *that are located on some coordinate plane.*

The following two measures 5), 6) are to be taken to cope with repetitive appearances of A_γ's mentioned in the Proposition 6.2, 3).

5) *Suppose that* $\Delta_1, \cdots, \Delta_k$ *are* $(n-1)$ *simplices from a simplex subdivision of faces of* $\Gamma(f)$ *such that* $\hat{\Delta}_1 \cap \cdots \cap \hat{\Delta}_k \neq \emptyset$. *To choose a canonical copy* $c_{\Delta_i}(\sigma^{int})$ *of each open skeleton* σ^{int} *of* $\hat{\Delta}_1 \cap \cdots \cap \hat{\Delta}_k$ *with respect to a simplex* Δ_i *that is to be chosen in dependence of* σ^{int}. *If the open skeleton* σ^{int} *has another expression like* $\sigma^{int} \subset \hat{\gamma}_1 \cap \cdots \cap \hat{\gamma}_{k'}$ *for another pair of simplices of a simplex subdivision* $\{\Delta_1, \cdots, \Delta_k\} \neq \{\gamma_1, \cdots, \gamma_{k'}\}$, *we do not add any of canonical copies* $c_{\gamma_j}(\sigma^{int})$, $j \in [1, k']$.

This procedure is necessary to recover these integer points that are located on the intersection $\hat{\Delta}_1 \cap \cdots \cap \hat{\Delta}_k$ on the level of $F^0/F^1(A_{\Delta_i})$ for some unique $i \in [1, k]$.

For example, in f_3 case below (see **7.3**) $(1,1,1) \in (0, \vec{v}_0)^{int}$ contained in $\hat{\Gamma}_1 \cap \hat{\Gamma}_2$, $\hat{\Gamma}_2 \cap \hat{\Gamma}_3$ and $\hat{\Gamma}_3 \cap \hat{\Gamma}_1$. The canonical copy $c_{\Gamma_2}((0, \vec{v}_0)^{int}) = (\vec{v}_1 + \vec{v}_2, \vec{v}_0 + \vec{v}_1 + \vec{v}_2)^{int}$ shall be added to G_{Γ_2}

6) *Furthermore if* $dim(\hat{\Delta}_1 \cap \cdots \cap \hat{\Delta}_k) = dim\ \sigma^{int}$ *we shall add other not canonical copies* $c^2(\sigma^{int}), \cdots, c^{k-1}(\sigma^{int})$ *(in understanding* $c^1(\sigma^{int}) = \sigma^{int}$, $c^k(\sigma^{int}) = c_{\Delta_i}(\sigma^{int})$ *of the procedure 5) above) such that*

$$c^{[\frac{k+1}{2}]+j}(\sigma^{int}) \in F^{[\frac{n+1}{2}]+j}/F^{[\frac{n+1}{2}]+j+1}(A_{\Delta_{i_j}}) \tag{6.19}$$

for $2 - [\frac{k+1}{2}] \leq j \leq k - [\frac{k+1}{2}] - 1$ *such that they produce a symmetry with respect to the Hodge filtration* $F^{\cdot}$.

In the case of simplicial Newton boundary $\Gamma(f)$ we have $k \leq n$ thus the above procedure can be realized so that (6.19) holds in such a way that $c^j(\sigma^{int}) \in G_{\Delta_{i_j}}$ and $\Delta_{i_j} \neq \Delta_{i'_j}$ for all pairs $j \neq j'$. On the contrary, if $\Gamma(f)$ is not simplicial, such a simple construction is already impossible. This situation explains why Danilov restricted himself to the simplicial Newton boundary case in [3].

For example, see (7.1.1), (7.1.2) and (7.1.3) below.

7) *Add zero dimensional faces (i.e. vertices) of* Δ_j *not belonging to the coordinate plane and their canonical copies with respect to* Δ_j *only once for each.*

Making use of the above basis, one can calculate the MHS of $A(f)$.

