On the field of definition for modularity of CM elliptic curves

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1 Introduction

Let $E$ be a CM elliptic curve defined over an algebraic number field $F \subseteq \mathbb{C}$ whose $\mathbb{Q}$-algebra of endomorphisms defined over $\overline{\mathbb{Q}}$, denoted by $\text{End}_{\overline{\mathbb{Q}}}(E)$, is isomorphic to an imaginary quadratic field $K \subseteq \mathbb{C}$. We take an integral ideal $m$ in $K$ and denote by $I_K(m)$ the group of fractional ideals in $K$ prime to $m$. We consider a homomorphism $\lambda : I_K(m) \rightarrow \mathbb{C}^\times$ such that (i) $\lambda((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \mod^\times m$; (ii) $\lambda$ is primitive, i.e. there is no proper divisor $n$ of $m$ such that $\lambda$ has a extension $\tilde{\lambda}$ to $I_K(n)$ with the property: $\tilde{\lambda}((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \mod^\times n$. Then we put

$$f_\lambda(z) := \sum_{a \in I_K(m) \atop a: \text{integral}} \lambda(a)e^{2\pi i N(a)z} \quad (z \in \mathfrak{H}, \text{ the complex upper plane}),$$

where $N(a)$ denotes the absolute norm of an ideal $a$. Let $-D$ be the discriminant of $K$ and put $N := DN(m)$. We define a Dirichlet character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ by

$$\overline{a} \mapsto \left(\frac{-D}{a}\right)^{\lambda((a))} \quad (a \in \mathbb{Z}, \ (a, \ N) = 1),$$

where if $a = p_1^{e_1} \cdots p_r^{e_r}$ is the factorization of $a$ into prime factors,

$$\left(\frac{-D}{a}\right) = \prod_{i=1}^r \left(\frac{-D}{p_i}\right)^{e_i}, \quad \left(\frac{-D}{p_i}\right) = \begin{cases} 1 & \text{if } p_i \text{ splits in } K/\mathbb{Q} \\ -1 & \text{if } p_i \text{ is inert in } K/\mathbb{Q} \end{cases}.$$

By Hecke-Shimura, we have the following:

Fact 1. $f_\lambda$ is a normalized newform of weight two on $\Gamma_1(N)$ and $\varepsilon$ is the Nebentypus of $f_\lambda$. 
By the Eichler-Shimura theory, for any normalized newform \( f \) of weight two on \( \Gamma_1(M) \), we can associate the abelian variety \( J_f \) defined over \( \mathbb{Q} \) which is a \( \mathbb{Q} \)-simple factor of \( J_1(M) \), the jacobian variety of the modular curve \( X_1(M) \). Shimura proved the following (see Proposition 1.6 and Remark 1.7 in [5]):

**Fact 2.** \( \text{Hom}_{\mathbb{Q}}(E, J_f) \neq \{0\} \) if and only if there exists an above \( \lambda \) such that \( f = f_{\lambda} \), where \( \text{Hom}_{\mathbb{Q}}(E, J_f) \) denotes the additive group of homomorphisms from \( E \) to \( J_f \) defined over \( \overline{\mathbb{Q}} \).

For any imaginary quadratic field \( K \), if we take an integral ideal \( m_0 \) in \( K \) such that
\[
\zeta \in K, \quad \zeta \text{ is a root of unity, } \zeta \equiv 1 \mod m_0 \implies \zeta = 1
\]
holds (we can always do so), there exists a homomorphism \( \lambda : I_K(m_0) \to \mathbb{C}^{\times} \) satisfying the condition (i). Replacing \( m_0 \) by the minimal divisor \( m \) of \( m_0 \) such that \( \lambda \) has an extension \( \tilde{\lambda} \) to \( I_K(m) \) and \( \tilde{\lambda} \) has also the property (i), we may assume that \( \lambda \) is primitive. Therefore we have

**Fact 3.** For any CM elliptic curve \( E \) defined over an algebraic number field \( F \), there exists a newform \( f \) such that a non-zero homomorphism \( \varphi : E \to J_f \) defined over \( \overline{\mathbb{Q}} \) exists, that is, \( E \) is modular over \( \overline{\mathbb{Q}} \).

