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AN INTRODUCTION TO ARC SPACES

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ABSTRACT. This paper is an introduction to the structure of the arc space of an algebraic variety.
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1. INTRODUCTION

The concept jet space and arc space over an algebraic variety or an analytic space is introduced by Nash in his preprint in 1968 which is later published as [20]. The study of these spaces was further developed by Kontsevich, Denef and Loeser as the theory of motivic integration, see [14, 5]. These spaces are considered as something to represent the nature of the singularities of the base space. In fact, papers [7], [18], [19] by Mustaţă, Ein and Yasuda show that geometric properties of the jet schemes determine certain properties of the singularities of the base space.

In this paper, we provide the beginners with the basic knowledge of these spaces. One of the most powerful arms to work on these space is the motivic integration. But this paper does not step into this theory, as there are already very good introduction papers on the motivic integration by A. Craw [4] and W. Veys [24]. We devote into the basic study of geometric structure of arc spaces. We also give the introduction to the Nash problem which was posed in [20]

Throughout this paper the base field k is algebraically close field of arbitrary characteristic and a variety is an irreducible reduced scheme of finite type over k.

2. CONSTRUCTION OF JET SPACES AND ARC SPACES

Definition 2.1. Let X be a scheme of finite type over k and $K \supset k$ a field extension. A morphism $\operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ is called an m-jet of X and $\operatorname{Spec} K[[t]] \longrightarrow X$ is called an *arc* of X. We denote the closed point of $\operatorname{Spec} K[[t]]$ by 0 and the generic point by η.

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Proposition 2.2. *Let X be a scheme of finite type over k. Let Sch/k be the category of k-schemes and Set the category of sets. Define a contravariant functor $F_m^X : Sch/k \longrightarrow Set$ by*

$$F_m^X(Y) = \mathrm{Hom}_k(Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X).$$

Then, F_m^X is representable by a scheme X_m of finite type over k, that is

$$\mathrm{Hom}_k(Y, X_m) \simeq \mathrm{Hom}_k(Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X).$$

This X_m is called the space of m-jets of X.

Proof. This proposition is proved in [3, p. 276]. In this paper, we prove this by a concrete construction for affine X first and then patching them together for a general X.

Let X be $\mathrm{Spec}\, R$, $R = k[x_1, \dots, x_n]/(f_1, .., f_r)$. It is sufficient to prove the representability for an affine variety $Z = \mathrm{Spec}\, A$. Then, we obtain that

$$(2.2.1) \qquad \mathrm{Hom}(Z \times \mathrm{Spec}\, k[t]/(t^{m+1}), X) \simeq \mathrm{Hom}(R, A[t]/(t^{m+1}))$$

$$\simeq \{\varphi \in \mathrm{Hom}(k[x_1, ., x_n], A[t]/(t^{m+1})) \mid \varphi(f_i) = 0 \text{ for } i = 1, .., r\}.$$

If we write $\varphi(x_j) = a_j^{(0)} + a_j^{(1)}t + a_j^{(2)}t^2 + .. + a_j^{(m)}t^m$, it follows that

$$\varphi(f_i) = F_i^{(0)}(a_j^{(l)}) + F_i^{(1)}(a_j^{(l)})t + .. + F_i^{(m)}(a_j^{(l)})t^m$$

for polynomials $F_i^{(s)}$ in $a_j^{(l)}$'s. Then the above set (2.2.1) is as follows:

$$= \{\varphi \in \mathrm{Hom}(k[x_j, x_j^{(1)}, ., x_j^{(m)} \mid j = 1, ., n], A) \mid \varphi(x_j^{(l)}) = a_j^{(l)},\ F_i^{(s)}(a_j^{(l)}) = 0\}$$

$$= \mathrm{Hom}(k[x_j, x_j^{(1)}, ., x_j^{(m)}]/(F_i^{(s)}(x_j^{(l)})), A).$$

If we write $X_m = \mathrm{Spec}\, k[x_j, x_j^{(1)}, ., x_j^{(m)}]/(F_i^{(s)}(x_j^{(l)}))$, the above set (2.2.1) is

$$\mathrm{Hom}(Z, X_m).$$

For a general X, cover it by affine open subsets U_i $(i \in I)$. Then, we can patch $(U_i)_m$'s together by the following lemma. □