8) *We classify all points from* $x^{\vec{\alpha}} \in A(f)$ *according to their position with respect to faces of simplex subdivision* $\delta_1, \cdots, \delta_m$. *That is to say to find* δ_i *such that*

$$x^{\vec{\alpha}+\mathbf{1}} \in B_{\bar{\delta}_i},$$

where $x^{\mathbf{1}} = x_1 \cdots x_n$.

9) *To evaluate* $h(\vec{\alpha}+\mathbf{1})$ *by means of the piecewise linear function* h *such that* $h|_{\delta_i} = 1$ *introduced just after* (6.6).

10) *(*$\chi \neq 1$ *case) If* $h(\vec{\alpha}+\mathbf{1}) = n-1-\beta-p$ *for* $0 \leq p \leq n-1$, $0 < \beta < 1$, *then* $x^{\vec{\alpha}} \in H^{p,q}_{\chi\neq 1}$. *Here the index* q *can be chosen in the following way. For* $p < [\frac{n-1}{2}]$ *the index* q *is to be chosen* $q = dim\ \sigma^{int} - 1 > 0$ *if* $\vec{\alpha}+\mathbf{1}$ *belongs to one of the copies of* σ^{int}. *While for* $p > [\frac{n-1}{2}]$ *the index* q *is to be chosen* $q = n - dim\ \sigma^{int} \geq 0$ *under a parallel situation. All other cases* $(h^{p,q}_{\chi\neq 1}\ )$ *except* $H^{\frac{n-1}{2},\frac{n-1}{2}}_{\chi\neq 1}$, *(*$n$ *: odd) can be recovered from the above data making use of the relation* (6.15) $h^{p,q}_{\chi} = h^{n-1-p,n-1-q}_{\chi^{-1}}$ *realized by taking proper copies. The exceptional case has a following expression,*

$$H^{\frac{n-1}{2},\frac{n-1}{2}}_{\chi\neq 1} \cong \{\vec{\alpha}+\mathbf{1} \in \bigcup_{i=1}^{m} B^{int}_{\delta_i}; \frac{n-1}{2} < h(\vec{\alpha}+\mathbf{1}) < \frac{n+1}{2}\}.$$

11) *(*$\chi = 1$ *case) If* $h(\vec{\alpha}+\mathbf{1}) = n-1-p$ *for* $0 \leq p \leq n-1$ *and* $\vec{\alpha}+\mathbf{1}$ *belongs to one of the copies of* σ^{int}, *then* $x^{\vec{\alpha}} \in H^{p+1,q}_{\chi=1}$. *Here the index* q *can be chosen as* $q = dim\ \sigma^{int} > 0$ *if* $\vec{\alpha}+\mathbf{1}$ *belongs to one of the copies of* σ^{int}, *while for* $p > [\frac{n-1}{2}]$ *the index* q *is to be chosen* $q = n - dim\ \sigma^{int} > 0$ *under a parallel situation. All cases can be recovered from the above data making use of the relation* (6.15) $h^{p,q}_{\chi=1} = h^{n-p,n-q}_{\chi=1}$ *realized by taking proper copies.*

Remark 2 *The choice of the representative* $modJ_{f,\Delta}$ *in* B_Δ *does effect not only on the weight filtration but also on the Hodge filtration (see examples below).*

7 Examples

We show examples of calculus by means of the computer algebra system for computation SINGULAR. One can find an introduction to algorithms to compute monodromy related invariants (namely spectral pairs) of isolated hypersurface singularities in [15]. In the sequence, we use the notation $[i]' = [i]xy$ for **7.1** and $[i]' = [i]xyz$ for **7.2**, **7.3**. In the description of the spectral pairs we use the convention $((\alpha, w), m_{\alpha,w})$ under the notation of (6.12). We see that the rational monodromy α_i of the basis $[i]$ is expressed as $\alpha_i = h([i]') - 1$ for piecewise linear function $h(\cdot)$ introduced just after (6.6).