In this paper we will consider the following questions.

**Question 1.** Let \( E/F \) be as above. Under what condition does there exist a newform \( f \) such that a non-zero homomorphism \( \varphi : E \to J_f \) defined over \( F \) exists, that is, when \( E \) is modular over \( F \)?

**Question 2.** Assume that \( E/F \) is modular over \( F \). Therefore there exists a newform \( f \) with \( \text{Hom}_F(E, J_f) \neq \{0\} \). Then, how large is \( \text{Hom}_F(E, J_f) \)? In other words, decide the multiplicity of \( E \) as \( F \)-simple factor of \( J_f \).

## 2 Preliminaries

Let \( E/F, K, \lambda : I_K(m) \to \mathbb{C}^{\times} \), and \( f = f_{\lambda} \) be as in the introduction. Let \( f = \sum_{m \geq 1} a_m q^m \) (\( q = e^{2\pi i\tau} \)) be the Fourier expansion at \( \infty \) and put \( H := \mathbb{Q}(a_m|m \geq 1) \) \((\subseteq \mathbb{C})\). Let \( n \) be the dimension of \( J_f \), then \( H \) is an algebraic number field with \([H : \mathbb{Q}] = n\). A \( \mathbb{Q} \)-algebra isomorphism \( \theta : H \cong \text{End}_Z^0(J_f) = \text{End}_Z(J_f) \otimes \mathbb{Z} \mathbb{Q} \) is defined by

\[
a_m \mapsto \text{the endomorphism of } J_f \text{ induced by the } m\text{-th Hecke operator w.r.t. } \Gamma_1(N)
\]

\((m = 1, 2, \ldots)\). In [3] Shimura proved that \( J_f \) is isogenous to \( E^n = E \times \cdots \times E \) (\( n \) terms) over \( \mathbb{Q} \), expressed by \( J_f \sim_{\mathbb{Q}} E^n \). So we have \( \text{End}_Z^0(J_f) \cong M_n(K) \), the algebra
of $n \times n$-matrices with entries in $K$. Let $Z$ be the center of $\text{End}_K^0(J_f)$. Then we have $Z \cong K$. We denote by $T$ the sub $\mathbb{Q}$-algebra of $\text{End}_K^0(J_f)$ generated by $Z$ and $\theta(H)$. Shimura used the following facts in the proof of Proposition 1.6 in [5] and we state them as a lemma without proof.

**Lemma 2.1.** (1) $Z \cap \theta(H) = \mathbb{Q}$. Especially this implies that $\dim_H T = 2$.
(2) $\text{End}_K^0(J_f) = T$.

Therefore, as for the structure of $T$, we have the possibility of the following two cases:

**Case 1:** $T$ is isomorphic to an algebraic number field with degree $2n$ (over $\mathbb{Q}$)
$$
\iff K \not\subseteq H;
$$

**Case 2:** $T \cong H \oplus H \iff K \subseteq H$.

Let $F' = \langle F, K \rangle$ be the subfield of $\mathbb{C}$ generated by $F$ and $K$. It is well known that $\text{End}_K^0(E) = \text{End}_{F'}(E) (\cong K)$. We put $\mathcal{M} := \text{Hom}_\mathbb{Q}(E, J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/F)$ over $F$ acts on $\mathcal{M}$ by the action on coefficients of homomorphisms. If we know the structure of $\mathcal{M}$ as Galois module, we will be able to answer Questions 1 and 2. Therefore our purpose in this paper is to determine the structure of $\mathcal{M}$ as $\text{Gal}(\overline{\mathbb{Q}}/F')$-module. On the other hand we have the following.

**Lemma 2.2.** $\text{Hom}_{F'}(E, J_f) \neq \{0\} \iff \text{Hom}_F(E, J_f) \neq \{0\}$.

By this lemma, for answer to Question 1, it is enough to study the structure of $\mathcal{M}$ as $\text{Gal}(\overline{\mathbb{Q}}/F')$-module. But, for answer to Question 2, this does not seem to be enough. Nevertheless, as we will see later, under assumption $\text{Hom}_{F'}(E, J_f) \neq \{0\}$ the structure of $\mathcal{M}$ as $\text{Gal}(\overline{\mathbb{Q}}/F)$-module can be easily recovered from that of $\mathcal{M}$ as $\text{Gal}(\overline{\mathbb{Q}}/F')$-module. Therefore, in the following we will study the $\text{Gal}(\overline{\mathbb{Q}}/F')$-module structure.

