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Persistence of Termination for Overlay Term Rewriting Systems

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Abstract

A property P is called persistent if for any many-sorted term rewriting system $\mathcal{R}$, $\mathcal{R}$ has the property P if and only if term rewriting system $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, has the property P. In this paper, we show that termination is persistent for locally confluent overlay term rewriting systems and we give the example as application of this result. Furthermore we show that termination is persistent for right-linear overlay term rewriting systems and we obtain that termination is modular for right-linear overlay term rewriting systems.

1 Introduction

Term rewriting systems (TRSs) can offer both flexible computing and effective reasoning with equations. TRSs have been widely used as a model of functional and logic programming languages and as a basis of theorem provers, symbolic computation, algebraic specification and software verification [3, 4, 9, 12].

A rewrite system is called *terminating (strongly normalizing)* if there exists no infinite reduction sequence. In theory and practice, one of the most important properties of TRSs is the termination property. It is well-known that termination is undecidable for TRSs in general [3, 5]. Thus, several semi-automated techniques for proving termination of TRSs have been successfully developed [4, 5, 12]. In particular, simplification ordering, like recursive path ordering [5], are widely used.

Zantema [17] introduced the notion of *persistence* as follows: A property P is called persistent if for any many-sorted TRS $\mathcal{R}$, $\mathcal{R}$ has the property P if and only if unsorted TRS $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, has the property P. Zantema [17] showed that termination is persistent for TRSs without collapsing or duplicating rules. However termination is *not* persistent in general [17]. The basic counterexample from Toyama [15] leads to the following many-sorted TRS $\mathcal{R}$:

$$\mathcal{R} = \begin{cases} f(0,1,x) \to f(x,x,x) \\ g(y,z) \to y \\ g(y,z) \to z \end{cases}$$

where the set of sorts $\mathcal{S} = \{\alpha, \beta\}$ and the function symbols and variables are defined as follows: $f : \alpha \times \alpha \times \alpha \to \alpha$, $0 : \alpha$, $1 : \alpha$, $g : \beta \times \beta \to \beta$, $x : \alpha$, $y : \beta$, $z : \beta$.

The many-sorted TRS $\mathcal{R}$ is terminating. Let Θ be a sort elimination function. Then unsorted TRS $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, is not terminating.

$$\begin{array}{ll} & f(\underline{g(0,1)}, g(0,1), g(0,1)) \\ \to_{\Theta(\mathcal{R})} & f(0, \underline{g(0,1)}, g(0,1)) \\ \to_{\Theta(\mathcal{R})} & \underline{f(0, 1, g(0,1))} \\ \to_{\Theta(\mathcal{R})} & \cdots \end{array}$$

is an infinite reduction in $\Theta(\mathcal{R})$. In each step the contracted *redex* is underlined. Aoto and Toyama showed the persistence of confluence [1]. Ohsaki and Middeldorp [13] studied the persistence of termination, acyclicity and non-loopingness on equational many-sorted TRSs. Aoto proved that the persistence of termination for TRSs in which all variables are of the same sort [2]. Furthermore we showed that the persistence of termination for non-overlapping TRSs [8].

In this paper, we show the persistence of termination for locally confluent overlay TRSs and we give the example as application of this result. Zantema's result can *not* be applied to our example. Furthermore we show that termination is persistent for right-linear overlay term rewriting systems and we obtain that termination is modular for right-linear overlay term rewriting systems.

In section 2, many-sorted TRS is formulated and well-sortedness is characterized in section 3. First, we show the persistence of local confluence and strong innermost normalization. Next, we show the persistence of termination for locally confluent overlay TRSs and we give the example as an application of this result in section 4. We show the persistence of termination for right-linear overlay TRSs in section 5. Furthermore, we obtain the persistence of completeness for right-linear overlay TRSs. In section 6, we show the modularity of termination for right-linear overlay TRSs. Furthermore, we obtain the modularity of completeness for right-linear overlay TRSs.