7.1 Let us begin with a polynomial in two variables,

$$f_1 = x^{15} + x^6y^4 + x^3y^6 + y^{12}.$$

Here and further on, we shall make use of the notational convention $xiyjzk = x^iy^jz^k$. The algebra $A(f_1)$ (rank $A(f_1) = 94$) has the following basis, $[1] = xy13, [2] = y13, [3] = xy12, [4] = y12, [5] = xy11, [6] = y11$, $[7] = xy10, [8] = y10, [9] = xy9, [10] = y9, [11] = xy8, [12] = y8$, $[13] = xy7, [14] = y7, [15] = xy6, [16] = y6, [17] = x2y5, [18] = xy5$, $[19] = y5, [20] = x5y4, [21] = x4y4, [22] = x3y4, [23] = x2y4, [24] = xy4$, $[25] = y4, [26] = x8y3$, $[27] = x7y3, [28] = x6y3, [29] = x5y3$, $[30] = x4y3, [31] = x3y3$, $[32] = x2y3, [33] = xy3, [34] = y3$, $[35] = x16y2, [36] = x15y2$, $[37] = x14y2, [38] = x13y2, [39] = x12y2$, $[40] = x11y2, [41] = x10y2, [42] = x9y2, [43] = x8y2, [44] = x7y2, [45] = x6y2, [46] = x5y2, [47] = x4y2, [48] = x3y2, [49] = x2y2, [50] = xy2, [51] = y2, [52] = x19y, [53] = x18y, [54] = x17y, [55] = x16y, [56] = x15y, [57] = x14y, [58] = x13y, [59] = x12y, [60] = x11y, [61] = x10y, [62] = x9y, [63] = x8y, [64] = x7y, [65] = x6y, [66] = x5y, [67] = x4y, [68] = x3y, [69] = x2y, [70] = xy$, [71]=y , [72]=x22 , [73]=x21 , [74]=x20 , [75]=x19 , [76]=x18 , [77]=x17 , [78]=x16 , [79]=x15 , [80]=x14 , [81]=x13 , [82]=x12 , [83]=x11 , [84]=x10 , [85]=x9 , [86]=x8 , [87]=x7 , [88]=x6 , [89]=x5 , [90]=x4 , [91]=x3 , [92]=x2 ,$[93] = x, [94] = 1$.

The spectral pairs are calculated as follows,

((-19/24,1),1),((-43/60,1),1),((-2/3,2),1),((-13/20,1),1),((-7/12,1),3) , ((-31/60 , 1) , 1) , ((-1/2 , 2) , 1) , ((-1/2 , 1) , 1) , ((-11/24 , 1) , 1) , ((-9/20 , 1) , 1) , ((-13/30 , 1) , 1) , ((- 5/12 , 1) , 1) , ((-23/60 , 1) , 1) , ((-3/8 , 1) , 1) , ((-11/30 , 1) , 1) , ((-1/3 , 2) , 1) , ((-1/3 , 1) , 1) , ((-19/60,1),1),((-3/10,1),1),((-7/24,1),1),((-1/4,1) , 4) , ((-7/30 , 1) , 1) , ((-13/60 , 1) , 1) , ((-11/60 , 1) , 1) , ((-1/6 , 1) , 4) , ((-3/20 , 1) , 1) , ((-1/8 , 1) , 1) , ((-7/60 , 1) , 1) , ((-1/1 0 , 1) , 1) , ((-1/12 , 1) , 4) , ((-1/20 , 1) , 1) , ((-1/24 , 1) , 1) , ((-1/30 , 1) , 1) , ((-1/60 , 1) , 1) , ((0 , 1) , 4) , ((1/60 , 1) , 1) , ((1/30 , 1) , 1) , ((1/24 , 1) , 1) , ((1/20 , 1) , 1) , ((1/12 , 1) , 4) , ((1/10 , 1) , 1) , ((7/60 , 1) , 1) , ((1/8 , 1) , 1) , ((3/20 , 1) , 1) , ((1/6 , 1) , 4) , ((11/60 , 1) , 1) , ((13/60 , 1) , 1) , ((7/30 , 1) , 1) , ((1/4 , 1) , 4) , ((7/24 , 1) , 1) , ((3/10 , 1) , 1) , ((19/60 , 1) , 1) , ((1/3 , 1) , 1) , ((1/3 , 0) , 1) , ((11/30 , 1) , 1) , ((3/8 , 1) , 1) , ((23/60 , 1) , 1) , ((5/12 , 1) , 1) , ((13/30 , 1) , 1) , ((9/20 , 1) , 1) , ((11/24 , 1) , 1) , ((1/2 , 1) , 1) , ((1/2 , 0) , 1) , ((31/60 , 1) , 1) , ((7/12 , 1) , 3) , ((13/20 , 1) , 1) , ((2/3 , 0) , 1) , ((43/60 , 1) , 1) , ((19/24 , 1) , 1).