By composition of homomorphisms, $\mathcal{M}$ has the structure of left $T$- and right $K$-module:

$$
T = \text{End}_K^0(J_f) \hookrightarrow \mathcal{M} \hookrightarrow \text{End}_{F'}^0(E) \cong K.
$$

As $J_f \sim_{\mathbb{Q}} E^n$, we have

$$
\mathcal{M} \cong \text{Hom}_{\mathbb{Q}}(E, E^n) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K^n
$$

as $\mathbb{Q}$-vector space. In particular we have $\dim_{\mathbb{Q}} \mathcal{M} = n \times \dim_{\mathbb{Q}} K = 2n$. On the other hand $H \hookrightarrow \theta(H) \subseteq T$, we can view $\mathcal{M}$ as $H$-vector space. Since $[H : \mathbb{Q}] \times \dim_H \mathcal{M} = \dim_{\mathbb{Q}} \mathcal{M} = 2n$, we have $\dim_H \mathcal{M} = 2$.

**Proposition 2.3.** $\mathcal{M}$ is a free left $T$-module of rank 1.
Let $\ell$ be a prime number and put

\[
V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad V_\ell(J_f) := T_\ell(J_f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad \mathcal{M}_\ell := \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell
\]

where $T_\ell(E)$ and $T_\ell(J_f)$ are Tate modules. We can consider the following actions:

- $\text{Gal}(\overline{\mathbb{Q}}/F) \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by diagonal;
- $H \rightarrow T \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by the action on $\mathcal{M}$.

We define a homomorphism $\nu: \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E) \rightarrow V_\ell(J_f)$ by

\[
(\varphi \otimes a) \otimes x \mapsto a\varphi(x).
\]

**Proposition 2.4.** $\nu$ is an isomorphism of (left) $H \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell \text{-modules}$ and is also an isomorphism of (left) $T \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell \text{-modules}$, where $H \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell \text{-modules}$ (resp. $T \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell \text{-modules}$) denotes the group algebra of $\text{Gal}(\overline{\mathbb{Q}}/F)$ (resp. $\text{Gal}(\overline{\mathbb{Q}}/F')$) over $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ (resp. $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$).

### 3 The action of $\text{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$

We review the known results about the structure of $V_\ell(E)$ as $\text{Gal}(\overline{\mathbb{Q}}/F')$-module. By changing $\iota: K \rightarrow \text{End}_{F'}^0(E)$ if necessary, we may assume that the CM-type of $(E, \iota)$ is $(K; \{\text{id}\})$. Then there exists a lattice $\mathfrak{a}$ of $K$ such that the following commutative diagram holds:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathfrak{a} & \rightarrow & K_R & \rightarrow & K_R/\mathfrak{a} & \rightarrow & 0 \quad \text{(exact)} \\
& & \downarrow q & & \downarrow r & & \\
0 & \rightarrow & q(\mathfrak{a}) & \rightarrow & \mathbb{C} & \rightarrow & E(\mathbb{C}) & \rightarrow & 0 \quad \text{(exact)},
\end{array}
\]

where $K_R := K \otimes_{\mathbb{Q}} \mathbb{R}$ and $q(\mathfrak{a} \otimes x) = ax$. By the theory of complex multiplication, the following is well known (see Theorem 19.8, p. 134 in [6]).

**Theorem 3.1.** (1) Every point of $E(\mathbb{C})$ with finite order is $F'_{ab}$-rational, where $F'_{ab}$ denotes the maximal abelian extension of $F'$.