2 Preliminaries

We mainly follow basic definitions in the literature [1, 9].

2.1 Sorted term rewriting systems

In this subsection, we introduce the basic notions of sorted term rewriting systems. Usual term rewriting systems [3] are considered as special cases of sorted term rewriting systems.

Let $\mathcal{S}$ be a set of *sorts* and $\mathcal{V}$ be a set of countably infinite *sorted variables*. We assume that there are countably infinite variables of sort α for each sort $\alpha \in \mathcal{S}$. Let $\mathcal{F}$ be a set of sorted function symbols. We assume that each sorted function symbol $f \in \mathcal{F}$ is given with the sorts of its arguments and the sort of its value, all of which are included in $\mathcal{S}$. We write $f{:}\alpha_1 \times \ldots \times \alpha_n \to \beta$ if and only if f takes n arguments of sorts $\alpha_1,\ldots,\alpha_n$ respectively to a value of sort β. Function symbol of with no arguments is *constant*.

The set $\mathcal{T}(\mathcal{F},\mathcal{V}) = \cup_{\alpha\in\mathcal{S}} \mathcal{T}(\mathcal{F},\mathcal{V})^\alpha$ of all *sorted terms* built from $\mathcal{F}$ and $\mathcal{V}$ is defined as follows: (1) $\mathcal{V}^\alpha \subseteq \mathcal{T}(\mathcal{F},\mathcal{V})^\alpha$, (2) $f{:}\alpha_1 \times \ldots \times \alpha_n \to \alpha$, $t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})^{\alpha_i}$ $(i = 1,\ldots,n)$ then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})^\alpha$. Here $\mathcal{T}(\mathcal{F},\mathcal{V})^\alpha$ denotes the set of all sorted terms of sort α. A sorted term t is *linear* if t does not contain multiple occurrences of the same variable. We write $t : \alpha$ if t is of sort α. $\mathcal{V}(t)$ denotes the set of all variables in t. $\mathcal{T}(\mathcal{F},\mathcal{V})^\alpha$ and $\mathcal{T}(\mathcal{F},\mathcal{V})$ are abbreviated as $\mathcal{T}^\alpha$ and $\mathcal{T}$, respectively. Let $\Box^\alpha$ be a special constant (*hole*) of sort α. Elements of $\mathcal{T}(\mathcal{F} \cup \{\Box^\alpha \mid \alpha \in \mathcal{S}\},\mathcal{V})$ are called *contexts* over $\mathcal{T}(\mathcal{F},\mathcal{V})$. We write $C{:}\alpha_1 \times \ldots \times \alpha_n \to \alpha$ if and only if the sort of context C is α and it has n holes $\Box^{\alpha_1},\ldots,\Box^{\alpha_n}$. If $C{:}\alpha_1 \times \ldots \times \alpha_n \to \alpha$ and $t_i{:}\alpha_i$ $(i = 1,\ldots,n)$ then $C[t_1,\ldots,t_n]$ denotes the term obtained from C by replacing holes with $t_1,\ldots,t_n$ from left to right. A context that contains precisely one hole is denoted by $C[\,]$. A term t is said to be a ***subterm*** of s if and only if $s = C[t]$ for some context C. A ***substitution*** θ is a mapping from $\mathcal{V}$ to $\mathcal{T}$ such that $x \in \mathcal{V}^\alpha$ implies $\theta(x) \in \mathcal{T}^\alpha$. A substitutions over terms is defined as a homomorphic extension. $\theta(t)$ is usually written as $t\theta$. A *sorted rewrite rule* on $\mathcal{T}$ is a pair $l \to r$ such that $l \notin \mathcal{V}$, $\mathcal{V}(r) \subseteq \mathcal{V}(l)$, sorted terms l and r have the same sort. A ***sorted term rewriting system*** (STRS, for short) is a pair $(\mathcal{F},\mathcal{R})$ where $\mathcal{F}$ is a set of sorted function symbols and $\mathcal{R}$ is a set of sorted rewrite rules on $\mathcal{T}$. $(\mathcal{F},\mathcal{R})$ is often abbreviated as $\mathcal{R}$ and in that case $\mathcal{F}$ is defined to be the set of function symbols that appear in $\mathcal{R}$.