Let us use the notation $\vec{v}_1 = (0,12)$, $\vec{v}_2 = (3,6)$, $\vec{v}_3 = (6,4)$, $\vec{v}_4 = (5,0)$, $\tau_1 =$ *convex hull*$\{\vec{v}_1,\vec{v}_2\}$, $\tau_2 =$ *convex hull*$\{\vec{v}_2,\vec{v}_3\}$, $\tau_3 =$ *convex hull*$\{\vec{v}_3,\vec{v}_4\}$. Then we have

$$supp(V_{\tau_1}) = \mathbf{Z}^2 \cap \{convex\ hull\{\vec{v}_1,\vec{v}_2,\vec{v}_1+\vec{v}_2\}^{int} \cup convex\ hull\{0,\vec{v}_2\}^{int}\}.$$

$$supp(V_{\tau_2}) = \mathbf{Z}^2 \cap \{convex\ hull\{\vec{v}_2,\vec{v}_3,\vec{v}_2+\vec{v}_3\}^{int} \cup convex\ hull\{0,\vec{v}_2\}^{int} \cup convex\ hull\{0,\vec{v}_3\}^{int} \cup 2\{0\}\}.$$

$$supp(V_{\tau_3}) = \mathbf{Z}^2 \cap \{convex\ hull\{\vec{v}_3,\vec{v}_4,\vec{v}_3+\vec{v}_4\}^{int} \cup convex\ hull\{0,\vec{v}_3\}^{int}\}.$$

As we see there are repetitive appearances of *convex hull*$\{0,\vec{v}_2\}^{int}$, *convex hull*$\{0,\vec{v}_3\}^{int}$ and $\{0\}$ each of them twice. Thus the summation (6.5) must be taken in the following way,

$$A(f_1) \cong \mathbf{Z}^2 \cap \{convex\ hull\{\vec{v}_1,\vec{v}_2,\vec{v}_1+\vec{v}_2\}^{int} \cup convex\ hull\{\vec{v}_2,\vec{v}_3,\vec{v}_2+\vec{v}_2\}^{int} \tag{7.1.1}$$

$$\cup convex\ hull\{\vec{v}_3,\vec{v}_4,\vec{v}_3+\vec{v}_4\}^{int} \cup convex\ hull\{0,2\vec{v}_2\}^{int} \cup convex\ hull\{0,2\vec{v}_3\}^{int}\}. \tag{7.1.2}$$

Here it is worthy to notice that

$$\begin{aligned} convex\ hull\{0,2\vec{v}_2\}^{int} &\cong convex\ hull\{0,\vec{v}_2\}^{int} \cup \{\vec{v}_2\} \cup c_{\tau_1}(convex\ hull\{0,\vec{v}_2\}^{int}), \\ &\cong convex\ hull\{0,\vec{v}_2\}^{int} \cup \{\vec{v}_2\} \cup c_{\tau_2}(convex\ hull\{0,\vec{v}_2\}^{int}), \\ convex\ hull\{0,2\vec{v}_3\}^{int} &\cong convex\ hull\{0,\vec{v}_3\}^{int} \cup \{\vec{v}_3\} \cup c_{\tau_2}(convex\ hull\{0,\vec{v}_3\}^{int}), \\ &\cong convex\ hull\{0,\vec{v}_3\}^{int} \cup \{\vec{v}_3\} \cup c_{\tau_3}(convex\ hull\{0,\vec{v}_3\}^{int}). \end{aligned} \tag{7.1.3}$$

