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Kyoto University
EVERSION OF A FOLD MAP OF $S^2$ TO $\mathbb{R}^2$
WITH ONE SINGULAR SET

MINORU YAMAMOTO (山本 梓 (旧) 九州大学. (現) 北海道大学)
(OLD ADDRESS) GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY
(CURRENT ADDRESS) DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY

1. INTRODUCTION

In the following, all manifolds and maps are differentiable of class $C^\infty$.

Let $M$ be an $n$-dimensional closed manifold, $N$ an $n$-dimensional manifold and $f : M \to N$ a map of $M$ into $N$. We denote by $S(f)$ the set of the points in $M$ where the rank of the differential of $f$ is strictly less than $n$. We call $S(f)$ the singular set of $f$ and $f(S(f))$ the singular value set of $f$. We say that a map $f : M \to N$ is a fold map if there exist local coordinate systems $(x_1, x_2, \ldots, x_n)$ around $q \in M$ and $(y_1, y_2, \ldots, y_n)$ around $f(q) \in N$ such that $f$ has one of the following forms:

$$(y_1 \circ f, y_2 \circ f, \ldots, y_{n-1} \circ f, y_n \circ f) = \begin{cases} (x_1, x_2, \ldots, x_{n-1}, x_n), & \text{q: regular point}, \\ (x_1, x_2, \ldots, x_{n-1}, x_n^2), & \text{q: fold point}. \end{cases}$$

Note that for a fold map $f : M \to N$, $S(f)$ is an $(n-1)$-dimensional submanifold of $M$. If the restricted map $f|S(f) : S(f) \to N$ is an immersion with normal crossings, we call $f$ a stable fold map.

Let $V$ be an $(n-1)$-dimensional submanifold of $M$ and $f : M \to N$ a fold map such that $S(f) = V$. We denote by $\mathcal{F}(M, N; V)$ the set of such fold maps. Note that $\mathcal{F}(M, N; V)$ is the subspace of $C^\infty(M, N)$ having the Whitney $C^\infty$-topology. Let $T$ be a tubular neighborhood of $V$ in $M$ such that there exists a fiber involution of it, $h : T \to T$, whose fixed points set is $V$ and the composition $(f|T) \circ h$ coincides with $f|T$. Note that for any $f \in \mathcal{F}(M, N; V)$, we may assume that $T$ does not depend on $f$ but depends on $M$ and $V$. For $\overline{M} = \text{cl}(M \setminus T)$, the closure of $M \setminus T$, $f|\overline{M} : \overline{M} \to N$ is an immersion.

In [1, 2], Eliashberg studied the existence of a fold map $f : M \to N$. In the appendix of [2], he proved that the number of the connected components of $\mathcal{F}(S^2, \mathbb{R}^2; S^0_0)$ is strictly four, where $S^2$ is an oriented 2-dimensional sphere, $\mathbb{R}^2$ is the oriented plane and $S^0_0$ is the equator of $S^2$. We denote by $S_1, S_2, \mathcal{E}_1$ and $\mathcal{E}_2$ the connected components of $\mathcal{F}(S^2, \mathbb{R}^2; S^0_0)$. We call a fold map $f$ in $\mathcal{S}_i$ a standard fold map and in $\mathcal{E}_i$ an exotic fold map ($i = 1, 2$). In the same paper, he showed the representative elements of each connected components of $\mathcal{F}(S^2, \mathbb{R}^2; S^0_0)$. Let $e : S^2 \to \mathbb{R}^2$ be the representative element of $\mathcal{E}_1$ such that Eliashberg gave this map in [2] ([1]). This fold map is constructed by using two immersed disks called Milnor’s examples. We can construct

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another exotic fold map $\tilde{e} \in E_1$ by using these two Milnor’s examples. Then, there exists a homotopy \( E : S^2 \times [-1, 1] \to \mathbb{R}^2 \) such that \( e_{-1} = e, e_1 = \tilde{e} \) and \( e_t \in E_1 \), where \( e_t \) is defined by \( e_t(x) = E(x, t) \ (x \in S^2, t \in [-1, 1]) \). We call such a homotopy a fold eversion between \( e \) and \( \tilde{e} \).

In [2], Eliashberg only stated the existence of a fold eversion. As the theorem of sphere eversion [7], it is difficult to give a fold eversion at first glance. In this report, we construct a fold eversion between \( e \) and \( \tilde{e} \) concretely as Morin and Petit constructed a sphere eversion concretely (see [6]).