Lemma 2.3. *Assume the functor F_m^X is representable by X_m for a k-scheme X. Let $\pi_m : X_m \longrightarrow X$ be the morphism induced from the canonical surjection $k[t]/(t^{m+1}) \longrightarrow k$. Then for an open subset $U \subset X$, F_m^U is representable by $X_m|_U := \pi_m^{-1}(U)$.*

This lemma follows immediately from a more general statement Proposition 3.3.

Remark 2.4. The defining equations $F_i^s(x_j^{(l)})$'s of X_m are obtained as follows: Let D be a derivation of $k[x_j, x_j^{(1)}, ., x_j^{(m)}]$ defined by $D(x_j^{(l)}) = x_j^{(l+1)}$, where we define $x_j^{(l)} = 0$ for $l > m$. Then, it follows that $F_i^s(x_j^{(l)}) = D^s(f_i)$.

Example 2.5. For $X = \mathbb{A}_k^n$, it follows $X_m = \mathbb{A}_k^{n(m+1)}$ from the proof of Proposition 2.2.

Example 2.6. Let X be a hypersurface in $\mathbb{A}_k^3$ defined by $f = 0$. Then, X_m is defined in $\mathbb{A}_k^{3(m+1)}$ by $f = D(f) = \ldots = D^m(f) = 0$. For example, if $f = xy + z^2$ and $m = 2$, we obtain that X_2 is defined by $xy + z^2 = x^{(1)}y + xy^{(1)} + 2zz^{(1)} = x^{(2)}y + 2x^{(1)}y^{(1)} + xy^{(2)} + 2z^{(1)}z^{(1)} + 2zz^{(2)} = 0$. One can see that X_2 is irreducible and not normal.

2.7. The canonical surjection $k[t]/(t^{m+1}) \longrightarrow k[t]/(t^m)$ induces a morphism $\phi_m : X_m \longrightarrow X_{m-1}$. Define $\pi_m = \phi_1 \circ \cdots \circ \phi_m : X_m \longrightarrow X$. A point of X_m gives an m-jet $\alpha : \operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$. We denote the point of X_m corresponding to $\alpha : \operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ by the same symbol α. Then, $\pi_m(\alpha) = \alpha(0)$,

Let $X_\infty = \varprojlim_m X_m$ and call it the *space of arcs* of X. X_∞ is not of finite type over k but it is a scheme, see [5].

Using the representability of F_m we obtain the following universal property of X_∞:

Proposition 2.8. *Let X be a scheme of finite type over k. Then*

$$\operatorname{Hom}_k(Y, X_\infty) \simeq \operatorname{Hom}_k(Y \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]], X)$$

for an arbitrary k-scheme Y, where $Y \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]]$ means the formal completion of $Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]]$ along the subscheme $Y \times_{\operatorname{Spec} k} \{0\}$.

Corollary 2.9. *There is a universal family of arcs*

$$X_\infty \hat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X.$$

2.10. Denote the canonical projection $X_\infty \longrightarrow X_m$ by η_m and the composite $\pi_m \circ \eta_m$ by π. When we need to specify the base space X, we write it by π_X. A point $x \in X_\infty$ gives an arc $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ and $\pi(x) = \alpha_x(0)$, where K is the residue field at x. As an m-jet we denote $x \in X_\infty$ and α_x by the same symbol α.

Example 2.11. If $X = \mathbb{A}_k^n$, then $X_\infty = \operatorname{Spec} k[x_j, x_j^{(1)}, x_j^{(2)} \ldots]$ which we denote by $\mathbb{A}_k^\infty$. Here, we note that the set of closed points of $\mathbb{A}_k^\infty$ is not necessarily in the set

$$k^\infty := \{(a_1, a_2, \ldots) \mid a_i \in k\},$$

see the following theorem.