We can calculate by hands the spectral pairs above in evaluating the monomials $[i], 1 \leq i \leq 94$ modulo Jacobian ideal of f_1 by means of a piecewise linear function,

$$\begin{aligned} h(i,j) &= \tfrac{i}{6} + \tfrac{j}{12} \quad for\ (i,j) \in \bar{B}_{\tau_1} \\ &= \tfrac{i}{12} + \tfrac{j}{8} \quad for\ (i,j) \in \bar{B}_{\tau_2} \\ &= \tfrac{i}{15} + \tfrac{3j}{20} \quad for\ (i,j) \in \bar{B}_{\tau_3}, \end{aligned}$$

according to their classification into $\bar{B}_{\tau_1}$, $\bar{B}_{\tau_2}$, $\bar{B}_{\tau_3}$ (closures of parallelepipeds introduced in (6.17)).

For example

$$h([69]') - 1 = \frac{3}{12} + \frac{2}{8} - 1 = \frac{3}{6} + \frac{2}{12} - 1 = -\frac{1}{2},$$

which gives the spectral pair $((-\frac{1}{2},2),1)$. Here the weight filtration index 2 indicates that $[69]' \in cone(\tau_2 \cap \tau_3)$. In a similar way

$$h([71]') - 1 = \frac{1}{6} + \frac{2}{12} - 1 = \frac{1}{12} + \frac{2}{8} - 1 = -\frac{2}{3},$$

that gives the spectral pair $((-\frac{2}{3}, 2), 1)$.

7.2 Let us treat the case studied by [3] as an example.

$$f_2 = x4 + y4 + z8 + x2z2 + y2z2.$$

The ring $A(f_2)$ with rank 31 has the following basis,
[1]= $x4y2$, [2]= $xy2z3$, [3]= $z8$, [4]= $z7$, [5]=z6 , [6]=z5 , [7]=yz4 , [8]=xz4 , [9]=z4 , [10]=y3z2 , [11]=x3y2 , [12]=z3 , [13]=y2z2 , [14]=xyz2 , [15]=yz2 , [16]=xz2 , [17]=z2 , [18]=y2z , [19]=xyz , [20]=yz , [21]=xz , [22]=z , [23]=x2y2 , [24]=xy2 , [25]=y2 , [26]=x2y , [27]=xy , [28]=y , [29]=x2 , [30]=x , [31] = 1.

This result is slightly different from what SINGULAR gives us due to the reason mentioned in Remark 2.

SINGULAR calculates the spectral pairs as follows,
((−1/4, 2), 1), ((0, 3), 1), ((0, 2), 2), ((1/8, 2), 1), ((1/4, 2), 6), ((3/8, 2), 1), ((1/2, 2), 7), ((5/8, 2), 1), ((3/4, 2), 6), ((7/8, 2), 1), ((1, 2), 2), ((1, 1), 1), ((5/4, 2), 1).

Let us denote by Γ_1 the convex hull of $\{(0,0,0),(0,0,8),(2,0,2),(0,2,2)\}$, Γ_2 that of $\{(0,0,0),(2,0,2),(0,2,2),(0,4,0)\}$, Γ_3 that of $\{(0,0,0),(2,0,2),(4,0,0),(0,4,0)\}$. The piecewise linear function $h(i_1,i_2,i_3)$ is given by the following,

$$\begin{aligned} h(i_1,i_2,i_3) &= \tfrac{i_1+i_2+i_3}{4} \quad for\ (i_1,i_2,i_3) \in \bar{B}_{\Gamma_2} \cup \bar{B}_{\Gamma_3} \\ &= \tfrac{3(i_1+i_2)+i_3}{8} \quad for\ (i_1,i_2,i_3) \in \bar{B}_{\Gamma_1}. \end{aligned}$$