Given a STRS $\mathcal{R}$, a sorted term s is *reduced* to a sorted term t ($s \to_{\mathcal{R}} t$, in symbol) when $s = C[l\theta]$ and $t = C[r\theta]$ for some rewrite rule $l \to r \in \mathcal{R}$, context C and substitution θ. We call $s \to_{\mathcal{R}} t$ a *rewrite step* or *reduction* from s to t of $\mathcal{R}$. $l\theta$ is called *redex* of this rewrite step. One can easily check that sorted terms s and t have the same sort whenever $s \to_{\mathcal{R}} t$.

The transitive reflexive closure of $\to_{\mathcal{R}}$ is denoted by $\to^*_{\mathcal{R}}$. Terms t_1 and t_2 are *joinable* if there exists some term t' such that $t_1 \to^*_{\mathcal{R}} t' \leftarrow^*_{\mathcal{R}} t_2$. A term t is *confluent* if for any terms t_1 and t_2, t_1 and t_2 are joinable whenever $t_1 \leftarrow^*_{\mathcal{R}} t \to^*_{\mathcal{R}} t_2$. A STRS $\mathcal{R}$ is confluent if every term is confluent to $\to_{\mathcal{R}}$. A term t is ***locally confluent*** if for any terms t_1 and t_2, t_1 and t_2 are joinable whenever $t_1 \leftarrow_{\mathcal{R}} t \to_{\mathcal{R}} t_2$. A STRS $\mathcal{R}$ is locally confluent if every term is locally confluent to $\to_{\mathcal{R}}$. A term t is a *normal form* if there is no term t' such that $t \to_{\mathcal{R}} t'$. A term t is *terminating (strongly normalizing)* (SN) if there is no infinite reduction sequence starting from term t. A STRS $\mathcal{R}$ is terminating if every term is terminating to $\to_{\mathcal{R}}$. A reduction step $s \to_{\mathcal{R}} t$ is *innermost*, denoted by $s \to_{i\mathcal{R}} t$, if no proper subterm of the contracted redex is itself a redex. An *innermost derivation* consists solely of innermost reduction steps. A STRS $\mathcal{R}$ is *strongly innermost normalizing* (SIN) if there is no infinite innermost derivation. A STRS $\mathcal{R}$ is *complete* if $\mathcal{R}$ is confluent and terminating. Every terminating STRS is strongly innermost normalizing. The set of all position of a term s is denoted by $O(s)$. The root position of a term is denoted by ϵ. A rewrite step of the form $s \to^{\epsilon} t$ is said to be a *root reduction*.

A rewrite rule $l \to r$ is a *collapsing* rule if r is a variable. A rewrite rule $l \to r$ is a *duplicating* rule if some variable has more occurrences in r than in l. Let $l_1 \to r_1$ and $l_2 \to r_2$ be renamed versions of rewrite rules in a STRS $\mathcal{R}$ such that they have no variables in common. Suppose $l_1 = C[t]$ with $t \notin \mathcal{V}$ such that t and l_2 are unifiable, i.e. $t\theta = l_2\theta$ for a most general unifier θ. The term $l_1\theta = C[l_2]\theta$ is subject to the rewrite steps $l_1\theta \to_{\mathcal{R}} r_1\theta$ and $l_1\theta \to_{\mathcal{R}} C[r_2]\theta$. Then the pair of reducts $\langle C[r_2]\theta, r_1\theta\rangle$ is called a ***critical pair*** of $\mathcal{R}$. A STRS $\mathcal{R}$ is *non-overlapping* if there is no critical pair between rules of $\mathcal{R}$. If every critical pair of a STRS $\mathcal{R}$ is obtained by an *overlay*, i.e. by overlapping left-hand sides of rules at root position, then $\mathcal{R}$ is said to be an *overlay system* (OS). A STRS $\mathcal{R}$ is *right-linear* (RL) if r is linear for any $l \to r \in \mathcal{R}$.