The report is organized as follows.

In Section 2, we characterize each connected component of \( F(S^2, \mathbb{R}^2; S_0^1) \) and construct fold maps \( e \) and \( \tilde{e} \). We observe local behaviors of a homotopy of fold maps.

In Section 3, we give a fold eversion between \( e \) and \( \tilde{e} \) concretely.

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2. Preliminaries

In this section, we state each connected component of \( F(S^2, \mathbb{R}^2; S_0^1) \) precisely. We also see local behaviors of a homotopy of fold maps.

Let \( S^2 \) be an oriented 2-dimensional sphere, \( \mathbb{R}^2 \) the oriented plane and \( S_0^1 \) the equator of \( S^2 \). Let \( T \) be a tubular neighborhood of \( S_0^1 \) in \( S^2 \) and we fix a trivialization \( T \cong S_0^1 \times [-1, 1] \) such that \( S_0^1 = S_0^1 \times \{0\} \). We have a fiber involution \( h : S_0^1 \times [-1, 1] \to S_0^1 \times [-1, 1] \) such that \( h(x, t) = (x, -t) \). Then we may assume that for any \( f \in F(S^2, \mathbb{R}^2; S_0^1) \), we have

\[
(2.1) \quad (f|S_0^1 \times [-1, 1])(x, t) = (f|S_0^1 \times [-1, 1])(x, -t)
\]

and

\[
(2.2) \quad f|S_0^1 \times \{t\} \text{ is sufficiently close to } f|S_0^1 \times \{1\}.
\]

Here, \( \mathbb{R}^2 \) has the Euclidean metric. We denote by \( D_N^2 \) and \( D_S^2 \) each connected component of \( \text{cl}(S^2 \setminus T) \).

**Definition 2.1.** Let \( f : S^2 \to \mathbb{R}^2 \) be a fold map in \( F(S^2, \mathbb{R}^2; S_0^1) \). We say that \( f|D_N^2 \) and \( f|D_S^2 \) are the same extensions of \( f|\partial T \) if there exists an orientation reversing diffeomorphism \( k : D_N^2 \to D_S^2 \) such that \( k|\partial D_N^2 = h \) and \( f|D_S^2 \circ k = f|D_N^2 \). Otherwise, we say that \( f|D_N^2 \) and \( f|D_S^2 \) are the different extensions of \( f|\partial T \).

Then, Eliashberg’s theorem is stated as follows.

**Theorem 2.2 (Eliashberg [2]).** Each connected component of \( F(S^2, \mathbb{R}^2; S_0^1) = S_1 \cup S_2 \cup E_1 \cup E_2 \) consists of all fold maps satisfying the following properties.

1. The connected component \( S_1 \) (resp. \( S_2 \)) consists of all fold maps \( f : S^2 \to \mathbb{R}^2 \) in \( F(S^2, \mathbb{R}^2; S_0^1) \) such that \( f|D_N^2 \) and \( f|D_S^2 \) are the same extensions of \( f|\partial T \). We set the
orientation of $S^2$ so that $f|D^2_N$ is the orientation preserving (resp. reversing) immersion and $f|D^2_S$ is the orientation reversing (resp. preserving) immersion.

(2) The connected component $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) consists of all fold maps $f : S^2 \to \mathbb{R}^2$ in $\mathcal{F}(S^2, \mathbb{R}^2; S^0_1)$ such that $f|D^2_N$ and $f|D^2_S$ are the different extensions of $f|\partial T$. We set the orientation of $S^2$ so that $f|D^2_N$ is the orientation preserving (resp. reversing) immersion and $f|D^2_S$ is the orientation reversing (resp. preserving) immersion.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the canonical projection defined by $\pi(x_1, x_2, x_3) = (x_1, x_2)$ and $i : S^2 \to \mathbb{R}^3$ the inclusion defined by $i(S^2) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ and $i(S^1_0) = \{(x_1, x_2, x_3) \in i(S^2) \mid x_3 = 0\}$. If we choose a suitable orientation on $S^2$, $s = \pi \circ i$ is a stable fold map in $S_1$. The images of $s(D^2_N)$ and $s(D^2_S)$ are depicted as in FIGURE 1.

