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格子から切り取った平面曲線と Dehn 手術の係数
(subtitle: Plane slalom curves of a certain type, pretzel links and Kirby-Melvin's Grapes)

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Abstract

We are concerned with plane curves of type $C(p,q,r)$ as in Figure 1 and 2, and their corresponding links $L(C(p,q,r))$ via A'Campo's divide theory, where p,q,r are positive integers with $1 \leq p \leq q \leq r$. We will point out that 2-fold covering spaces of the 3-dimensional sphere S^3 branched along $L(C(p,q,r))$ (2-branched coverings, for short) is represented by Kirby-Melvin's grapes. We will also refer to some other related topics.

1 Introduction

The *divide* is a relative, generic immersion of a 1-manifold in a unit disk D in $\mathbf{R}^2$. N. A'Campo formulated the way to associate to each divide C a link $L(C)$ in the 3-dimensional sphere S^3 ([A1, A2, A3, A4]):

$$L(C) = \{(u,v) \in TD | u \in C,\, v \in T_uC,\, |u|^2 + |v|^2 = 1\} \subset S^3.$$

The class of links of divides properly contains the class of the links arising from isolated singularities of complex curves. In this paper, we draw only curves C but the disk. Note that the number of components of $L(C)$ is $\sharp_a(C) + 2\sharp_c(C)$, where $\sharp_a(C)$ (and $\sharp_c(C)$, respectively) is the number of immersed components of arcs (and circles) in C. We say that C is in *arc case* if $\sharp_a(C) = 1$ and $\sharp_c(C) = 0$.

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Here the author focus on a part of (not whole) his talk on Nov.28, 2003. Thus he has add a subtitle of this article as above.

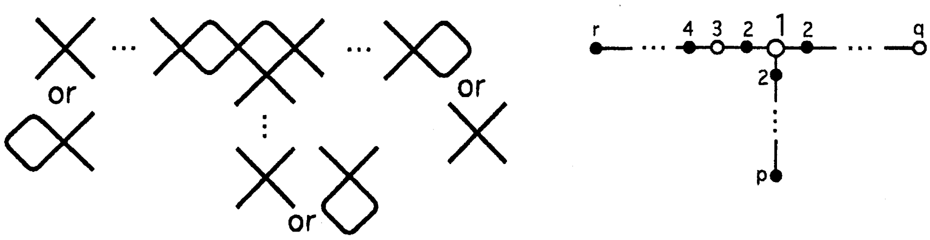

Figure 1: $C(p,q,r)$ and $D(p,q,r)$

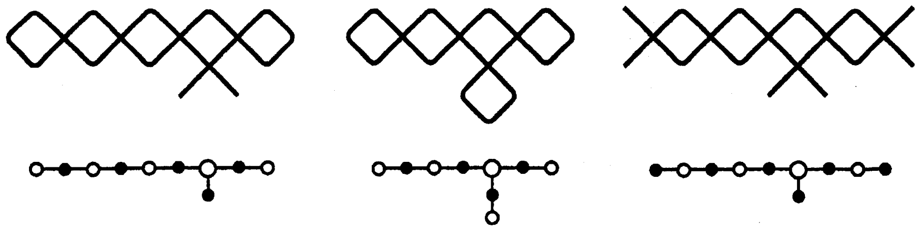

Figure 2: $C(2,3,7)$, $C(3,3,5)$ and $C(2,4,6)$

For a plane curve of type $C(p,q,r)$, by $D(p,q,r)$, we denote the corresponding diagram in Figure 1. Numbers in the diagram are written only for counting. Note that each odd number in $\{p,q,r\}$ corresponds to white point and "α" at the terminal, and that $\sharp_a(C)$ and $\sharp_c(C)$ are given by;

$$\sharp_a(C(p,q,r)) = e(p,q,r), \qquad \sharp_c(C(p,q,r)) = \begin{cases} 1 & \text{if} \;\; e(p,q,r) = 0 \\ 0 & \text{if} \;\; e(p,q,r) \geq 1 \end{cases},$$

where $e(p,q,r)$ is the number of even number(s) in $\{p,q,r\}$. Curves $C(p,q,r)$ in arc cases are included in the class of *slalom curves*, which was studied by N. A'Campo in [A2] (Theorem 4.1 in Section. 4 is one of his results). We will study the links $L(C(p,q,r))$ from mainly the point of view of 4-dimensional topology, branched coverings, Kirby-Melvin's grapes and moves of framed links.

