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A spectral sequence for Iwasawa adjoints

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Abstract

This article is, apart from some small corrections, an unchanged version of a note written in 1994. It was circulated as a manuscript and quoted in several articles. Its aim was to provide a purely algebraic tool for the Iwasawa theory of arbitrary \( p \)-adic Lie groups, by constructing a spectral sequence which relates the generalized 'Iwasawa adjoints' \( E^i(M) \) of certain Iwasawa modules \( M \) to projective limits often encountered in the applications. A somewhat extended version will be published elsewhere.

Let \( k \) be a number field, fix a prime \( p \), and let \( k_\infty \) be some Galois extension of \( k \) such that \( G = \text{Gal}(k_\infty/k) \) is a \( p \)-adic Lie-group (e.g., \( G \cong \mathbb{Z}_p^r \) for some \( r \geq 1 \)). Let \( S \) be a finite set of primes containing all primes above \( p \) and \( \infty \), and all primes ramified in \( k_\infty/k \), and let \( k_S \) be the maximal \( S \)-ramified extension of \( k \); by assumption, \( k_\infty \subseteq k_S \). Let \( G_S = \text{Gal}(k_S/k) \) and \( G_{\infty,S} = \text{Gal}(k_S/k_\infty) \).

Let \( A \) be a discrete (left) \( G_S \)-module which is isomorphic to \( (\mathbb{Q}_p/\mathbb{Z}_p)^r \) for some \( r \geq 1 \) as an abelian group (e.g., \( A = \mathbb{Q}_p/\mathbb{Z}_p \) with trivial action, or \( A = E[p^\infty] \), the group of \( p \)-power torsion points of an elliptic curve \( E/k \) with good reduction outside \( S \)). We are not assuming that \( G_{\infty,S} \) acts trivially.

Let \( \Lambda = \mathbb{Z}_p[[G]] \) be the completed group ring. For a finitely generated \( \Lambda \)-module \( M \) we put

\[
E^i(M) = \text{Ext}_\Lambda^i(M, \Lambda).
\]

Hence \( E^0(M) = \text{Hom}_\Lambda(M, \Lambda) =: M^+ \) is just the \( \Lambda \)-dual of \( M \). This has a natural structure of a \( \Lambda \)-module, by letting \( \sigma \in G \) act via

\[
\sigma f(m) = f(m) \cdot \sigma^{-1}
\]

for \( f \in M^+, m \in M \). It is known that \( \Lambda \) is a noetherian ring (here we use that \( G \) is a \( p \)-adic Lie group), by results of Lazard [La]. Hence \( M^+ \) is a finitely generated \( \Lambda \)-module again (choose a projection \( \Lambda^r \rightarrow M; \) then we have an injection \( M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^* \)). By standard homological algebra, the \( E^i(M) \) are finitely generated \( \Lambda \)-modules for all \( i \geq 0 \).
Examples a) If \( \mathcal{G} = \mathbb{Z}_p \), then \( \Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]] \) is the classical Iwasawa algebra, and, for a \( \Lambda \)-torsion module \( M \), \( E^1(M) \) is isomorphic to the Iwasawa adjoint, which can be defined as

\[
\text{ad} \,(M) = \lim_{n \to \infty} (M/\alpha_n M)^\vee
\]

where \( (\alpha_n)_{n \in \mathbb{N}} \) is a sequence of elements in \( \Lambda \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( (\alpha_n) \) is prime to the support of \( M \) for every \( n \geq 1 \), and where

\[
N^\vee = \text{Hom} \,(N, \mathbb{Q}_p/\mathbb{Z}_p)
\]
is the Pontrjagin dual of a compact \( \mathbb{Z}_p \)-module \( N \). For any finitely generated \( \Lambda \)-module \( M \), \( E^1(M) \) is quasi-isomorphic to \( \text{Tor}_\Lambda(M)^\sim \), where \( \text{Tor}_\Lambda(M) \) is the \( \Lambda \)-torsion submodule of \( M \), and \( M^\sim \) is the "Iwasawa twist" of a \( \Lambda \)-module \( M \): the action of \( \gamma \in \mathcal{G} \) is changed to the action of \( \gamma^{-1} \).

b) If \( \mathcal{G} = \mathbb{Z}_p^f \), \( r \geq 1 \), then the \( E^i(M) \) are the standard groups considered in local duality. By duality for the ring \( \mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \ldots, x_r]] \), they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex.