FIGURE 1

In [2] ([1]), Eliashberg gave a representative element, $e : S^2 \to \mathbb{R}^2$, of $\mathcal{E}_1$. Let $D^2$ be an oriented 2-dimensional disk. Let $m_1$ and $m_2 : D^2 \to \mathbb{R}^2$ be two orientation preserving immersions called Milnor's examples (see FIGURE 2). Note that $m_1$ and $m_2$ are the different extensions of $m_1|\partial D^2 = m_2|\partial D^2$.

FIGURE 2

Let $g_N : D^2_N \to D^2$ be an orientation preserving diffeomorphism and $g_S : D^2_S \to D^2$ an orientation reversing diffeomorphism such that $g_S \circ h|\partial D^2_N = g_N|\partial D^2_N$ holds. Then, we have the desired fold map $e \in \mathcal{E}_1$ such that $e|D^2_N = m_1 \circ g_N$, $e|D^2_S = m_2 \circ g_S$ and $e|T$ satisfies the conditions (2.1) and (2.2). The image of $e(D^2_N)$ is depicted as in FIGURE 3 (a) and $e(D^2_S)$ is depicted as in FIGURE 3 (b).

FIGURE 3

If we exchange these two Milnor's examples on $D^2_N$ and $D^2_S$, we have another exotic fold map $\tilde{e} \in \mathcal{E}_1$ such that $\tilde{e}|D^2_N = m_2 \circ g_N$, $\tilde{e}|D^2_S = m_1 \circ g_S$ and $\tilde{e}|T$ satisfies the conditions (2.1) and (2.2). The image of $\tilde{e}(D^2_N)$ is depicted as in FIGURE 4 (a) and $\tilde{e}(D^2_S)$ is depicted as in FIGURE 4 (b).

FIGURE 4

Note that $e$ and $\tilde{e}$ are stable fold maps. In FIGURES 3 and 4, gray strips are the image of rectangles properly embedded in $D^2_N$ and $D^2_S$, respectively. We draw these gray strips so that they help the readers to understand how to extend $e|\partial T$ (resp. $\tilde{e}|\partial T$) to $e|D^2_N$ and $e|D^2_S$ (resp. $\tilde{e}|D^2_N$ and $\tilde{e}|D^2_S$). They also help the readers to understand $e|D^2_N$ and $e|D^2_S$ (resp. $\tilde{e}|D^2_N$ and $\tilde{e}|D^2_S$) are the different extensions of $e|\partial T$ (resp. $\tilde{e}|\partial T$).

Let $f$ and $g$ be fold maps in $\mathcal{F}(S^2, \mathbb{R}^2; S^0_1)$ such that $f$ is a stable fold map. Let $y_g \in g(S(g))$ be a singular value of $g$. Suppose that there exists a singular value $y_f \in f(S(f))$ such that a map germ $g : (S^2, g^{-1}(y_g) \cap S(g)) \to (\mathbb{R}^2, y_g)$ is $\mathcal{A}$-equivalent to a map germ $f : (S^2, f^{-1}(y_f) \cap S(f)) \to (\mathbb{R}^2, y_f)$. Then, we call $y_g$ a stable fold singular value of $g$.

Let $f$ and $g : S^2 \to \mathbb{R}^2$ be two stable fold maps such that they are in the same connected component of $\mathcal{F}(S^2, \mathbb{R}^2; S^0_1)$. By the relative version of the parameterized multi-transversality
theorem, there exists a homotopy $F : S^2 \times [-1, 1] \to \mathbb{R}^2$ such that $F$ satisfies the following properties.

1. For any $t \in [-1, 1]$, $f_t : S^2 \to \mathbb{R}^2$ is a fold map such that $f_{-1} = f$, $f_1 = g$ and $f_t$ and $f$ are in the same connected component of $\mathcal{F}(S^2, \mathbb{R}^2; S_0^1)$. Here, $f_t$ is defined by $f_t(x) = F(x, t)$.

2. There is a finite set of parameter values $\{t_i \mid -1 < t_1 < t_2 < \cdots < t_l < 1\}$ (possibly empty) in the open interval $(-1, 1)$ such that the following conditions hold.