The author would like to sincere gratitude to Professor N. A'Campo for his kind encouragement by e-mail. The author would like to thank to Professor Masaharu Ishikawa for many valuable advice ([GI1, GI2]) on A'Campo's theory and to Professor Mikami Hirasawa, who informed him the starting example Pr$(-2,3,7)$ and checked some examples of Theorem 2.1 by more knot-theoretical and visualized method in [H]. The author would like also to thank knot theorists Prof. Koya Shimokawa, Dr. Kazuhiro Ichihara and Dr. Takuji Nakamura for their valuable comments from their own recent research.

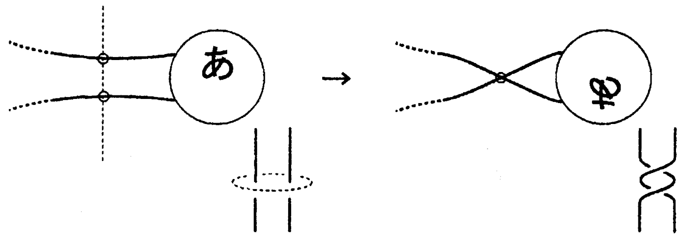

Figure 3: Blow-down

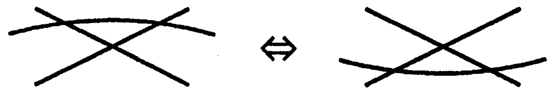

Figure 4: A triangle move

2 Pretzel links

First, we give an answer to the question "what link is $L(C(p,q,r))$?"

Theorem 2.1 *The link $L(C(p,q,r))$ is a pretzel link of type $(-1,p,q,r)$.*

Proof. In the small cases $(p,q,r) = (1,1,1),(1,1,2),(1,2,2)$ and $(2,2,2)$, it is easily checked by the standard singularity theory, or by [H]. In fact, the link is A_1:*a Hopf link,* A_2:*a trefoil knot,* A_3:*a torus link T(2,4) or* D_4:*a torus link* $T(3,3)$, respectively. In general cases, it is proved by some blow-down's, i.e., full-twistings, see Figure 3. Note that one blow-down increase one of p,q,r by two. □

2-branched coverings of S^3 along such pretzel links are known to be Seifert manifolds. Akbulut-Kirby's algorithm [AK] is useful.

Corollary 2.2 *The 2-fold covering space $M^3(p,q,r)$ of S^3 along $L(C(p,q,r))$ is a Seifert manifold of type $\{-1;(o,0);(p,1),(q,1),(r,1)\}$ in Orlik's notation [Or].*

The 3-manifold $M(p,q,r)$ (as a boundary of the 4-manifold $W^4(p,q,r)$) is represented by a framed link in Figure 5, where every framing is -2, thus omitted. Note that the 4-manifold $W(p,q,r)$ directly corresponds to the diagram $D(p,q,r)$, see [HKK, p.13 and p.25]. Such special framed links are represented by Kirby-Melvin's useful method "grapes" [KM]: *A grapes* is a configuration of hexagonally packed circles. Each individual circle will be called a *grape.* For the way to construct from a grapes to its framed link and more detail, the author strongly recommend to the readers to see [KM]. The advantage of representation by grapes is *slip of a grape*, i.e., that we can move a grape under a certain conditions without changing of the 4-manifold. On the other hand, by similar Kirby calculus to that in [K, p.15], it is proved that $M(p,q,r)$ is also represented by a framed link in Figure 7 (of course, $W(p,q,r)$ has been changed).