(2) There exists a unique homomorphism $\alpha_{E/F'}: F'^{\times}_A \rightarrow K^{\times}$ (where $F'^{\times}_A$ denotes the idele group of $F'$) such that

- $\text{Ker}(\alpha_{E/F'})$ is open in $F'^{\times}_A$;

- For any $x \in F'^{\times}_A$, $\alpha_{E/F'}(x)N_{F'/K}(x)^{-1}a = a$, where $N_{F'/K}$ is the norm map from $F'^{\times}_A$ to $K^{\times}_A$;
- For any $x \in F'_{A}^{\times}$, $\alpha_{E/F'}(x)\rho(\alpha_{E/F'}(x)) = N(il(x))$, where $\rho(v)$ is the complex conjugate of a complex number $v$ and $il(x)$ is the fractional ideal of $F'$ associated to an idele element $x$;

- For any $x \in F'_{A}^{\times}$ and $w \in K/a$, $r(x) = r(\alpha_{E/F'}(x)N_{F'/K}(x)^{-1}w)$, where $[x, F']$ is the element of $\text{Gal}(F'_{ab}/F')$ corresponding to $x$ by the reciprocity law of class field theory.

Since $V_{\ell}(E)$ is viewed as free $K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$-module of rank 1 by $\iota$, the action of $\text{Gal}(\bar{\mathbb{Q}}/F')$ on $V_{\ell}(E)$ determines the homomorphism

$$\vartheta : \text{Gal}(\bar{\mathbb{Q}}/F') \to (K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times}.$$ Then $\vartheta$ factors through the restriction map to $F'_{ab}$. So we denote by $\overline{\vartheta}$ the induced map from $\text{Gal}(F'_{ab}/F')$ to $(K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times}$ and by $\vartheta$ the composition of the reciprocity map for $F'$ and $\overline{\vartheta}$. Thus we have the following commutative diagram:

$$\text{Gal}(\bar{\mathbb{Q}}/F') \xrightarrow{\vartheta} (K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times} \xrightarrow{\text{restriction}} F'_{ab}^{\times} \xrightarrow{\overline{\vartheta}} (K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times}.$$ Then Theorem 3.1 implies the following:

**Corollary 3.2.** For any $x \in F_{A}^{\times}$, $\overline{\vartheta}(x) = (\alpha_{E/F'}(x)N_{F'/K}(x)^{-1})_{\ell}$, where $(\ )_{\ell}$ denotes the $\ell$-component.

By Proposition 2.3, the action of $\text{Gal}(\bar{\mathbb{Q}}/F')$ on $\mathcal{M}$ determines the homomorphism

$$\chi : \text{Gal}(\bar{\mathbb{Q}}/F') \to T^{\times}.$$ Let $\chi_{\ell}$ be the composition of $\chi$ and the canonical map $T^{\times} \to (T \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times}$, then $\chi_{\ell}$ corresponds to the action of $\text{Gal}(\bar{\mathbb{Q}}/F')$ on $\mathcal{M}_{\ell}$. In other words, taking a basis $\eta$ of $\mathcal{M}$ over $T$, we have $\sigma \eta = \chi_{\ell}(\sigma) \eta$ for any $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/F')$.

Firstly we consider Case 1. Since $K$ acts $T$-linearly on $\mathcal{M}$, we can take a $Q$-algebra isomorphism $\kappa : K \rightarrow \sim Z \subseteq T$ such that $\eta \circ \iota(a) = \kappa(a) \circ \eta$ for any $a \in K$, denoted by $\eta a = a \eta$ for short. We take a basis $v$ of $V_{\ell}(E)$ over $K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$. Then $\omega := \eta \otimes v$ becomes a free basis of $\mathcal{M}_{\ell} \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}} V_{\ell}(E)$ over $T \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$ and it holds that

$$\sigma \omega = (\sigma \eta \otimes \sigma v = (\chi_{\ell}(\sigma) \circ \eta) \otimes (\vartheta(\sigma)v) + (\chi_{\ell}(\sigma) \sigma \eta \circ (\iota \otimes 1)(\vartheta(\sigma))) \otimes v = (\chi_{\ell}(\sigma)\vartheta(\sigma) \eta \otimes v = \chi_{\ell}(\sigma)\vartheta(\sigma) \omega$$
for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$.

