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The main conjectures of non-commutative Iwasawa theory

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1 Introduction

The lecture reported on joint work with T. Fukaya, K. Kato, R. Sujatha, and O. Venjakob [1] on the formulation of the main conjectures of non-commutative Iwasawa theory. The general methods developed in [1] were inspired by the Heidelberg Habilitation Thesis of Venjakob [2].

Let G be a compact p-adic Lie gorup. We assume throughout that G has no element of order p, so that G has finite p-homological dimension. Let $\Lambda(G)$ denote the Iwasawa algebra of G. Let M be a finitely generated torsion $\Lambda(G)$-module. How can we define a characteristic element for M, and relate it to the Euler characteristic of M and its twists? In the classical case, when $G=\mathbb{Z}_{p}^{d}$ for some integer $d\geqslant 1$, such characteristic elements are defined via the structure theory of such modules up to pseudo-isomorphism. In fact, an analogue of the structure theorem is proven in [3] for all non-commutative G which are p-valued. However, in the non-commutative theory this does not seem to yield characteristic elements, both because reflexive ideals of $\Lambda(G)$ are not, in general, principal, and because pseudo-null modules with finite G-Euler characteristic do not, in general, have Euler characteristic 1[4]. The goal of [1] is to use localization techniques to find a way out of this dilemma for an important class of p-adic Lie groups G and a class of finitely generated torsion $L(G)$-modules which we optimistically hope includes all modules which occur in arithmetic at ordinary primes.

2 Algebraic theory

From now on, we assume that G satisfies the following:

Hypothesis on G *There is no element of order p in G, and G has a closed normal subgroup H such that $\Gamma = G/H$ is isomorphic to $\mathbb{Z}_p$.*

For example, if G is the Galois group of a p-adic Lie extension of a number field F which contains the cyclotomic $\mathbb{Z}_p$-extension of F, then G satisfies the second part of our hypotheses. We do not consider the category of all finitely generated torsion $\Lambda(G)$-modules, but rather the full subcategory $\mathfrak{M}_H(G)$ consisting of all finitely generated $\Lambda(G)$-modules M such that $M/M(p)$ is finitely generated over $\Lambda(H)$; here $M(p)$ denotes the p-primary submodule of M. In the special case when $H = 1$, $\mathfrak{M}_H(G)$ is indeed the category of all finitely generated torsion $\Lambda(G)$-modules. We define S to be the set of all f in $\Lambda(G)$ such that $\Lambda(G)/\Lambda(G)f$ is a finitely generated $\Lambda(H)$-module, and put

$$S^* = \bigcup_{n \geqslant 0} p^n S.$$

Theorem 2.1 *The set S^* is a multiplicatively closed left and right Ore set in $\Lambda(G)$, all of whose elements are non-zero divisors. A finitely generated $\Lambda(G)$-module M is S^*-torsion if and only if it belongs to the category $\mathfrak{M}_H(G)$.*

Thus S^* is a canonical Ore set in $\Lambda(G)$, and we write $\Lambda(G)_{S^*}$ for the localization of $\Lambda(G)$ at S^*. If R is any ring with unit, we write $K_m R$ ($m = 0, 1$) for the m-th K-group of R, and $R^\times$ for the group of units of R.

Theorem 2.2 *The natural map*

$$\Lambda(G)_{S^*}^\times \longrightarrow K_1(\Lambda(G)_{S^*})$$

is surjective.

Let $K_0(\mathfrak{M}_H(G))$ denote the Grothendieck group of the category $\mathfrak{M}_H(G)$. We recall that $\Lambda(G)$ has finite global dimension because G has no element of order p.

Theorem 2.3 *We have an exact sequence of localization*

$$K_1(\Lambda(G)) \longrightarrow K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_G} K_0(\mathfrak{M}_H(G)) \longrightarrow 0.$$

If $M \in \mathfrak{M}_H(G)$, we write $[M]$ for the class of M in $K_0(\mathfrak{M}_H(G))$. We then define a characteristic element of M to be any element ξ_M of $K_1(\Lambda(G)_{S^*})$ such that

$$\partial_G(\xi_M) = [M].$$

It is shown in [1] that ξ_M has all the good properties we would expect of characteristic elements. Most important amongst these for arithmetic applications is its behaviour under twisting. Let

$$\rho : G \longrightarrow GL_n(O)$$

be any continuous homomorphism, where O denotes the ring of integers of a finite extension of $\mathbb{Q}_p$. Of course, ρ induces a ring homomorphism

$$\rho : \Lambda(G) \longrightarrow M_n(O),$$

where $M_n(O)$ denotes the ring of $n \times n$ matrices with entries in O. If f is any element of $\Lambda(G)$, we define $f(\rho)$ to be the determinant of $\rho(f)$. Although it is far from obvious, it is shown in [1] that one can extend this notion to define $\xi_M(\rho)$ to be either ∞ or a. If M is any module in $\mathfrak{M}_H(G)$, we can also define

$$tw_\rho(M) = M \underset{\mathbf{Z}_p}{\otimes} O^n$$

where G acts on the second factor via ρ, and on the whole tensor product via the diagonal action. Again we have $tw_p(M)$ belongs to $\mathfrak{M}_H(G)$. We define

$$\chi(G, tw_\rho(M)) = \prod_{i \geqslant o} \sharp(H_i(G, tw_\rho(M)))^{(-1)^i},$$

saying that the Euler characteristic is finite if all the homology groups $H_i(G, tw_\rho(M))$ are finite. We write $\widehat{\rho}$ for the contragredient representation of ρ, i.e. $\widehat{\rho}(g) = \rho(g^{-1})^t$, where the $'t'$ denotes the transpose matrix.