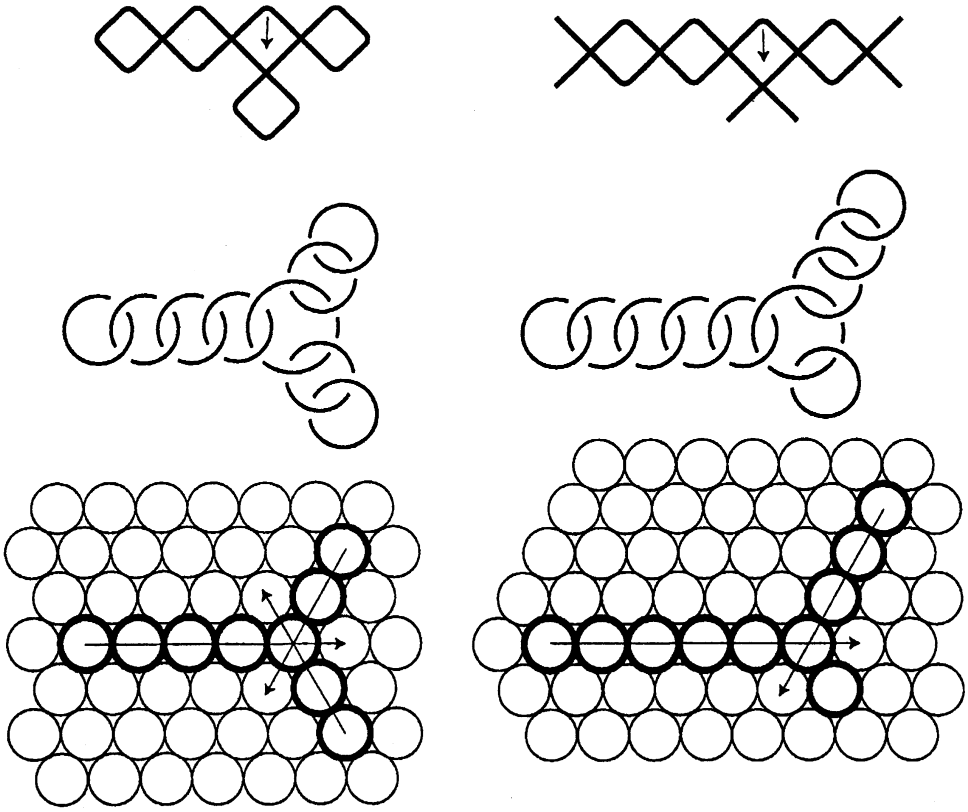

Figure 5: Framed links and Grapes

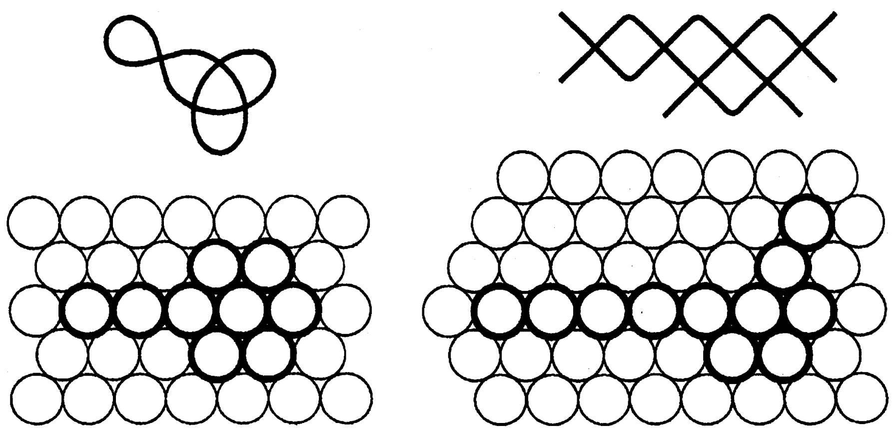

Figure 6: Triangle Moves and Grape Slips from Fig.5

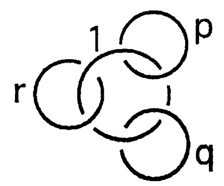

Figure 7: $M(p,q,r)$

In divide theory, *triangle moves* on divides in Figure 4 do not change the corresponding links [2]. See the moves from Figure 5 to Figure 6 (and see [AGV, p.117]). There might be a relationship between triangle moves and slips of grapes, but maybe indirectly, since the former is local and the latter is global.

3 Triangle singularities

Each Seifert manifold of type $\{-1;(o,0);(p,1),(q,1),(r,1)\}$ for 14 triples (p,q,r) in Table 1 is known to be a *link* of Arnold's triangle singularities ([Ar]) (exceptional singularities or unimodal singularities) $D_{p,q,r}$ in $\mathbf{C}^3$, i.e., an intersection of the complex algebraic surface and a small 5-sphere centered at the singularity. Here we copy the list as Table 1 from [D, p.63] (see [Ar], [AGV, p.110] and also [Mz]).

Question 1. Are there any topological or algebraic-geometrical relationship between the plane curves $C(p,q,r)$ and the singularities $D_{p,q,r}$?