The topic of this note is the following observation.

**Theorem 1** There is a spectral sequence

\[
E_2^{p,q} = E^p(H^q(G_{\infty,S}, A)^\vee) \Rightarrow \lim_{\kappa, m} H^{p+q}(G_{\kappa}', A[p^m]_\kappa, m).
\]

Here the limit runs through the natural numbers \( m \) and the finite extensions \( k'/k \) contained in \( k_\infty \), via the natural maps

\[
H^n(G_s(k'), A[p^{m+1}]) \to H^n(G_s(k'), A[p^m]) \quad (m \geq 1)
\]

and the corestrictions.

Before we give the proof, we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

\[
0 \to E^1(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}_1^f} \lim_{\kappa} H^1(G_s(k'), T_p A) \to (H^1(G_{\infty,S}, A)^\vee)^+ \to E^2(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}_2^f} \lim_{\kappa} H^2(G_s(k'), T_p A).
\]

To say more, we consider some assumptions.

**A.1** Assume that \( p > 2 \) or that \( k_\infty \) is totally imaginary. Then

\[
H^r(G_{\infty,S}, A) = 0 = \lim_{\kappa} H^r(G_s(k'), T_p A)
\]

for all \( r > 2 \).

**Corollary 2** Assume that \( H^2(G_{\infty,S}, A) = 0 \) (This is the so-called "weak Leopoldt conjecture" for \( A \). It is stated classically for \( A = \mathbb{Q}_p/\mathbb{Z}_p \), and there are precise conjectures when this is expected for representations coming from algebraic geometry, cf. [Ja2]).
Then the cokernel of $\text{inf}^2$ is

$$\ker (E^1(H^1(G_{\infty,s}, A)^\vee)) \rightarrow E^3(H^0(G_{\infty,s}, A)^\vee),$$

and there are isomorphisms

$$E^i(H^1(G_{\infty,s}, A)^\vee) \sim E^{i+2}(H^0(G_{\infty,s}, A)^\vee)$$

for $i \geq 2$.

Proof This comes from A.1 and the following picture of the spectral sequence

\[ \begin{array}{cccccc}
1 \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times \\
\end{array} \]

Corollary 3 Assume that $H^0(G_{\infty,s}, A) = 0$. Then

(a) $$\lim_{k'} H^1(G_{S}(k'), T_pA) \sim H^1(G_{\infty,s}, A)^\vee$$

(b) There is an exact sequence

$$0 \rightarrow E^1(H^1(G_{\infty,s}, A)^\vee) \rightarrow \lim_{k'} H^2(G_{S}(k'), T_pA)$$

$$\rightarrow (H^2(G_{\infty,s}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty,s}, A)^\vee) \rightarrow 0$$

(c) There are isomorphisms

$$E^i(H^2(G_{\infty,s}, A)^\vee) \sim E^{i+2}(H^1(G_{\infty,s}, A)^\vee)$$

for $i \geq 1$.