1.1. For any $t \in [-1, 1] \setminus \{t_1, \ldots, t_l\}$, $f_t : M \to \mathbb{R}^2$ is a stable fold map.

1.2. For each $t_i (i = 1, \ldots, l)$, $f_{t_i}$ has the unique singular value $y_i \in f_{t_i}(S(f_{t_i}))$ which is not stable fold singular value of $f_{t_i} (i = 1, \ldots, l)$. The map germ $F : (S^2 \times (t_i - \varepsilon, t_i + \varepsilon), (f_{t_i}^{-1}(y_i) \cap S_0^1) \times \{t_i\}) \to (\mathbb{R}^2, y_i)$ is $\mathcal{A}$-equivalent to one of the 1-parameter unfoldings in TABLE 1, where $\varepsilon$ is a sufficiently small positive real number.

We call such an $F$ a generic fold homotopy between $f$ and $g$. We say that each $t_i$ a codimension 1 bifurcation value of $F$ and each $f_{t_i}$ a codimension 1 fold map in $\mathcal{F}(S^2, \mathbb{R}^2; S_0^1) (i = 1, \ldots, l)$. We say that $f$ is the initial stable fold map of $F$ and $g$ is the terminal stable fold map of $F$. We denote by $\Gamma_i$ the set of all codimension 1 fold maps in $\mathcal{F}(S^2, \mathbb{R}^2; S_0^1)$. By using local normal forms in TABLE 1, $\Gamma_i$ is classified into five strata, $J_i^*$ and $T_i (\ast = +, -$ and $\ast = 1, 2$). Note that each stratum may not necessarily be connected.

Remark 2.3. The relative multi-transversality theorem is stated and proved in [4] and the parameterized relative multi-transversality theorem is stated in [8]. The $\mathcal{A}$-equivalence classification of map germs $g : (\mathbb{R}^2, S) \to (\mathbb{R}^2, 0)$ and their 1-parameter unfoldings has been studied by Gibson and Hobbs [3]. Here, $S$ consists of finitely many isolated points of $g^{-1}(0)$.

<table>
<thead>
<tr>
<th>type</th>
<th>normal form $G(x, y, t)$</th>
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<tr>
<td>$J^+$</td>
<td>$(x_1, y_1^2 + t), (x_2, x_2^2 + y_2^2)$</td>
</tr>
<tr>
<td>$J_1$</td>
<td>$(x_1, -y_1^2 + t), (x_2, x_2^2 + y_2^2)$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$(x_1, y_1^2 + t), (x_2, x_2^2 - y_2^2)$</td>
</tr>
<tr>
<td>$T_1$</td>
<td>$(x_1 + y_1^2, x_1 - y_1^2 + t), (x_2, y_2^2), (-y_2^2, x_3)$</td>
</tr>
<tr>
<td>$T_2$</td>
<td>$(x_1 + y_1^2, x_1 - y_1^2 + t), (x_2, y_2^2), (y_2^2, x_3)$</td>
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TABLE 1. 1-parameter unfoldings

Let $G : (\mathbb{R}^2 \times \mathbb{R}, S \times \{0\}) \to (\mathbb{R}^2, 0)$ be a 1-parameter unfolding in TABLE 1. We define $g_t : \mathbb{R}^2 \to \mathbb{R}^2$ by $g_t(x) = G(x, t)$ and suppose that $S \subset S(g_0)$. Using the local normal forms in TABLE 1, we see that the deformations of set germs $g_t(\mathbb{R}^2)$ around $0 \in \mathbb{R}^2$ are as depicted in FIGURE 5.
Let \( F : S^2 \times [-1, 1] \to \mathbb{R}^2 \) be a generic fold homotopy such that \( 0 \in [-1, 1] \) is the unique codimension 1 bifurcation value of \( F \). Then we say that \( F \) crosses \( \Gamma_1 \) positively at \( f_0 \) if one of the following holds.

1. When \( f_0 \in J^+, J_1^- \) and \( J_2^- \), the number of normal crossing points of \( f_1(S(f_1)) \) is greater than that of \( f_1(S(f_1)) \).
2. When \( f_0 \in T_1 \) and \( T_2 \), the number of preimage over a point in the new-born triangle of \( f_1(S(f_1)) \) is greater than that over a point in the vanishing triangle of \( f_1(S(f_1)) \).

If a generic fold homotopy \( F \) does not satisfy the above property, then we say that \( F \) crosses \( \Gamma_1 \) negatively at \( f_0 \).

3. Fold eversion between \( e \) and \( \tilde{e} \)

In this section, we concretely construct a fold eversion \( E : S^2 \times [-1, 1] \to \mathbb{R}^2 \) between \( e \) and \( \tilde{e} \) such that \( E \) is a generic fold homotopy.