Notation	p,q,r	Equation in $\mathbf{C}^3$	p',q',r'
Q_{10}	2, 3, 9	$x^3+y^4+yz^2$	3, 3, 4
Q_{11}	2, 4, 7	$x^3+y^2z+xz^3$	3, 3, 5
Q_{12}	3, 3, 6	$x^3+y^5+yz^2$	3, 3, 6
Z_{11}	2, 3, 8	$x^3y+y^5+z^2$	2, 4, 5
Z_{12}	2, 4, 6	$x^3y+xy^4+z^2$	2, 4, 6
Z_{13}	3, 3, 5	$x^3y+y^6+z^2$	2, 4, 7
S_{11}	2, 5, 6	$x^4+y^2z+xz^2$	3, 4, 4
S_{12}	3, 4, 5	$x^2y+y^2z+xz^3$	3, 4, 5
W_{12}	2, 5, 5	$x^4+y^5+z^2$	2, 5, 5
W_{13}	3, 4, 4	$x^4+xy^4+z^2$	2, 5, 6
E_{12}	2, 3, 7	$x^3+y^7+z^2$	2, 3, 7
E_{13}	2, 4, 5	$x^3+xy^5+z^2$	2, 3, 8
E_{14}	3, 3, 4	$x^3+y^8+z^2$	2, 3, 9
U_{12}	4, 4, 4	$x^3+y^3+z^4$	4, 4, 4

Table 1: List of Triangle singularities

[2]Recently, Prof. Masaharu Ishikawa has pointed out that the converse is not true and given infinitely many examples of different types after earlier discoveries in [GI1, GI2].

There is a symmetry called "Arnold's strange duality" between Dolgachev numbers (p, q, r) and Gabrielov numbers (p', q', r') in the list. The resolution space of the singularity $D_{p,q,r}$ is orientation-reversingly [3] diffeomorphic to the 4-manifold described by the framed link in Figure 7.

4 Related studies

Here we refer to some related works.

We start with knot theory on A, D, E-singularity. The link of A_{2k}, E_6 and E_8 singularity in $\mathbf{C}^2$ is torus knot of type $(2, 2k+1)$, $(3, 4)$ and $(3, 5)$ respectively. For divide theory on torus links $T(a, b)$ (singularities of type $z^a - w^b = 0$), see [AGV, GZ] and [GHY]. For $n \geq 4$,

$$L(C(2,2,n-2)) = \mathrm{Pr}(-1,2,2,n-2) = \mathrm{Pr}(-2,2,n-2) \qquad (D_n).$$

The link of D_4-singularity $x^2 + y^3 + z^3 = 0$ in $\mathbf{C}^3$ is the 2-branched covering of S^3 along $L(C(2,2,2))$ (= Torus link $T(3,3)$) is a quotient space of S^3 by the quaternion group G_8 of order 8, called "quarternionic space" Q_8. In [Y1], we studied a certain surgery along Q_8, from the view point of 4-manifold theory. On E-singularities,

$$\begin{aligned} L(C(2,3,3)) &= \mathrm{Pr}(-2,3,3) = T(3,4) \qquad (E_6), \\ L(C(2,3,5)) &= \mathrm{Pr}(-2,3,5) = T(3,5) \qquad (E_8). \end{aligned}$$

Next, we study the links from the view point of Dehn surgery on hyperbolic knots. A curve of type $C(2,3,n)$ with $n \geq 5$ is moved by triangle moves as in Figure 8 ($n = 7$ case). These three curves are obtained by "cutting out from a lattice X" as $X \cap \mathcal{R}$, where $\mathcal{R}$ is a union of rectangles in the plane. In [Y2, Y3] we pointed out that, in such curves of type $X \cap \mathcal{R}$, the area $\mathcal{R}$ is related to coefficient of *finite Dehn surgery*, i.e. surgery yielding 3-manifolds whose fundamental group is finite. Mainly hyperbolic knots have been researched ([CGLS] and many works). From such a view point, the following result by N. A'Campo is important:

Theorem 4.1 ([A2]) *For a slalom curves in arc cases, if the corresponding diagram is neither Dynkin diagram of type A_{2k} with $k \geq 1$, E_6 nor E_8, then the corresponding divide knot is hyperbolic.*

In the triangle moves in Figure 8, the area of $\mathcal{R}$ changes from $2n+8$ to $2n+6$ and to $2n+5$. Koya Shimokawa and Kazuhiro Ichihara pointed out to the author that these numbers are near the special numbers (slopes) of the knots $Pr(-2,3,n)$ with odd $n \geq 7$ for Dehn surgery and informed M. Dunfield's program to calculate *boudary slopes* of the knots. See Table 2. The data in the first four lines were picked up from K. Shimokowa's OHP-sheat used in his talk in Kobe, in Sep. 2003.