Next we consider Case 2. $\sqrt{-D} (\in K)$ acts $T$-linearly on $\mathcal{M}$, so there exists some $t \in T$ such that $\eta' \circ \iota(\sqrt{-D}) = t \circ \eta'$ for any $\eta' \in \mathcal{M}$. We will show that $t \in Z$ (one should note that in Case 2, $T$ has two $\mathbb{Q}$-subalgebras isomorphic to $K$, so it is not trivial that $t \in Z$). For any $\varphi \in \text{End}_{\mathbb{Q}}(J_f)$ and $\eta' \in \mathcal{M}$, we have

$$(\varphi \circ t) \circ \eta' = \varphi \circ (t \circ \eta') = (\varphi \circ \eta') \circ (\sqrt{-D}) = t \circ (\varphi \circ \eta') = (t \circ \varphi) \circ \eta',$$

therefore $t \circ \varphi = \varphi \circ t$ in $\text{End}_{\mathbb{Q}}(J_f)$, hence $t \in Z$. This concludes that similarly with Case 1, there exists a $\mathbb{Q}$-algebra isomorphism $\kappa : K \sim Z \subseteq T$ with the same property. Let $\gamma_1 : K \hookrightarrow H$ be the map induced by the inclusion $K \subseteq H$ and $\gamma_2 : K \hookrightarrow H$ be the other homomorphism. We define an isomorphism of $\mathbb{Q}$-algebras $\varepsilon : T \sim H \oplus H$ by

$$z \ (\in Z) \mapsto (\gamma_1(\kappa^{-1}(z)), \gamma_2(\kappa^{-1}(z))), \quad \theta(a) \ (\in \theta(H)) \mapsto (a, a).$$

For $k = 1, 2$, we set

$$\chi_{\ell}^{(k)} : \text{Gal}(\overline{\mathbb{Q}}/F') \rightarrow (T \otimes \mathbb{Q}_{\ell})^x \xrightarrow{\varepsilon \otimes 1} (H \otimes \mathbb{Q}_{\ell})^x \oplus (H \otimes \mathbb{Q}_{\ell})^x \xrightarrow{\text{projection to } k\text{-th component}} (H \otimes \mathbb{Q}_{\ell})^x.$$

These arguments imply the following:

**Proposition 3.3.** Let the notations be as above. We regard $K \otimes \mathbb{Q}_{\ell} \subseteq T \otimes \mathbb{Q}_{\ell}$ by injection $\kappa \otimes 1$.

(1) In Case 1, it holds that for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$,

$$\sigma \omega = \chi_{\ell}(\sigma) \theta(\sigma) \omega.$$

(2) In Case 2, identifying $T \otimes \mathbb{Q}_{\ell}$ with $(H \otimes \mathbb{Q}_{\ell})^{\otimes 2}$ by $\varepsilon \otimes 1$, it holds that for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$,

$$\sigma \omega = (\chi_{\ell}^{(1)}(\sigma) \gamma_1(\theta(\sigma))), \chi_{\ell}^{(2)}(\sigma) \gamma_2(\theta(\sigma))) \omega,$$

where we denote $\gamma_k \otimes 1 : K \otimes \mathbb{Q}_{\ell} \hookrightarrow H \otimes \mathbb{Q}_{\ell}$ by $\gamma_k \ (k = 1, 2)$ for simplicity.