We remark that $[1]' = x5y3z \in F^0/F^1(A_{\Gamma_1})$, while $[31]' = xyz \in F^2(A_{\Gamma_1})$. The point $[1]'$ is the canonical copy of $[31]'$ with respect to Γ_1. The point $[2]' \in H^{1,1}_{\chi=1}$ is located on $cone(\Gamma_1 \cap \Gamma_2)$ with spectral pair $(1,1)$ which is the canonical copy of $[22]' \in H^{2,2}_{\chi=1}$ with respect to Γ_1 whose spectral pair is $(0,3)$. The points $[10]'$, $[11]' \in H^{1,2}_{\chi=1}$ are located on the 2-dimensional open skeleton of $2\hat{\Gamma}_1$ and they give spectral pairs $((1,2),2)$. They are the canonical copies of $[28]'$, $[30]' \in H^{2,1}_{\chi=1}$ with spectral pairs $((0,2),2)$. All other integer points are located in the interior of $cone(\Gamma_1) \cup cone(\Gamma_1 \cup \Gamma_2)$ with weight filtration $w = 2$ and they correspond to $H^{1,1}_{\chi\neq1} \oplus H^{2,0}_{\chi\neq1} \oplus H^{0,2}_{\chi\neq1}$. Here we recall that $\chi = e^{-2\pi h([i]')\sqrt{-1}}$ for each basis element $[i]$.

7.3 Next we consider the case that B.Malgrange (in a letter to the editor of Inventiones Mathematicae) used to demonstrate that a maximum size Jordan cell (=the dimension n) of the monodromy T (or equivalently that of T_u) really appears,

$$f_3 = x8 + y8 + z8 + x2y2z2.$$

The MHS and essentially the spectral pairs of $Spp(f_3)$ are described in detail in [16], (3.15).

The ring $A(f_3)$ with rank 215 has the following basis,
[1]= $z16$, [2]= $z15$, [3]= $z14$, [4]= $z13$, [5]= $z12$, [6]= $z11$, [7]= $z10$, [8]= $xyz9$, [9]= $yz9$, [10]= $xz9$, [11]= $z9$, [12]= $y7z8$, [13]= $y6z8$, [14]= $y5z8$, [15]= $x12$, [16]= $y3z8$, [17]= $y2z8$, [18]= $xyz8$, [19]= $yz8$, [20]= $x7z8$, [21]= $x6z8$, [22]= $x5z8$, [23]= $y12$, [24]= $x3z8$, [25]= $x2z8$, [26]= $xz8$, [27]= $z8$, [28]= $y7z7$, [29]= $y6z7$, [30]= $y5z7$, [31]= $y4z7$, [32]= $y3z7$, [33]= $y2z7$, [34]= $xyz7$, [35]= $yz7$, [36]= $x7z7$, [37]= $x6z7$, [38]= $x5z7$, [39]= $x4z7$, [40]= $x3z7$, [41]= $x2z7$, [42]= $xz7$, [43]= $z7$, [44]= $y7z6$, [45]= $y6z6$, [46]= $y5z6$, [47]= $y4z6$, [48]= $y3z6$, [49]= $y2z6$, [50]= $xyz6$, [51]= $yz6$, [52]= $x7z6$, [53]= $x6z6$, [54]= $x5z6$, [55]= $x4z6$, [56]= $x3z6$, [57]= $x2z6$, [58]= $xz6$, [59]= $z6$, [60]= $y7z5$, [61]= $y6z5$, [62]= $y5z5$, [63]= $y4z5$, [64]= $y3z5$, [65]= $y2z5$, [66]= $xyz5$, [67]= $yz5$, [68]= $x7z5$, [69]= $x6z5$, [70]= $x5z5$, [71]= $x4z5$, [72]= $x3z5$, [73]= $x2z5$, [74]= $xz5$, [75]= $z5$, [76]= $y7z4$, [77]= $y6z4$, [78]= $y5z4$, [79]= $y4z4$, [80]= $y3z4$, [81]= $y2z4$, [82]= $xyz4$, [83]= $yz4$, [84]= $x7z4$, [85]= $x6z4$, [86]= $x5z4$, [87]= $x4z4$, [88]= $x3z4$, [89]= $x2z4$, [90]= $xz4$, [91]= $z4$, [92]= $y7z3$, [93]= $y6z3$, [94]= $y5z3$, [95]= $y4z3$, [96]= $y3z3$, [97]= $y2z3$, [98]= $xyz3$, [99]=