To describe a variant of such a fold eversion \( E \), we describe a finite series of images of stable fold map, \( e_t(D_N^2) \) and \( e_t(D_S^2) \), through which the reader can imagine the smooth fold eversion. Note that for any \( t \in [-1, 1] \), \( e_t|T \) satisfies the conditions (2.1) and (2.2) and \( e_t \) is defined by \( e_t(x) = E(x, t) \).

The initial stable fold map of \( E \) is \( e = e_{-1} \) and see FIGURE 6.

\( \text{FIGURE 6} \)
The stable fold map \( e_{s_1} \), FIGURE 7, is obtained from \( e \) by crossing \( J^+ \) positively four times.

\( \text{FIGURE 7} \)
The stable fold map \( e_{s_2} \), FIGURE 8, is obtained from \( e_{s_1} \) by crossing \( J^+ \), \( J_1^- \) and \( T_1 \) positively twice, \( T_2 \) positively four times and \( J_2^- \) negatively twice.

\( \text{FIGURE 8} \)
The stable fold map \( e_{s_3} \), FIGURE 9, is obtained from \( e_{s_2} \) by crossing \( J^+ \) and \( T_1 \) positively twice and \( J_2^- \) negatively once.

\( \text{FIGURE 9} \)
The stable fold map \( e_{s_4} \), FIGURE 10, is obtained from \( e_{s_3} \) by crossing \( T_2 \) positively twice.

\( \text{FIGURE 10} \)
The stable fold map \( e_{s_5} \), FIGURE 11, is obtained from \( e_{s_4} \) by crossing \( J^+ \) positively twice, \( T_2 \) positively four times and \( J_2^- \) negatively twice.

\( \text{FIGURE 11} \)
The stable fold map \( e_{s_6} \), FIGURE 12, is obtained from \( e_{s_5} \) with the rotation of \( \pi /2 \). We see that \( e_{s_5}(D_N^2) = e_{s_6}(D_N^2) \) and \( e_{s_5}(D_S^2) = e_{s_6}(D_S^2) \) hold if we ignore the orientations on \( D_N^2 \) and \( D_S^2 \).

\( \text{FIGURE 12} \)

We obtain the terminal stable fold map, \( e_1 = \tilde{e} \), of \( E \) (FIGURE 13) from \( e_{s_6} \) by reversing the generic fold homotopy between \( e \) and \( e_{s_6} \) constructed in FIGURES 6–11.
Then, we have the desired fold eversion $E : S^2 \times [-1, 1] \to \mathbb{R}^2$ between $e$ and $\bar{e}$. In Figures 6–13, we omit the orientations on $e_t(D_N^2), e_t(D_S^2)$ and $\mathbb{R}^2$. Gray strips are the image of rectangles properly embedded in $D_N^2$ and $D_S^2$, respectively. We draw the gray strips so that they help the readers to understand how to extend $e_t|\partial T$ to $e_t|D_N^2$ and $e_t|D_S^2$ ($t = \{-1, s_1, s_2, \ldots, s_6, 1\}$).

REFERENCES


E-mail address: (old) minomoto@math.kyushu-u.ac.jp, (current) minomoto@math.sci.hokudai.ac.jp
FIGURE 1. The stable fold map $s$

FIGURE 2. Milnor's examples $m_1$ and $m_2$
FIGURE 3. The stable fold map $e$

FIGURE 4. The stable fold map $\tilde{e}$
(1) if 0 corresponds to $J^+$

(2) if 0 corresponds to $J_1^-$

(3) if 0 corresponds to $J_2^-$

(4) if 0 corresponds to $T_1$

(5) if 0 corresponds to $T_2$

FIGURE 5
$e(D_N^2)$

$e(D_S^2)$

**Figure 6.** The stable fold map $e$

$e_{s_1}(D_N^2)$

$e_{s_1}(D_S^2)$

**Figure 7.** The stable fold map $e_{s_1}$
**FIGURE 8.** The stable fold map $e_{s_2}$

**FIGURE 9.** The stable fold map $e_{s_3}$
FIGURE 10. The stable fold map $e_{s_4}$

$e_{s_4}(D^2_N)$  
$e_{s_4}(D^2_S)$

FIGURE 11. The stable fold map $e_{s_5}$

$e_{s_5}(D^2_N)$  
$e_{s_5}(D^2_S)$
Figure 12. The stable fold map $e_{s_6}$

Figure 13. The stable fold map $\tilde{e}$