## 4 On relation between Eichler-Shimura theory and complex multiplication theory about $J_f$

In this section we will describe a relation between $\lambda$ in $f = f_{\lambda}$ and the homomorphism corresponding to $\alpha_{n, \lambda'}$ in higher dimensional case. The content of this section is essentially stated in the proof of Proposition 1.6 in [5] without detailed proof. We present the results in a slightly different form to be convenient to our purpose.
Firstly we consider Case 1. Then $L := \langle K, H \rangle (\subseteq \mathbb{C})$ is a CM-field with $[L : \mathbb{Q}] = 2n$. We define an isomorphism of $\mathbb{Q}$-algebras $\iota' : L \sim T = \text{End}_K^0(J_f)$ by
\[
\alpha (\in K) \mapsto \kappa(\alpha) (\in Z), \quad x (\in H) \mapsto \theta(x).
\]
Then $(J_f, \iota')$ is an abelian variety with complex multiplication defined over $K$ in the sense of Shimura (see §19.7 in [6]). Since $\theta(H) \subseteq \text{End}_0^0(J_f)$, the characteristic polynomial of any element of $H$ acting on $H^0(J_f, \Omega^1_{J_f}) = H^0(J_f, \Omega^1_{\mathbb{Q} \otimes \mathbb{C}})$ has $\mathbb{Q}$-rational coefficients. Therefore, by Lemma 1 in [7] (p. 38), the representation of $H$ on $H^0(J_f, \Omega^1_{J_f})$ is equivalent to the regular representation of $H$ over $\mathbb{Q}$. It is also proved that $Z$ acts on $H^0(J_f, \Omega^1_{J_f})$ by scalar multiple. Let $(L, \{\varpi_1, \ldots, \varpi_n\})$ be the CM-type of $(J_f, \iota')$, then we have
\[
\{\varpi_1|H, \ldots, \varpi_n|H\} = \{\varpi : H \hookrightarrow \mathbb{C}\}, \quad \varpi_{i|K} = id_K \quad (i = 1, \ldots, n)
\]
by changing the identification of $K$ as subfield of $\mathbb{C}$ if necessary. Hence the reflex of $(L, \{\varpi_1, \ldots, \varpi_n\})$ is $(K, \{id_K\})$. Let $g' : K^\times_A \rightarrow L^\times_A$ be the canonical map induced from the inclusion $K \subseteq L$. Similarly with case of $E/F'$, the action of $\text{Gal}(\overline{\mathbb{Q}}/K)$ on $V_{\iota}(J_f)$ determines the homomorphism
\[
\delta : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow (L \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^\times
\]
and we define $\tilde{\delta} : K^\times_A \rightarrow (L \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^\times$ by the same manner as defining $\tilde{\theta}$. The theory of complex multiplication also implies the following:

**Corollary 4.1.** For any $x \in K^\times_A$, $\tilde{\delta}(x) = (\alpha_{J_f/K}(x)g'(x)^{-1})_{\iota}$, where $\alpha_{J_f/K} : K^\times_A \rightarrow L^\times$ is the homomorphism corresponding to $\alpha_{E/F'}$ in higher dimensional case.

Let $\{p_1, \ldots, p_s\}$ be the set of all bad primes of $J_f/K$. For every $p_k \ (1 \leq k \leq s)$, we take the least positive integer $t_k$ such that
\[
x \in K^\times_{p_k} \subseteq K^\times_A, \quad x - 1 \in p_k^{t_k} \quad \Rightarrow \quad \alpha_{J_f/K}(x) = 1.
\]
We set $n := p_1^{t_1} \cdots p_s^{t_s}$, $G_K(n) := \{x \in K^\times_A | x_{\infty} = 1, x_{p_k} = 1 \ (1 \leq k \leq s)\}$, $U_K := \{x \in K^\times_A | x_p \in \mathcal{O}_{K_p}^\times \text{ for any finite prime } p\}$, and $U_K(n) := G_K(n) \cap U_K$. We consider the canonical isomorphism $G_K(n)/U_K(n) \sim I_K(n)$ by which the class represented by $x \in G_K(n)$ is sent to $il(x) \in I_K(n)$. Since $U_K(n) \subseteq \text{Ker}(\alpha_{J_f/K})$, we obtain the homomorphism
\[
\tilde{\alpha}_{J_f/K} : I_K(n) \rightarrow L^\times
\]
induced from $\alpha_{J_f/K}$. By the two properties of $\alpha_{J_f/K}$: (i) $x \in K^\times_{\infty} = \mathbb{C}^\times \subseteq K^\times_A \Rightarrow \alpha_{J_f/K}(x) = 1$; (ii) $x \in K^\times \subseteq K^\times_A \Rightarrow \alpha_{J_f/K}(x) = g'(x) = x$, it holds that
\[
\alpha \in K^\times_A, \quad \alpha \equiv 1 \text{ mod } n \quad \Rightarrow \quad \tilde{\alpha}_{J_f/K}(\alpha) = \alpha.
\]
It is clear that $\alpha_{J_{f}/K} : I_{K}(n) \to L^{\times} \subseteq \mathbb{C}^{\times}$ is primitive.