Theorem 2.12 ([11]). *Every closed point of $\mathbb{A}_k^\infty$ is a k-valued point if and only if $\#k$ is not countable.*

3. Miscellaneous properties of jet spaces and arc spaces

Proposition 3.1. *Let $f : X \longrightarrow Y$ be a morphism of k-schemes of finite type. Then the canonical morphism $f_m : X_m \longrightarrow Y_m$ is induced for every $m \in \mathbb{N} \cup \{\infty\}$.*

Proof. For an m-jet (or arc) $\alpha \in X_m$ of X, the composite $f \circ \alpha$ is an m-jet (or arc) of Y. This map $X_m \longrightarrow Y_m$, $\alpha \mapsto f \circ \alpha$ is our required morphism. $\square$

Proposition 3.2. *Let $f : X \longrightarrow Y$ be a proper birational morphism of k-schemes such that $f|_{X \setminus W} : X \setminus W \simeq Y \setminus V$, where $W \subset X$ and $V \subset Y$ are closed. Then f_∞ gives a bijection*

$$X_\infty \setminus W_\infty \longrightarrow Y_\infty \setminus V_\infty.$$

Proof. Let $\alpha \in Y_\infty \setminus V_\infty$, then $\alpha(\eta) \in X \setminus V$. As $X \setminus W \simeq Y \setminus V$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec} K((t)) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} K[[t]] & \xrightarrow{\alpha} & Y \end{array}.$$

Then, as f is a proper morphism, there is a unique morphism $\tilde{\alpha}$: $\operatorname{Spec} K[[t]] \longrightarrow X$ such that $f \circ \tilde{\alpha} = \alpha$. This shows the bijectivity as required. $\square$

The following is the generalization of Lemma 2.3.

Proposition 3.3. *If $f : X \longrightarrow Y$ is an étale morphism, then $X_m \simeq Y_m \times_Y X$, for every $m \in \mathbb{N} \cup \{\infty\}$.*

Proof. As $\varprojlim_m (Y_m \times_Y X) = (\varprojlim_m Y_m) \times_Y X$, it is sufficient to prove the assertion for $m \in \mathbb{N}$. By the commutative diagram:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array},$$

we obtain a morphism $\varphi : X_m \longrightarrow Y_m \times_Y X$. On the other hand, the projection $Y_m \times_Y X \longrightarrow Y_m$ corresponds to a morphism $Y_m \times_Y X \times_{\operatorname{Spec} k}$

$\operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow Y$ which completes the following commutative diagram:

$$\begin{array}{ccc} Y_m \times_Y X & \hookrightarrow & Y_m \times_Y X \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

As f is formally étale, there is a morphism $Y_m \times_Y X \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow X$ which make the diagram commutative. The corresponding morphism $Y_m \times_Y X \longrightarrow X_m$ is the inverse morphism of φ. □

Proposition 3.4. *There is a canonical isomorphism:*

$$(X \times Y)_m \simeq X_m \times Y_m,$$

for every $m \in \mathbb{N} \cup \{\infty\}$.

Proof. For an arbitrary k-scheme Z,

$$\operatorname{Hom}_k(Z, X_m \times Y_m) \simeq \operatorname{Hom}_k(Z, X_m) \times \operatorname{Hom}_k(Z, Y_m),$$

and the right hand side is isomorphic to

$$\operatorname{Hom}_k(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X) \times \operatorname{Hom}_k(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), Y)$$

$$\simeq \operatorname{Hom}_k(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X \times Y).$$

$$\simeq \operatorname{Hom}_k(Z, (X \times Y)_m).$$

For $m = \infty$, the proof is similar. □

Proposition 3.5. *Let $f : X \longrightarrow Y$ is an open immersion (resp. closed immersion) of k-schemes of finite type. Then the induced morphism $f_m : X_m \longrightarrow Y_m$ is also an open immersion (resp. closed immersion) for every $m \in \mathbb{N} \cup \{\infty\}$.*