For properties P and Q of STRSs we write $P \wedge Q$ for denoting the conjunction of P and Q. By $P(\mathcal{R})$ we denote that the STRS $\mathcal{R}$ has the property P.

When $\mathcal{S} = \{*\}$, an STRS is called a term rewriting system (TRS, for short). Given an arbitrary STRS $\mathcal{R}$, by identifying each sort with $*$, we obviously obtain a TRS $\Theta(\mathcal{R})$ - called the underlying TRS of $\mathcal{R}$.

2.2 Sorting of term rewriting systems

Aoto and Toyama [1] defined the notion of *sort attachment* and formulated the notion of persistence using sort attachment. We mainly follow basic definitions in [1] in this subsection.

Let $\mathcal{F}$ and $\mathcal{V}$ be sets of function symbols and variables, respectively, on a trivial set $\{*\}$ of sorts. Terms built from this language are called *unsorted terms*. Let $\mathcal{S}$ be another set of sorts. A *sort attachment* τ on $\mathcal{S}$ is a mapping from $\mathcal{F} \cup \mathcal{V}$ to the set $\mathcal{S}^*$ of finite sequences of elements from $\mathcal{S}$ such that $\tau(x) \in \mathcal{S}$ for any $x \in \mathcal{V}$ and $\tau(f) \in \mathcal{S}^{n+1}$ for any n-ary function symbol $f \in \mathcal{F}$. We write $\tau(f) = \alpha_1 \times \ldots \times \alpha_n \to \beta$. Without loss of generality we assume that there are countably infinite variables x with $\tau(x) = \alpha$ for each $\alpha \in \mathcal{S}$. The set of τ-sorted function symbols from $\mathcal{F}$ is denoted by $\mathcal{F}^\tau$.

A term t is said to be *well-sorted* under τ with sort α if $t : \alpha$ is derivable in the following rules: (1) $\tau(x) = \alpha$ implies $x{:}\alpha$, (2) $\tau(f) = \alpha_1 \times \ldots \times \alpha_n \to \beta$, $t_1{:}\alpha_1, \ldots, t_n{:}\alpha_n$ imply $f(t_1, \ldots, t_n){:}\beta$.

The set of well-sorted terms under τ is denoted by $\mathcal{T}^\tau$, i.e. $\mathcal{T}^\tau = \{t \in \mathcal{T} | t : \alpha \text{ for some } \alpha \in \mathcal{S}\}$. Clearly, $\mathcal{T}^\tau \subseteq \mathcal{T}$. For a context C, we write $C{:}\alpha_1 \times \ldots \times \alpha_n \to \beta$ if $C[\square^{\alpha_1}, \ldots, \square^{\alpha_n}]{:}\beta$ is derivable by rules (1), (2) with an additional rule: (3)$\alpha \in \mathcal{S}$ implies $\square^\alpha : \alpha$.

Let $\mathcal{R}$ be a TRS. A sort attachment τ is said to be *consistent* with $\mathcal{R}$ if for any rewrite rule $l \to r \in \mathcal{R}$, l and r are well-sorted under τ with the same sort. Note that $\mathcal{R}^\tau$ acts on $\mathcal{T}^\tau$, i.e. well-sorted terms $s, t \in \mathcal{T}^\tau$ whenever $s \to_{\mathcal{R}^\tau} t$; and that for any $s, t \in \mathcal{T}^\tau$, $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}^\tau} t$.

From a given TRS $\mathcal{R}$ and a sort attachment τ consistent with $\mathcal{R}$, by regarding each function symbol f to be of sort $\tau(f)$ and each variable x to be of sort $\tau(x)$, we get a STRS $\mathcal{R}^\tau$ - called a STRS induced from $\mathcal{R}$ and τ.

Using the sort attachment, persistence can be alternatively formulated as follows. It is clear that definition of Zantema [17] and the following definition are equivalent.