[3] The author's orientation may be opposite from the ordinary one here.

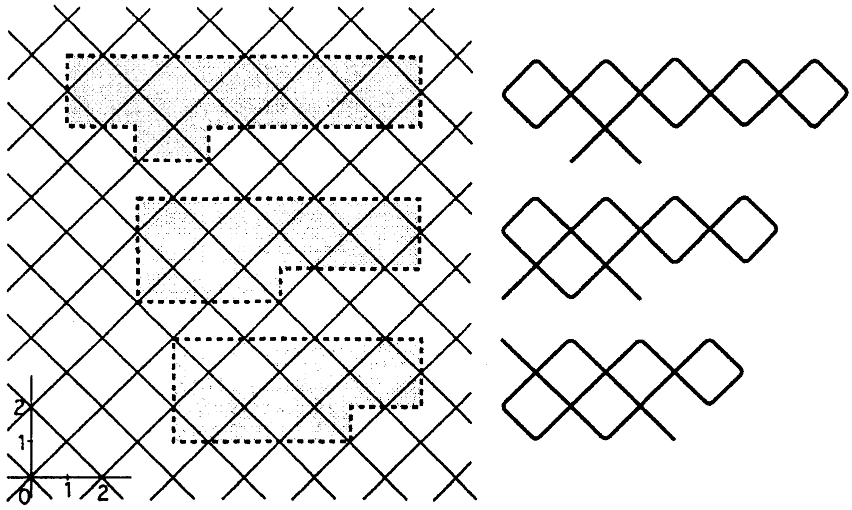

Figure 8: Triangle moves as curves in the lattice

n (odd)	$2n+3$	$2n+4$	$2n+5$	$2n+6$	$2n+7$	$2n+8$
7	17:Fin.	18: lens	19 : lens	20 : Tor.	–	–
9	–	22: Fin.	23 : Fin.	24 : Bdr.	–	–
$n \geq 11$	–	Seif.	Seif.	Bdr.	–	–
area of $\mathcal{R}$	–	–	•	•	–	•
$A'(\mathcal{R})$ below	–	∘	∘	∘	–	–

Table 2: Special slopes for $Pr(-2,3,n)$

where

Fin. = finite (but non-cyclic) surgery, i.e.,
yielding a 3-manifold whose fundamental group is non-cyclic finite,
lens = yielding a lens space,
(ex. 19-surgery on $Pr(-2,3,7)$ is $L(19,8)$, see [FS] and also [Y2])
Seif. = yielding a Seifert manifold,
Tor. = toroidal surgery, i.e.,
yielding a 3-manifold that contains an essential torus,
Bdr. = boundary slope, i.e.,
there exists an essential surface in the knot exterior
whose boundary curves has the slope,

but we do not refer to these terminologies in detail here.

Back to Figure 8 again, we set

$$A'(\mathcal{R}) := \text{(the area of } \mathcal{R}) - \text{“the number of } 270^\circ\text{-corner (i.e. concave ones)”}.$$

Then, in the triangle moves, $A'(\mathcal{R})$ changes from $2n+6$ to $2n+5$ and to $2n+4$.

Question 2. Does the number $A'(\mathfrak{R})$ for the curves or the corresponding knots for general $\mathfrak{R}$ have mathematical meanings ?

Finally, we give one more information from knot theory. Any divide knot is known to be a closure of strongly quasi-positive braid, i.e., of a composite of special conjugation of positive generators. Takuji Nakamura pointed out that any $Pr(-1, p, q, r)$ with $p, q, r > 0$ is a closure of positive braid, i.e., of a composite of positive generators, of index 3. According to the author's knowledge ([Y1, Y2], and [B]), it seems that any divide knot yielding finite surgery is a closure of positive braid. It seems also that any knot yielding lens spaces is a closure of positive braid, of course up to mirror image.

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