Proof. The open case follows from Lemma 2.3. For the closed case, we may assume that Y is affine. If Y is defined by f_i $(i = 1,.,r)$ in an affine space, then X is defined by f_i $(i = 1,.,r,.,u)$ in the same affine space. Then, Y_m is defined by $D^s(f_i)$ $(i = 1,.,r,\ s \leq m)$ and X_m is defined by $D^s(f_i)$ $(i = 1,.,r,.,u,\ s \leq m)$ in the corresponding affine space. This shows that X_m is a closed subscheme of Y_m. □

Remark 3.6. In the above proposition we see that the property open or closed immersion of the base spaces is inherited by the morphism of the space of jets and arcs. But some properties are not inherited. For example, surjectivity and closedness are not inherited.

Example 3.7. There is an example that $f : X \longrightarrow Y$ is surjective (resp. closed) but $f_\infty : X_\infty \longrightarrow Y_\infty$ is not surjective (resp. closed). Let $X = \mathbb{A}^2_{\mathbb{C}}$ and $G = \langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{n-1} \end{pmatrix} \rangle$ be a finite cyclic subgroup in $\mathrm{GL}(2, \mathbb{C})$ acting on X, where $n \geq 2$ and ϵ is a primitive n-th root of unity. Let $Y = X/G$ be the quotient of X by the action of G. Then, it is well known that the singularity appeared in Y is A_{n-1}-singularity. Then the canonical projection $f : X \longrightarrow Y$ is closed and surjective. We will see that these two properties are not inherited by $f_\infty : X_\infty \longrightarrow Y_\infty$. Let p be the image $f(0) \in Y$. Then, by the commutativity

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y, \end{array}$$

we obtain $\pi_X^{-1}(0) = f_\infty^{-1} \circ \pi_Y^{-1}(p)$. Here, $\pi_X^{-1}(0)$ is irreducible, since X is non-singular. On the other hand $\pi_Y^{-1}(p)$ has $(n-1)$-irreducible components by [20], [12]. Therefore the morphism f_∞ is not surjective. As $X \setminus \{0\} \longrightarrow Y \setminus \{p\}$ is étale, Proposition 3.3 yields the surjectivity of the morphism $(X \setminus \{0\})_\infty \longrightarrow (Y \setminus \{p\})_\infty$. Since Y_∞ is irreducible, f_∞ is dominant. Therefore, f_∞ is not closed.

Next we discuss about the irreducibility of the arc space or jet spaces.

Theorem 3.8. *If characteristic of k is zero, then the space of arcs of a variety X is irreducible.*

Proof. This is proved in [13]. The lemma [12, 2.12] also gives a proof of this statement. It proves that every arc $\alpha \in (\mathrm{Sing}\, X)_\infty$ is in the closure of an arc $\beta \in X_\infty \setminus (\mathrm{Sing}\, X)_\infty$. We can see that $X_\infty \setminus (\mathrm{Sing}\, X)_\infty$ is irreducible, since this is the image of $Y_\infty \setminus (\varphi^{-1}(\mathrm{Sing}\, X))_\infty$ by a morphism φ_∞, where $\varphi : Y \longrightarrow X$ is a resolution of the singularities of X. (c.f., Proposition 3.2). As $Y_\infty \setminus (\varphi^{-1}(\mathrm{Sing}\, X))_\infty$ is an open subset of an irreducible Y_∞, it follows the irreducibility of X_∞. □

Example 3.9 ([12]). If the characteristic of k is $p > 0$, X_∞ is not necessarily irreducible. For example, the hypersurface X defined by $x^p - y^p z = 0$ has an irredicible component in $(\mathrm{Sing}\, X)_\infty$ which is not in the closure of $X_\infty \setminus (\mathrm{Sing}\, X)_\infty$.