Definition 2.1 A property P is called *persistent* if for any TRS $\mathcal{R}$ and any sort attachment τ that is consistent with $\mathcal{R}$ the following property holds:
$\mathcal{R}^\tau$ has the property $P \Leftrightarrow \mathcal{R}$ has the property P.

We consider the persistent property for TRSs using definition 2.1 in this paper instead of Zantema's definition. From now on, we assume that a set $\mathcal{S}$ of sorts, a TRS $\mathcal{R}$ are given. Then an attachment τ on $\mathcal{S}$ that is consistent with $\mathcal{R}$ is fixed.

3 Characterizations of well-sortedness

In this section we give a characterization of well-sortedness.

Definition 3.1 The *top sort* (under τ) of an unsorted term t is defined as follows:

- *top*$(t) = \tau(t)$ if $t \in \mathcal{V}$.
- *top*$(t) = \beta$ if $t = f(t_1, \ldots, t_n)$ with $\tau(f) = \alpha_1 \times \ldots \times \alpha_n \to \beta$.

Definition 3.2 Let $t = C[t_1, \ldots, t_n]$ $(n \geq 0)$ be an unsorted terms with $C[, \ldots,] \neq \square$. We write $t = C[\![t_1, \ldots, t_n]\!]$ if and only if

(1) $C{:}\alpha_1 \times \ldots \times \alpha_n \to \beta$ is a context that is well-sorted under τ.

(2) *top*$(t_i) \neq \alpha_i$ for $i = 1, \ldots, n$.

The $t_1, \ldots, t_n$ are said to be the *principal subterms* of t.

We consider the example of top sort, principal subterm of an unsorted term.

Example 3.3 Let $\mathcal{F} = \{f, g, h, A, B\}$, $\mathcal{S} = \{0, 1\}$ and $\tau = \{f : 0 \times 0 \to 1, g : 1 \to 0, h : 0 \times 1 \times 1 \to 1, A : 0, B : 0\}$.

We consider the unsorted term f $(g$ (A), h $(x, B, B))$.

- *top*$(f(g(A), h(x, B, B))) = 1$ because of $\tau(f) = 0 \times 0 \to 1$.
- $f(g(A), h(x, B, B)) = C[\![\, A,\ h(x, B, B)]\!]$ where $C[, \ldots,] = f(g(\square), \square)$. The principal subterms of $f(g(A), h(x, B, B))$ are A and $h(x, B, B)$.

4 Persistence of termination for locally confluent overlay TRSs

In this section we show the persistence of termination for locally confluent overlay TRSs. It is *main* *theorem* in this paper. First, we show the persistence of local confluence and strong innermost normalization. Next, we show the persistence of termination for locally confluent overlay TRSs. Furthermore we give the example as application of our main result.

Let $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ be terms. We write $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ if and only if for any $1 \leq i, j \leq n$, $s_i = s_j$ implies $t_i = t_j$. Moreover, we write $\langle s_1, \ldots, s_n \rangle \infty \langle t_1, \ldots, t_n \rangle$ if and only if $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ and $\langle t_1, \ldots, t_n \rangle \propto \langle s_1, \ldots, s_n \rangle$.

The following theorem was proved by Gramlich in [6].

Theorem 4.1 ([6]) *Let $\mathcal{R}$ be a locally confluent overlay TRS. Then, $\mathcal{R}$ is strongly innermost normalizing if and only if $\mathcal{R}$ is terminating.*

Lemma 4.2 *Let $\mathcal{R}^\tau$ be a locally confluent overlay STRS. Then, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}^\tau$ is terminating.*

Proof. For any well-sorted terms $s, t \in \mathcal{T}^\tau$, $s \to_{\mathcal{R}^\tau} t$ if and only if $s \to_{\mathcal{R}} t$. By theorem 4.1, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}^\tau$ is terminating. □

The following lemma was proved by Huet in [7].