Proof In this case, the spectral sequence looks like

\[ \begin{array}{cccccc}
2 \times & \times & \times & \times & \times & \times \\
1 \times & \times & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]
Corollary 4 Assume that $\mathcal{G}$ is a $p$-adic Lie group of dimension 1 (equivalently: an open subgroup is $\cong \mathbb{Z}_p$). Then $E^i(-) = 0$ for $i \geq 3$. Let

$$B = \text{im} \left( \inf^2 : E^2(H^0(G_{\infty}, s, A)^\vee) \to \varprojlim_{k'} H^2(G_{S}(k'), T_pA) \right)$$

Then $B$ is finite, and there is an exact sequence

$$0 \to E^1(H^1(G_{\infty}, A)^\vee) \to \varprojlim_{k'} H^2(G_{S}(k'), T_pA)/B \to (H^2(G_{\infty}, s, A)^\vee)^+ \to E^2(H^1(G_{\infty}, A)^\vee) \to 0,$$

and

$$E^1(H^2(G_{\infty}, s, A)^\vee) = 0 = E^2(H^2(G_{\infty}, s, A)^\vee),$$

i.e., $(H^2(G_{\infty}, s, A)^\vee$ is a projective $\Lambda$-module.

Proof Quite generally, for a $p$-adic Lie group $\mathcal{G}$ of dimension $n$ one has $vcd_p(\mathcal{G}) = n$ for the virtual cohomological $p$-dimension of $\mathcal{G}$, and hence $E^i(-) = 0$ for $i > n + 1$, cf. [Ja3]. The finiteness of $E^2(M)$ (for a finitely generated $\Lambda$-module $M$) in our case is well-known, cf. [Ja3]. The remaining claims follow from the following shape of the spectral sequence:

\begin{center}
\begin{tikzpicture}
\matrix (m) [row sep=4em, column sep=3em, text height=2.5ex, text depth=0.25ex]
{ 2 & \times & \times & \times & \times \cr
1 & \times & \times & \times & \times \cr
\times & \times & \times & \times & \times \cr
};
\end{tikzpicture}
\end{center}

Lemma 5 Assume that $\mathcal{G}$ is a $p$-adic Lie group of dimension $n$ (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^n$). Then $E^i(H^0(G_{\infty}, s, A)^\vee) = 0$ for $i \neq n, n + 1$.

(a) If $H^0(G_{\infty}, s, A)$ is divisible (e.g., if $G_{\infty,s}$ acts trivially on $A$), then

$$E^i(H^0(G_{\infty}, s, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n \\ \text{Hom} (D, H^0(G_{\infty}, s, A)) & \text{for } i = n, \end{cases}$$

where $D$ is the dualising module for $\mathcal{G}$ ($D = Q_p/\mathbb{Z}_p$ if $\mathcal{G} = \mathbb{Z}_p^n$).

(b) If $H^0(G_{\infty}, s, A)$ is finite, then

$$E^i(H^0(G_{\infty}, s, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n + 1 \\ \text{Hom} (H^0(G_{\infty}, s, A), D)^\vee & \text{for } i = n + 1 \end{cases}$$
Proof This is well-known, see [Ja3].

Corollary 6 Let $\mathcal{G}$ is a $p$-adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty,S}$ acts trivially on $A$, then there are exact sequences

$$0 \to \lim_{k'} H^1(G_S(k'), T_pA) \to (H^1(G_{\infty,S}, A)^\vee)^+$$

$$\to T_pA \overset{\text{inf}^2}{\to} \lim_{k'} H^2(G_S(k'), T_pA)$$

and

$$0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \lim_{k'} H^2(G_S(k'), T_pA) / \text{inf}^2$$

$$\to (H^1(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee) \to 0,$$

an isomorphism

$$E^1(H^2(G_{\infty,S}, A)^\vee) \cong E^3(H^1(G_{\infty,S}, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty,S}, A)^\vee) = 0 = E^3(H^2(G_{\infty,S}, A)^\vee).$$

Proof The spectral sequence looks like

$$\begin{array}{cccc}
\text{Corollary 7} & \text{Let } \mathcal{G} \text{ be a } p\text{-adic Lie group of dimension 2 (So } E^i(-) = 0 \text{ for } i \geq 4). \text{ If } H^0(G_{\infty,S}, A) \text{ is finite, then}
\lim_{k'} H^1(G_S(k'), T_pA) \cong (H^1(G_{\infty,S}, A)^\vee)^+ .
\end{array}$$