Example 3.10 ([11]). Let X be a toric variety over an algebraically closed field of arbitrary characterictic. Then, X_∞ is irreducible.

A space of m-jets is not necessarily irreducible even if the characteristic of k is zero.

Theorem 3.11 ([18]). *If X is locally a complete intersection variety over an algebraically closed field of characteristic zero, then X_m is irreducible for all $m \geq 1$ if and only if X has rational singularities.*

Another story in which a geometric property of space of jets determines the singularities on the base space is as follows:

Theorem 3.12 ([7]). *Let X be a reduced divisor on a nonsingular variety over $\mathbb{C}$. X has terminal singularities if and only if X_m is normal for every $m \in \mathbb{N}$.*

4. Introduction to the Nash problem

The Nash problem is a question about the Nash components and the essential divisors. First we introduce the concept of essential divisors.

Definition 4.1. Let X be a variety, $g : X_1 \longrightarrow X$ a proper birational morphism from a normal variety X_1 and $E \subset X_1$ an irreducible exceptional divisor of g. Let $f : X_2 \longrightarrow X$ be another proper birational morphism from a normal variety X_2. The birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$ is defined on a (nonempty) open subset E^0 of E. The closure of $(f^{-1} \circ g)(E^0)$ is well defined. It is called the *center* of E on X_2.

We say that E appears in f (or in X_2), if the center of E on X_2 is also a divisor. In this case the birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$ is a local isomorphism at the generic point of E and we denote the birational transform of E on X_2 again by E. For our purposes $E \subset X_1$ is identified with $E \subset X_2$. (Strictly speaking, we should be talking about the corresponding *divisorial valuation* instead.) Such an equivalence class is called an *exceptional divisor over* X.

Definition 4.2. Let X be a variety over k. In this paper, by a *resolution* of the singularities of X we mean a proper, birational morphism $f : Y \longrightarrow X$ with Y non-singular such that $Y \setminus f^{-1}(\operatorname{Sing} X) \longrightarrow X \setminus \operatorname{Sing} X$ is an isomorphism.

Definition 4.3. An exceptional divisor E over X is called an *essential divisor* over X if for every resolution $f : Y \longrightarrow X$ the center of E on Y is an irreducible component of $f^{-1}(\operatorname{Sing} X)$.

For a given resolution $f : Y \longrightarrow X$, the set

$$\mathcal{E} = \mathcal{E}_{Y/X} = \left\{ \begin{array}{c} \text{irreducible components of } f^{-1}(\operatorname{Sing} X) \\ \text{which are centers of essential divisors over } X \end{array} \right\}$$

corresponds bijectively to the set of all essential divisors over X.

Therefore we call an element of $\mathcal{E}$ an *essential component* on Y.

C. Bourvier and G. Gonzalez-Sprinberg also work on "essential divisors" and "essential components" in [1] and [2], but we should note that the definitions are different from ours. In order to distinguish them we give different names to their "essential divisors" and "essential components".

Definition 4.4 ([1], [2]). An exceptional divisor E over X is called a *BGS-essential divisor* over X if E appears in every resolution. An exceptional divisor E over X is called a *BGS-essential component* over X if the center of E on every resolution f of the singularity of X is an irreducible component of $f^{-1}(E')$, where E' is the center of E on X.

Proposition 4.5. *If X is a surface, then each set of "essential divisors", "BGS-essential divisors" and "BGS-essential components" are bijective to the set of the components of the fiber $f^{-1}(\mathrm{Sing}\, X)$, where $f : Y \longrightarrow X$ is the minimal resolution.*

Remark 4.6. Four concepts "essential divisor", "essential component", "BGS-essential divisor" and "BGS-essential component" are mutually different in general.

First, our essential componet is different from the others, because it is a closed subset on a specific resolution and the others are all equivalence class of divisors.