Lemma 4.3 ([7]) *A TRS is locally confluent if and only if all its critical pairs are joinable.*

The persistence of local confluence was conjectured by Zantema [17]. However Zantema did not give the proof of it. So, we give the proof of persistence of local confluence.

Lemma 4.4 *Local confluence is a persistent property of TRSs.*

Proof. Let $\mathcal{R}$ be a TRS. We show that $\mathcal{R}^\tau$ is locally confluent if and only if $\mathcal{R}$ is locally confluent.

- (if)-part: For well-sorted terms $s, t \in \mathcal{T}^\tau$, $s \to_{\mathcal{R}^\tau} t$ if and only if $s \to_{\mathcal{R}} t$. Hence, every well-sorted term is locally confluent.

- (only if)-part: Suppose $\mathcal{R}^\tau$ is locally confluent. Hence every well-sorted term is locally confluent. The set of critical pairs of $\mathcal{R}$ consists of the critical pairs of $\mathcal{R}^\tau$, since for any rewrite rule $l \to r \in \mathcal{R}$, l and r are well-sorted under τ with same sort. According to the lemma 4.3 these pairs are joinable. Another application of the lemma 4.3 yields the local confluence of $\mathcal{R}$. □

Next, We give the proof of persistence of strong innermost normalization.

Lemma 4.5 *Strong innermost normalization is a persistent property of TRSs.*

Proof. Let $\mathcal{R}$ be a *TRS*. We show that $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}$ is strongly innermost normalizing.

- (if)-part: For well-sorted terms $s, t \in \mathcal{T}^\tau$, $s \to_{\mathcal{R}^\tau} t$ if and only if $s \to_{\mathcal{R}} t$. Hence, every well-sorted term is strongly innermost normalizing.

- (only if)-part: We will show by structural induction on t that every unsorted term t is strongly innermost normalizing with respect to $\mathcal{R}$. If t is a variable then t is strongly innermost normalizing. If t is a constant then t is strongly innermost normalizing by assumption. If $t = f(t_1, \ldots, t_n)$ then we have t_i is strongly innermost normalizing for all $i = 1, \ldots, n$ by induction hypothesis. Now, if t is irreducible with respect to $\mathcal{R}$ we are done. Otherwise, we know by induction hypothesis that for every $i \in \{1, \ldots, n\}$ every innermost derivation of t_i is eventually terminating. This means that every innermost derivation starting with t is either terminating or has the form $t = f(t_1, \ldots, t_n) \to_{i\mathcal{R}}^* f(t'_1, \ldots, t'_n) \to_{i\mathcal{R}}^\epsilon \cdots$ where $t'_1, \ldots, t'_n$ are all irreducible with respect to $\mathcal{R}$. In the letter case we can denote $f(t'_1, \ldots, t'_n) = C[s_1, \ldots, s_m]$ for some context $C[, \ldots,]: \alpha_1 \times \ldots \times \alpha_m \to \alpha$. Choose fresh variables $x_i \in \mathcal{V}^{\alpha_i}$ for $i = 1, \ldots, m$ such that $\langle s_1, \ldots, s_m \rangle \infty \langle x_1, \ldots, x_m \rangle$. If we replace every principal subterm $s_1, \ldots, s_m$ of t by fresh variable x_i for $i = 1, \ldots, m$, then $C[x_1, \ldots, x_m]$ is not strongly innermost normalizing. Hence, $\mathcal{R}^\tau$ is not strongly innermost normalizing. □

We obtain the main theorem in this paper from theorem 4.1, lemma 4.2 ,lemma 4.4 and lemma 4.5.