Theorem 2.4 *Let $M \in \mathfrak{M}_H(G)$, and let ξ_M denote a characteristic element of M. For each continuous representation $\rho : G \to GL_n(\sigma)$ such that $\chi(G, tw_{\widehat{\rho}}(M))$ is finite, we have $\xi_M(\rho) \neq 0, \infty$ and*

$$\chi(G, tw_{\widehat{\rho}}(M)) = |\xi_M(G)|_p^{-m_\rho},$$

where m_ρ denotes the degree over $\mathbb{Q}_p$ of the quotient field of O.

3 Connexion with L-values

We only briefly discuss the main conjecture when E is an elliptic curve defined over $\mathbb{Q}$, $p \geqslant 5$ is a prime of good ordinary reduction, $F_\infty = \mathbb{Q}(E_{p^\infty})$, and G is the Galois group of F_∞ over $\mathbb{Q}$. Thus G has dimension 2 or 4 according as E does or does not have complex multiplication. Let $X(E/F_\infty)$ be the dual of the Selmer group of E over F_∞. Taking H to be the subgroup of G which fixes the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, the following conjecture (which can be proven in some cases) is made in [1].

Conjecture 3.1 *$X(E/F_\infty)$ belongs to $\mathfrak{M}_H(G)$.*

Now let ρ be a variable Artin representation of G, i.e. a representation which factors through a finite quotient of G. Let $L(\rho, s)$ denote the complex L-function of ρ, and $L(E, \rho, s)$ the complex L-function of E twisted by ρ. The L-functions $L(E, \rho, s)$ appear to have many interesting properties, but they appear to have been somewhat neglected by the experts on automorphic forms. The point $s = 1$ is critical for $L(E, \rho, s)$, and we assume in what follows the analytic continuation is known at $s = 1$. We fix a minimal Weierstrass equation for E over $\mathbb{Q}$, and let $\Omega_+(E)$ and $\Omega_-(E)$ denote generators of the groups of real and purely imaginary periods of the Néron differential of E. Let $d^+(\rho)$ (resp. $d^-(\rho)$) denote the dimension of the subspace of the realization of ρ which is fixed by complex multiplication (resp. on which complex conjugation acts like -1). A special case of Deligne's conjecture asserts that

$$\frac{L(E, \rho, 1)}{\Omega_+(E)^{d^+(\rho)}\Omega_-(E)^{d^-(\rho)}} \in \overline{\mathbb{Q}}.$$

Let p^{f_ρ} denote the p-part of the conductor of ρ. For each prime q, we let $P_q(\rho, X)$ be the polynomial such that the Euler factor of $L(\rho, s)$ at q is $P_q(\rho, q^{-s})^{-1}$. Also, since E is ordinary at p, we have

$$1 - a_p X + pX^2 = (1 - uX)(1 - wX),$$

where $u \in \mathbb{Z}_p^x$ and, as usual, $p + 1 - a_p$ is the number of points over $\mathbb{F}_p$ on the reduction of E module p. Let R be the finite set consisting of p and all primes q such that $\mathrm{ord}_q(j_E) < 0$. We write $L_R(E, \rho, s)$ for the complex L-function obtained by suppressing in $L(E, \rho, s)$ the Euler factors at the primes in R. The following two conjectures are made in [1].

Conjecture 3.2 *Assume that $p \geqslant 5$ and E has good ordinary reduction at p. Then there exists $\mathcal{L}_E$ in $K_1(\Lambda(G)_{S^*})$ such that, for all Artin representations ρ of G, we have $\mathcal{L}_E(\rho) \neq \infty$, and*

$$\mathcal{L}_E(\rho) = \frac{L_R(E,\rho,1)}{\Omega_+(E)^{d^+(\rho)}\Omega_-(E)^{d^-(\rho)}} \cdot e_p(\rho)u^{-f_\rho} \cdot \frac{P_p(\widehat{\rho},u^{-1})}{P_p(\rho,w^{-1})},$$

where $e_p(\rho)$ denotes the local ε-factor attached to ρ at p.

Conjecture 3.3 (The main conjecture) *Assume that $p \geqslant 5$, E has good ordinary reduction at p, and $X(E/F_\infty)$ belongs to $\mathfrak{M}_H(G)$. Granted Conjecture 2, the p-adic L-function $\mathcal{L}_E$ in $K_1(\Lambda(G)_{S^*})$ is a characteristic element of $X(E/F_\infty)$.*

Of course, when E does not admit complex multiplication, very little is known at present about Conjecture 3. However, when $E = X_1(11)$ and $p = 5$, some remarkable numerical calculations of T. Fisher and T. and V. Dokchitser provide fragmentary evidence in support of it.

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