If

$$d_2^{1,1} : E^1(H^1(G_{\infty,S}, A)^\vee) \to E^3(H^0(G_{\infty,S}, A)^\vee)$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$0 \to \ker d_2^{1,1} \to \lim_{k'} H^2(G_S(k'), T_pA) \to$$

$$\to \ker(a_2^{0,2}) \cong (H^2(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee)) \to \coker d_2^{1,1} \to 0,$$
an isomorphism
\[ \xymatrix{ E^1(H^2(G_{\infty,s}, A)^\vee) \ar[r] & E^3(H^1(G_{\infty,s}, A)^\vee) } \]
and the vanishing
\[ E^2(H^2(G_{\infty,s}, A)^\vee) = 0 = E^3(H^2(G_{\infty,s}, A)^\vee) . \]

**Proof** The spectral sequence looks like

\[
\begin{array}{cccc}
2 & \times & \times & \times \\
1 & \times & \times & \times \\
0 & 0 & 0 & \times \\
1 & 2 & 3
\end{array}
\]

**Remark** In the situation of Corollary 5, one has an exact sequence up to finite modules:
\[
0 \to E^1(H^1(G_{\infty,s}, A)^\vee) \to \lim_{k'} H^2(G_s(k'), T_pA) \\
\to (H^2(G_{\infty,s}, A)^\vee)^+ \to E^2(H^1(G_{\infty,s}, A)^\vee) \to 0 .
\]

**Corollary 8** Let $G$ be a $p$-adic Lie group of dimension $> 2$. Then
\[ (H^1(G_{\infty,s}, A)^\vee)^+ \cong \lim_{k'} H^1(G_s(k'), T_pA) . \]

**Proof** The first three columns of the spectral sequence look like

\[
\begin{array}{cccc}
2 & \times & \times & \times \\
1 & \times & \times & \times \\
0 & 0 & 0 & \\
1 & 2
\end{array}
\]
We now turn to the proof of Theorem 1:

Let $G$ be any profinite group, let $H \leq G$ be a closed normal subgroup, and denote $G = G/H$. Let $M_G$ be the category of discrete $G$-modules which are torsion as abelian groups, and let $M_G$ be the category of inverse systems $(A_n)$ in $M_G$ as in [Ja1]. For any $A$ in $M_G$, one gets an inverse system $T_p A := (A[p^n])$, where the transition maps $A[p^{n+1}] \to A[p^n]$ are induced by multiplication with $p$ in $A$. For reasons explained later, denote by $H^m_{\text{cont}}(G, H; RT_p A)$ the value at $A$ of the $m$-th derived functor of the left exact functor 

$$F : A \mapsto \lim_{n} \lim_{U} H^0(U, A[p^n])$$

where $U$ runs through all open subgroups $U \subset G$ containing $H$, and the transition maps are the corestriction maps and these coming from $A[p^{n+1}] \to A[p^n]$, respectively. We can write $F$ as the composition of the functors 

$$T_p : M_G \to M_G^n$$

and 

$$H^0_{\text{cont}}(G, H; -) : M_G^n \to Ab$$

$$(A_n) \mapsto \lim_{n} \lim_{U} H^0(U, A_n)$$

where $Ab$ is the category of abelian groups. Because $T_p$ maps injectives to $H^0_{\text{cont}}(G, H; -)$-acyclics we get a spectral sequence 

$$E^{p,q}_2 = H^p_{\text{cont}}(G, H; R^q T_p A) \Rightarrow H^{p+q}_{\text{cont}}(G, H; RT_p A).$$