$yz3$, [100]= $x7z3$, [101]= $x6z3$, [102]= $x5z3$, [103]= $x4z3$, [104]= $x3z3$, [105]= $x2z3$, [106]= $xz3$, [107]= $z3$, [108]= $y7z2$, [109]= $y6z2$, [110]= $y5z2$, [111]= $y4z2$, [112]= $y3z2$, [113]= $y2z2$, [114]= $xyz2$, [115]= $yz2$, [116]= $x7z2$, [117]= $x6z2$, [118]= $x5z2$, [119]= $x4z2$, [120]= $x3z2$, [121]= $x2z2$, [122]= $xz2$, [123]= $z2$, [124]= $xy7z$, [125]= $y7z$, [126]= $xy6z$, [127]= $y6z$, [128]= $xy5z$, [129]= $y5z$, [130]= $xy4z$, [131]= $y4z$, [132]= $xy3z$, [133]= $y3z$, [134]= $xy2z$, [135]= $y2z$, [136]= $x7yz$, [137]= $x6yz$, [138]= $x5yz$, [139]= $x4yz$, [140]= $x3yz$, [141]= $x2yz$, [142]= xyz, [143]= yz, [144]= $x7z$, [145]= $x6z$, [146]= $x5z$, [147]= $x4z$, [148]= $x3z$, [149]= $x2z$, [150]= xz, [151]= z, [152]= $x7y7$, [153]= $x6y7$, [154]= $x5y7$, [155]= $x4y7$, [156]= $x3y7$, [157]= $x2y7$, [158]= $xy7$, [159]= $y7$, [160]= $x7y6$, [161]= $x6y6$, [162]= $x5y6$, [163]= $x4y6$, [164]= $x3y6$, [165]= $x2y6$, [166]= $xy6$, [167]= $y6$, [168]= $x7y5$, [169]= $x6y5$, [170]= $x5y5$, [171]= $x4y5$, [172]= $x3y5$, [173]= $x2y5$, [174]= $xy5$, [175]= $y5$, [176]= $x7y4$, [177]= $x6y4$, [178]= $x5y4$, [179]= $x4y4$, [180]= $x3y4$, [181]= $x2y4$, [182]= $xy4$, [183]= $y4$, [184]= $x7y3$, [185]= $x6y3$, [186]= $x5y3$, [187]= $x4y3$, [188]= $x3y3$, [189]= $x2y3$, [190]= $xy3$, [191]= $y3$, [192]= $x7y2$, [193]= $x6y2$, [194]= $x5y2$, [195]= $x4y2$, [196]= $x3y2$, [197]= $x2y2$, [198]= $xy2$, [199]= $y2$, [200]= $x7y$, [201]= $x6y$, [202]= $x5y$, [203]= $x4y$, [204]= $x3y$, [205]= $x2y$, [206]= xy, [207]= y, [208]= $x7$, [209]= $x6$, [210]= $x5$, [211]= $x4$, [212]= $x3$, [213]= $x2$, [214]= x, [215]= 1. Here we have chosen [15], [23] differently from the computation given by SINGULAR due to the reason mentioned in Remark 2.