Theorem 4.6 *Termination is a persistent property of locally confluent overlay TRSs.*

Proof. Let $\mathcal{R}$ be a locally confluent overlay *TRS*. We have to show that $\mathcal{R}^\tau$ is terminating if and only if $\mathcal{R}$ is terminating. By theorem 4.1, $\mathcal{R}$ is strongly innermost normalizing if and only if $\mathcal{R}$ is terminating. By theorem 4.5, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}$ is strongly innermost normalizing. Hence, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}$ is terminating. Since TRS $\mathcal{R}$ is locally confluent overlay and lemma 4.4, STRS $\mathcal{R}^\tau$ is locally confluent overlay. By lemma 4.2, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}^\tau$ is terminating. Therefore, $\mathcal{R}^\tau$ is terminating if and only if $\mathcal{R}$ is terminating. □

Since non-overlapping is obviously persistent, we obtain the following corollary form above theorem.

Corollary 4.7 *Termination is a persistent property of non-overlapping TRSs.*

Example 4.8 We show that the following locally confluent overlay TRS $\mathcal{R}$ is terminating using theorem 4.6. To show the termination of the following TRS directly seems difficult from known results (E.g. recursive path ordering [5]). Also, we can *not* use the modularity results for composable systems [11, 12] and hierarchical combination and hierarchical combination with common subsystem [10, 12].

$$\mathcal{R} = \begin{cases} g(x,d(z,B)) \to g(x,d(z,A)) & (r1) \\ g(x,d(z,A)) \to x & (r2) \\ g(x,d(z,B)) \to x & (r3) \\ I(A,g(x,d(y,C))) & \\ \quad \to I(B,g(x,d(y,C))) & (r4) \\ d(z,D) \to e(z,z) & (r5) \end{cases}$$

Zantema's result [17] that termination is persistent for TRSs without collapsing or duplicating rules can *not* be applied, because the above TRS contains both collapsing rules $(r2)$ and $(r3)$ and duplicating rule $(r5)$. However, we can show the termination of the above TRS using our results in this paper.

Let $\mathcal{S} = \{0,1,2\}$. We give the following sort attachment τ.

$$\tau = \begin{cases} g : 1 \times 0 \to 1 \\ I : 0 \times 1 \to 2 \\ d : 0 \times 0 \to 0 \\ e : 0 \times 0 \to 0 \\ A : 0, B : 0, C : 0, D : 0 \end{cases}$$

Any well-sorted term in $\mathcal{T}^0$, $\mathcal{T}^1$ and $\mathcal{T}^2$ is terminating, i.e. any well-sorted term in $\mathcal{T}^\tau$ is terminating. We consider the following cases:

- $t \in \mathcal{T}^0$. Then $(r5)$ is the only applicable rule. A TRS $\{(r5)\}$ is terminating using recursive path ordering. Hence, t is terminating.
- $t \in \mathcal{T}^1$. Then $(r1)$, $(r2)$, $(r3)$ and $(r5)$ are the only applicable rules. A TRS $\{(r1)$, $(r2)$, $(r3)$, $(r5)\}$ is terminating using recursive path ordering. Hence, t is terminating.
- $t \in \mathcal{T}^2$. Then $(r1)$, $(r2)$, $(r3)$, $(r4)$ and $(r5)$ are the applicable rules. For any proper subterm s of t, $top(s) = 0$ or $top(s) = 1$. Since the above two cases, s is terminating. Since $top(t) = 2$, $(r4)$ is the only applicable rule to root position of term t. Hence, t is terminating.

Then, STRS $\mathcal{R}^\tau$ is terminating. Since $\mathcal{R}^\tau$ is locally confluent overlay TRS and theorem 4.6, TRS $\mathcal{R}$ is terminating.

Furthermore we obtain the persistence of completeness for locally confluent overlay TRSs.

The following theorem was given by Aoto and Toyama [1].

Theorem 4.9 ([1]) *Confluence is a persistent property of TRSs.*

Since a complete TRS is confluent and terminating, we obtain the following corollary from theorem 4.6 and theorem 4.9.

Corollary 4.10 *Completeness is a persistent property of locally confluent overlay TRSs.*

5 Persistence of termination for right-linear overlay TRSs

In this section we show the persistence of termination for right-linear overlay TRSs. It is *main theorem* in this paper.

The following theorem was proved by Sakai in [14].