One has 

$$R^q T_p A = \begin{cases} (A/p^m A) & q = 1 \\ 0 & q > 1 \end{cases}$$

and hence short exact sequences 

$$0 \to H^n_{\text{cont}}(G, H; T_p A) \to H^n_{\text{cont}}(G, H; RT_p A) \to H^{n-1}_{\text{cont}}(G, H; R^1 T_p A) \to 0$$

explaining the notation for $R^n F$. In fact, $H^n_{\text{cont}}(G, H; RT_p A)$ is the hypercohomology with respect to $H^0_{\text{cont}}(G, H; -)$ of a complex $RT_p A$ in $M_G$ computing the $R^q T_p A$. If $A$ is $p$-divisible, then $R^q T_p A = 0$ for all $q > 0$, the spectral sequence degenerates and gives isomorphisms $H^n_{\text{cont}}(G, H; R T_p A) \cong H^n_{\text{cont}}(G, H; T_p A)$. There is another spectral sequence from deriving the inverse limit, viz. 

$$E^{p,q}_2 = R^p \lim_{\nu \in \mathcal{U}} H^q(U, A_n) \Rightarrow H^{p+q}_{\text{cont}}(G, H; (A_n))$$

If $G/H$ has a countable basis of neighbourhoods of identity, i.e., if there is a countable family $U_\nu$ of open subgroup, $H \leq U_\nu \leq G$, with $\bigcup U_\nu = H$, then $R^p \lim_{\nu \in \mathcal{U}} = 0$ for $p > 1$, and $R^1 \lim_{\nu \in \mathcal{U}}$ has the usual description ([Ja1]). If, in addition, all $H^q(U, A_n)$ are finite, then $R^1 \lim_{\nu \in \mathcal{U}} H^q(U, A_n) = 0$, and we get isomorphisms 

$$H^n_{\text{cont}}(G, H; (A_n)) = \lim_{\nu \in \mathcal{U}} H^n(U, A_n).$$
All this applies in the situation of the theorem, so that
\[
H^n_{\text{cont}}(G_S, G_{\infty,S}; RT_pA) = \lim_{n} \lim_{U} H^n(G_S(k'), A[p^n])
\]
where \(k'\) runs through the finite extensions \(k'/k\) contained in \(k_{\infty}\).

On the other hand, we can write \(F\) as the composition of the left exact functors
\[
H^0(H, -) : M_G \rightarrow M_G
\]
and
\[
M_G \rightarrow Ab
\]
\[
B \mapsto \text{Hom}_\Lambda(B^\vee, \Lambda)
\]
where \(\Lambda = \mathbb{Z}_p[[G]]\) is the completed \(\mathbb{Z}_p\)-group ring of \(G\) and \(B^\vee\) is the Pontrjagin dual of \(B\), which is a compact \(\Lambda\)-module. In fact, one has (cf. [Ja3] p. 179).
\[
\text{Hom}_\Lambda(B^\vee, \Lambda) = \lim_{\mathcal{U}} \text{Hom}_G(B^\vee, \mathbb{Z}_p[G/U])
\]
\[
= \lim_{n} \lim_{\mathcal{U}} \text{Hom}(H^0(U/H, B)^\vee, \mathbb{Z}/p^n)\mathbb{Z}
\]
\[
= \lim_{n} \lim_{\mathcal{U}} \text{Hom}(H^0(U/H, B[p^n])^\vee, \mathbb{Z}/p^n)\mathbb{Z}
\]
\[
= \lim_{\mathcal{U}} H^0(U/H, B[p^n])
\]
and hence
\[
\text{Hom}_\Lambda(H^0(H, A)^\vee, \Lambda) = \lim_{n} \lim_{\mathcal{U}} (H^0(U, A[p^n]) = F(A).
\]
Since taking Pontrjagin duals is an exact functor
\[
M_G \rightarrow (\text{compact } \Lambda\text{-modules})
\]
taking injectives to projectives, the derived functors of the functor \(B \mapsto \text{Hom}_\Lambda(B^\vee, \Lambda)\)
are the functors \(B \mapsto \text{Ext}^i_A(B^\vee, \Lambda) = E^i(B^\vee)\), and we get a spectral sequence
\[
E^2_{p,q} = E^p(H^q(H, A)^\vee) \Rightarrow R^{p+q}F(A) = H^{p+q}_{\text{cont}}(G, H; RT_pA).
\]
The theorem follows.

References