The spectral pairs are calculated by SINGULAR as follows, ((-1/2 , 4) , 1) , ((-3/8 , 3) , 3) , ((-1/4 , 3) , 3) , ((-1/4 , 2) , 3) , ((-1/8 , 3) , 3) , ((-1/8 , 2) , 6) , ((0 , 3) , 4) , ((0 , 2) , 9) , ((1/8 , 3) , 3) , ((1/8 , 2) , 15) , ((1/4 , 3) , 3) , ((1/4 , 2) , 18) , ((3/8 , 3) , 3) , ((3/8 , 2) , 21) , ((1/2 , 2) , 25) , ((5/8 , 2) , 21) , ((5/8 , 1) , 3) , ((3/4 , 2) , 18) , ((3/4 , 1) , 3) , ((7/ 8 , 2) , 15) , ((7/8 , 1) , 3) , ((1 , 2) , 9) , ((1 , 1) , 4) , ((9/8 , 2) , 6) , ((9/8 , 1) , 3) , ((5/4 , 2) , 3) , ((5 /4 , 1) , 3) , ((11/8 , 1) , 3) , ((3/2 , 0) , 1).

Let us denote by Γ_1 the convex hull of $\{\vec{v}_3 = (0,0,8), \vec{v}_0 = (2,2,2), \vec{v}_1 = (8,0,0)\}$, Γ_2 that of $\{\vec{v}_0, \vec{v}_1, \vec{v}_2 = (0,8,0)\}$, Γ_3 that of $\{\vec{v}_0, \vec{v}_2, \vec{v}_3\}$. The most interesting monomials with spectral pairs $(0,3), (0,2)$ are the following

$$[91], [142], [183], [211] \in H^{2,2}_{\chi=1} \ \ with\ spp(f_3) = (0,3),$$

$$[99], [106], [112], [121], [132], [148], [190], [197], [204] \in H^{2,1}_{\chi=1} \ \ with\ spp(f_3) = (0,2).$$

Monomials of $H^{1,2}_{\chi=1}$ (with spectral pairs ((0,2),9)) are obtained as the canonical copies of $H^{2,1}_{\chi=1}$ (with spectral pairs ((1,2),9)) with respect to properly chosen 2-faces. Namely,

$$[30], [38], [45], [53], [60], [68], [154], [161], [168] \in H^{1,2}_{\chi=1} \ \ with\ spp(f_3) = (1,2).$$

We see also,

$$[5], [8], [15], [23] \in H^{1,1}_{\chi=1} \ \ with\ spp(f_3) = (1,1).$$

We see also $H^{2,2}_{\chi=-1} = \{[215]\}$ and $H^{0,0}_{\chi=-1} = \{[1]\}$. All other monomials are located in $B^{int}_{\Gamma_1} \cup B^{int}_{\Gamma_2} \cup B^{int}_{\Gamma_3}$.

8 Complete intersections

It is quite natural to extend the above approaches (to describe MHS of the cohomology group) to the case of complete intersections (CI). The research on CI is also divided into the global study (CI in a torus) and the local study (the Milnor fibre of CI).

We remember that one can find the calculus of the Hodge numbers of an IC in a torus already in [4]. Danilov and Khovanski made use of so called Cayley trick to reduce the computation of Hodge numbers for an IC to that for an hypersurface.

In [21], I made use of the isomorphism based on the Cayley trick to get concrete expressions of the fibre integrals associated to the non-degenerate affine CI variety. Further as an application, [20] verified that the fibre integral of certain Givental' type CI coincides with the Fourier transform of the quantum cohomology to the projective space and calculated the monodromy of the fibre integral in question.

As for the local Milnor fibre case, after studies initiated by [7] and [1], [8] which describe the MHS of quasihomogeneous isolated CI singularities, [6] gives a tentative description of MHS to isolated CI singularities (ICIS) not necessarily quasihomogeneous. In[19], I established an alogorithm to calculate Gauss-Manin system associated to the quasihomogeneous ICIS and described the poles of the Mellin transform of their fibre integrals in terms of the MHS of the Milnor fibre.

Though [6] succeeded to fabricate finite dimensional vector space of differential forms associated to each ICIS that possesses symmetry property similar to that for spectral pairs (6.12) in hypersurface case, still their method carries ad hoc character dependent on each type of singularity and it is still distant from a combinatorially universal description given by [3] for simplicial hypersurface singularities.

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