Next, a BGS-essential divisor is different from a BGS-essential component or a essential divisor. Indeed, for $X = (xy - zw = 0) \subset \mathbb{A}_k^4$, the exceptional divisor obtained by a blow-up at the origin is the unique essential divisor and also the unique BGS-essential component, while there is no BGS-essential divisor, since X has a small resolution whose exceptional set is $\mathbb{P}_k^1$.

Finally a BGS-essential component and an essential divisor are different. Indeed, consider a cone generated by $(0,0,1), (2,0,1), (1,1,1)$, $(0,1,1)$ in $N_{\mathbb{R}} = \mathbb{Z}^3$. Let X be the affine toric variety defined by this cone. Then the canonical subdivision adding a one dimensional cone $\mathbb{R}_{\geq 0}(1,0,1)$ is a resolution of X. As the singular locus of X is of dimension one, there is no small resolution. Therefore, the divisor $D_{(1,0,1)}$ is the unique essential divisor, while $D_{(1,1,2)}$ and $D_{(2,1,2)}$ are BGS-essential components by the criterion [1, Theorem 2.3].

Definition 4.7. Let X be a variety. An irreducible component C of $\pi^{-1}(\mathrm{Sing}\, X)$ is called a *Nash component* if it contains an arc α such that $\alpha(\eta) \notin \mathrm{Sing}\, X$. This is equivalent to that $C \not\subset (\mathrm{Sing}\, X)_\infty$.

The following lemma is already quoted for the irreducibility of the space of arcs (Theorem 3.8).

Lemma 4.8 ([12]). *If the characterisitic of the base field k is zero, then every irreducible component of $\pi^{-1}(\mathrm{Sing}\, X)$ is a Nash component.*

Example 4.9 ([12]). Let the characteristic of the base field k be $p > 0$. Let X be a hypersurface defined by $x^p - y^p z = 0$. Then, X has an irredicible component in $\pi^{-1}(\operatorname{Sing} X)$ contained in $(\operatorname{Sing} X)_\infty$.

Let $f : Y \longrightarrow X$ be a resolution of the singularities of X and E_l $(l = 1, .., r)$ the irreducible components of $f^{-1}(\operatorname{Sing} X)$. Now we are going to introduce a map $\mathcal{N}$ which is called the Nash map

$$\left\{ \begin{array}{c} \text{Nash components} \\ \text{of the space of arcs} \\ \text{through } \operatorname{Sing} X \end{array} \right\} \xrightarrow{\mathcal{N}} \left\{ \begin{array}{c} \text{essential} \\ \text{components} \\ \text{on } Y \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{essential} \\ \text{divisors} \\ \text{over } X \end{array} \right\}.$$

4.10 (construction of the Nash map). The resolution $f : Y \longrightarrow X$ induces a morphism $f_\infty : Y_\infty \longrightarrow X_\infty$ of schemes. Let $\pi_Y : Y_\infty \longrightarrow Y$ be the canonical projection. As Y is non-singular, $(\pi_Y)^{-1}(E_l)$ is irreducible for every l. Denote by $(\pi_Y)^{-1}(E_l)^o$ the open subset of $(\pi_Y)^{-1}(E_l)$ consisting of the points corresponding to arcs $\beta : \operatorname{Spec} K[[t]] \longrightarrow Y$ such that $\beta(\eta) \notin \bigcup_l E_l$. Let C_i $(i \in I)$ be the Nash components of X. Denote by C_i^o the open subset of C_i consisting of the points corresponding to arcs $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha(\eta) \notin \operatorname{Sing} X$. By restriction f_∞ gives $f'_\infty : \bigcup_{l=1}^r (\pi_Y)^{-1}(E_l)^o \longrightarrow \bigcup_{i \in I} C_i^o$. By Proposition 3.2, f'_∞ is surjective. Hence, for each $i \in I$ there is $1 \leq l_i \leq r$ such that the generic point β_{l_i} of $(\pi_Y)^{-1}(E_{l_i})^o$ is mapped to the generic point α_i of C_i^o. By this correspondence $C_i \mapsto E_{l_i}$ we obtain a map

$$\mathcal{N} : \left\{ \begin{array}{c} \text{Nash components} \\ \text{of the space of arcs} \\ \text{through } \operatorname{Sing} X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{irreducible} \\ \text{components} \\ \text{of } f^{-1}(\operatorname{Sing} X) \end{array} \right\}.$$