Theorem 5.1 ([14]) *Let $\mathcal{R}$ be a right-linear overlay TRS. Then, $\mathcal{R}$ is strongly innermost normalizing if and only if $\mathcal{R}$ is terminating.*

Lemma 5.2 *Let $\mathcal{R}^\tau$ be a right-linear overlay STRS. Then, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}^\tau$ is terminating.*

Proof. For any well-sorted terms $s, t \in \mathcal{T}^\tau$, $s \to_{\mathcal{R}^\tau} t$ if and only if $s \to_{\mathcal{R}} t$. By theorem 5.1, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}^\tau$ is terminating. □

We obtain the main theorem in this paper from lemma 4.5, theorem 5.1 and lemma 5.2.

Theorem 5.3 *Termination is a persistent property of right-linear overlay TRSs.*

Proof. Let $\mathcal{R}$ be a right-linear overlay *TRS*. We have to show that $\mathcal{R}^\tau$ is terminating if and only if $\mathcal{R}$ is terminating. By theorem 5.1, $\mathcal{R}$ is strongly innermost normalizing if and only if $\mathcal{R}$ is terminating. By lemma 4.5, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}$ is strongly innermost normalizing. Hence, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}$ is terminating. By lemma 5.2, $\mathcal{R}^\tau$ is strongly innermost normalizing if and only if $\mathcal{R}^\tau$ is terminating. Therefore, $\mathcal{R}^\tau$ is terminating if and only if $\mathcal{R}$ is terminating. □

Furthermore we obtain the persistence of completeness for right-linear overlay TRSs.

Since a complete TRS is confluent and terminating, we obtain the following corollary from theorem 4.9 and theorem 5.3.

Corollary 5.4 *Completeness is a persistent property of right-linear overlay TRSs.*

6 Modularity

In this section, we show the modularity of termination for right-linear overlay TRSs. Persistence is closely to the notion of modularity [12].

$(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ are *disjoint* if they do not share function symbols, that is, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. A property P is *modular* for disjoint TRSs if, for any disjoint TRSs $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ that have the property P, their union $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$ also has the property P.

The following theorem was showed by Gramlich [6].

Theorem 6.1 ([6]) *Strongly innermost normalizing is a modular property of TRSs.*

We obtain the modularity of termination for right-linear overlay TRSs.

Theorem 6.2 *Termination is a modular property of right-linear overlay TRSs.*

Proof. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be disjoint, right-linear and overlay TRSs. Hence we have $(OS \wedge RL \wedge SN)(\mathcal{R}_1)$ and $(OS \wedge RL \wedge SN)(\mathcal{R}_2)$ implying $(OS \wedge RL \wedge SIN)(\mathcal{R}_1)$ and $(OS \wedge RL \wedge SIN)(\mathcal{R}_2)$ by theorem 5.1. Since both OS and RL are obviously modular, and SIN, too, by theorem 6.1 we conclude $(OS \wedge RL \wedge SIN)(\mathcal{R}_1 \cup \mathcal{R}_2)$. By applying theorem 5.1 we finally obtain $(OS \wedge RL \wedge SN)(\mathcal{R}_1 \cup \mathcal{R}_2)$ as desired. The other direction is straightforward. □

Furthermore we obtain the modularity of completeness for right-linear overlay TRSs.

The following theorem was given by Toyama [16].

Theorem 6.3 ([16]) *Confluence is a modular property of TRSs.*

Since a complete TRS is confluent and terminating, we obtain the following corollary from theorem 6.2 and theorem 6.3.

Corollary 6.4 *Completeness is a modular property of right-linear overlay TRSs.*

7 Conclusion

In this paper, we have discussed the persistence of termination for overlay TRSs. We have given our main results in the following.

First, we have shown the persistence of local confluence and strong innermost normalization. Next, we have shown the persistence of termination for locally confluent overlay TRSs and we have given the example as application of our main result. Furthermore we have shown the persistence of termination for right-linear overlay TRSs. Finally, we have obtained the modularity of termination for right-linear overlay TRSs.

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