Lemma 4.11. *The map $\mathcal{N}$ is an injective map to the subset $\{$ essential components on $Y\}$.*

Proof. Let $\mathcal{N}(C_i) = E_{l_i}$. Denote the generic point of C_i by α_i and the generic point of $(\pi_Y)^{-1}(E_l)$ by β_l. If $E_{l_i} = E_{l_j}$ for $i \neq j$, then $\alpha_i = f'_\infty(\beta_{l_i}) = f'_\infty(\beta_{l_j}) = \alpha_j$, a contradiction.

To prove that the $\{E_{l_i} : i \in I\}$ are essential components on Y, let $Y' \longrightarrow X$ be another resolution and $\tilde{Y} \longrightarrow X$ a divisorial resolution which factors through both Y and Y'. Let $E'_{l_i} \subset Y'$ and $\tilde{E}_{l_i} \subset \tilde{Y}$ be the irreducible components of the exceptional sets corresponding to C_i. Then, we can see that E_{l_i} and E'_{l_i} are the image of $\tilde{E}_{l_i}$. This shows that $\tilde{E}_{l_i}$ is an essential divisor over X and therefore E_{l_i} is an essential component on Y. $\square$

Problem 4.12. *Is the Nash map*

$$\left\{\begin{array}{c}\text{Nash components}\\ \text{of the space of arcs}\\ \text{through } \operatorname{Sing} X\end{array}\right\} \xrightarrow{\mathcal{N}} \left\{\begin{array}{c}\text{essential}\\ \text{components}\\ \text{on } Y\end{array}\right\} \simeq \left\{\begin{array}{c}\text{essential}\\ \text{divisors}\\ \text{over } X\end{array}\right\}.$$

bijective?

After Nash's preprint which posed this problem was circulated in 1968, Bouvier, Gonzalez-Sprinberg, Hickel, Lejeune-Jalabert, Nobile, Reguera-Lopez and others (see, [1], [9], [10], [15], [16], [17], [21], [22]) worked on the arc space of a singular variety related to this problem.

Recently for a toric variety of arbitrary dimension the Nash problem is affirmatively answered but is negatively answered in general in [12].

Here, we introduce a brief history of this problem.

Theorem 4.13 ([20]). *The Nash problem holds true for an A_n-singularity ($n \in \mathbb{N}$), where a A_n-singularity is the hypersurface singularity defined by $xy - z^{n+1} = 0$ in $\mathbb{A}_k^3$.*

Theorem 4.14 ([22]). *The Nash problem holds true for a minimal surface singularity. Here, a minimal surface singularity means a rational surface singularity with the reduced fundamental cycle.*

Theorem 4.15 ([17], [23]). *The Nash problem holds true for a sandwiched surface singularity. Here, a sandwichted surface singularity means the formal neighborhood of a singular point on a surface obtained by blowing up a complete ideal in the local ring of a closed point on a non-singular algebraic surface.*

Theorem 4.16 ([12]). *The Nash problem holds true for a toric singularity of arbitrary dimension.*

So far we have seen the affirmative answers. The last year, negative examples are given in [12].

Example 4.17. Let X be a hypersurface defined by $x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$ in $\mathbb{A}_{\mathbb{C}}^5$. Then the number of the Nash components is one, while the number of the essential divisors is two. Therefore the Nash map is not bijective.

By the above example we can construct counter examples to the Nash problem for dimension greater than 3. At this moment the Nash problem is still open for two and three dimensional variety.

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AN INTRODUCTION TO ARC SPACES

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