**Proposition 4.2.** In Case 1, we have $\lambda = \alpha_{J_{f}/K}$ and $m = n$.

Next we investigate Case 2. Since $J_{f}$ is defined over $\mathbb{Q}$, $\rho_{|K}$ ($\in \text{Gal}(K/\mathbb{Q})$) acts on $T = \text{End}^{0}_{K}(J_{f})$. Identifying $T$ with $H \oplus H$ by $\epsilon$, this action corresponds to the automorphism of $H \oplus H$ defined by $(x, y) \mapsto (y, x)$. Let $\xi_{1}, \xi_{2}$ be the elements of $T$ which correspond to $(1, 0), (0, 1)$ respectively. We take a positive integer $r$ such that $r\xi_{k} \in \text{End}^{0}_{K}(J_{f}) (k = 1, 2)$ and set $\xi'_{k} := r\xi_{k}$. Then $C := \text{Im}(\xi'_{1})$ is an abelian subvariety of $J_{f}$ defined over $K$. Since $\rho\xi'_{1} = \xi'_{2}$, we have $\rho C = \text{Im}(\xi'_{2})$. So we can define an isogeny $\varphi : J_{f} \to C \times \rho C$ defined over $K$ by $x \mapsto (\xi'_{1}(x), \xi'_{2}(x))$ and this implies $J_{f} \sim_{K} C \times \rho C$.

**Lemma 4.3.** We have $J_{f} \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(C) \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(\rho C)$, where $R_{K/\mathbb{Q}}(C)$ denotes the Weil restriction from $K$ to $\mathbb{Q}$ of $C$.

To understand the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{\ell}(J_{f})$, it is sufficient to do so for that of $\text{Gal}(\overline{\mathbb{Q}}/K)$ on $V_{\ell}(C)$ by this lemma. Putting $R := \theta^{-1}(\text{End}_{\mathbb{Q}}(J_{f}))$, we define a ring homomorphism $\iota'' : R \to \text{End}_{K}(C)$ by

$$a \mapsto (C \ni x \mapsto (\theta(a))(x) \in C)$$

and denote $\iota'' \otimes 1 : H = R \otimes_{\mathbb{Q}} \mathbb{Q} \to \text{End}^{0}_{K}(C)$ by the same notation $\iota''$. In Case 2, $K \subseteq H$, so $H$ is a CM-field. Then $(C, \iota'')$ is an abelian variety with complex multiplication defined over $K$. Let $H_{0}$ be the maximal real subfield of $H$ and $(H, \{\tau_{1}, \ldots, \tau_{n}\})$ ($n_{1} := \frac{n}{2}$) be the CM-type of $(C, \iota'')$. Since $H_{0} \subseteq \text{End}^{0}_{K}(C)$, the characteristic polynomial of any element of $H_{0}$ acting on $H^{0}(C, \Omega^{1}_{/C})$ has $K$-rational coefficients. Since $H_{0}$ is totally real, its coefficients also lie in $\mathbb{R}$. So it has $\mathbb{Q}$-rational coefficients. It is also proved that $K \subseteq H$ acts on $H^{0}(C, \Omega^{1}_{/C})$ by scalar multiple because $\iota''(K)$ coincides with the center of $\text{End}^{0}_{K}(C) \cong M_{n_{1}}(K)$. Therefore we have

$$\{\tau_{1 \mid H_{0}}, \ldots, \tau_{n \mid H_{0}}\} = \{\tau \mid \tau : H_{0} \hookrightarrow \mathbb{R}\}, \quad \tau_{i \mid K} = id_{K} \quad (i = 1, \ldots, n_{1})$$

by changing the identification of $K$ as subfield of $\mathbb{Q}$ if necessary. Hence the reflex of $(H, \{\tau_{1}, \ldots, \tau_{n}\})$ is $(K, \{id_{K}\})$. Let $g'' : K^{x}_{A} \to H^{x}_{A}$ be the canonical map induced from $\gamma_{1} : K \hookrightarrow H$. Similary with Case 1, we have

$$\delta' : \text{Gal}(\overline{\mathbb{Q}}/K) \to (H \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{x}, \quad \tilde{\delta'} : K^{x}_{A} \to (H \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{x},$$

and the following:

**Corollary 4.4.** For any $x \in K^{x}_{A}$, $\tilde{\delta'}(x) = (\alpha_{c/K}(x)g''(x)^{-1})_{\ell}$. 
Let $n'$ be the one corresponding to $n$ in case of $C/K$. Then, as Case 1, we can define 
\[ \overline{\alpha_{C/K}} : I_K(n') \rightarrow H^\times. \]

**Proposition 4.5.** In Case 2, we have $\lambda = \overline{\alpha_{C/K}}$ and $m = n'$.

## 5 Main results

Let $\beta_{E/F'} : F'_{\lambda} \rightarrow C^\times$ be the Grössen-character of $E/F'$. (By definition, $\beta_{E/F'}(x) = (\alpha_{E/F'}(x)N_{F'/K}(x)^{-1})_{\infty}$.)

**Theorem 5.1.** Let $E$ be an elliptic curve with complex multiplication defined over an algebraic number field $F$ ($\subseteq C$) with $\text{End}_{\mathbb{Q}}(E) \cong K$ ($\subseteq C$). Put $F' := (F, K)$ ($\subseteq C$). Then the following three conditions are equivalent:

1. $E$ is modular over $F$.
2. There exists a Grössen-character $\gamma : K_{\lambda}^\times \rightarrow C^\times$ such that $\gamma \circ N_{F'/K} = \beta_{E/F'}$.
3. All the points of $E$ of finite order are rational over $\langle F', K_{ab} \rangle$. 

**Proof.** The equivalence of (2) and (3) is a special case of Theorem 4. p. 511 in [4].

We will prove that (1) implies (2). By assumption, there exists a normalized newform $f$ of weight two (obtained by some $\lambda : I_K(m) \rightarrow C^\times$ as $f = f_\lambda$) such that $\text{Hom}_{F'}(E, J_f) \neq \{0\}$. From $f$ we define $H$ as above. Firstly we consider Case 1. We define $\tilde{\chi}_L : F'_{\lambda}^\times \rightarrow (T \otimes_{\mathbb{Q}} \mathbb{Q}_L)^\times$ from $\chi_L$ by the same manner as defining $\tilde{\theta}$ from $\theta$ in Section 3. By the commutative diagram

\[
\begin{array}{ccc}
F'_{\lambda}^\times & \xrightarrow{\text{norm}} & K_{\lambda}^\times \\
\text{reciprocity law} \downarrow & & \downarrow \text{reciprocity law} \\
\text{Gal}(F_{ab}'/F') & \xrightarrow{\text{restriction}} & \text{Gal}(K_{ab}/K),
\end{array}
\]

Proposition 2.4, Corollary 3.2, Proposition 3.3, and Corollary 4.1, we have that
\[ \tilde{\chi}_L(x) = \alpha_{E/F'}(x)^{-1}\alpha_{J_f/K}(N_{F'/K}(x)) \text{ for any } x \in F'_{\lambda}^\times. \]

(We identify $L$ with $T$ by $\iota'$.) In Case 1, $T$ is a field, so we have
\[ \text{Hom}_{F'}(E, J_f) \neq \{0\} \iff \chi = 1 \iff \tilde{\chi}_L = 1 \iff \alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'}. \]

We note that Proposition 4.2 is rephrased to that the map
\[ G_K(m) \xrightarrow{u} I_K(m) \xrightarrow{\lambda} C^\times \]
can be continuously extended to $K_A^\times$ by the manner: any $x \in K^\times (\subseteq K_A^\times)$ is mapped to 1 and this extended map, denoted by $\overline{\lambda}$, coincides with $\beta_{J_f/K}$. Then it holds that

$$\alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} \iff \overline{\lambda} \circ N_{F'/K} = \beta_{E/F'},$$

so we can take $\overline{\lambda}$ as $\gamma$ in (2).

Next we consider Case 2. By the argument in the proof of Proposition 4.5, the action of $\text{Gal}(\overline{\mathbb{Q}}/F')$ on $V_\ell(J_f)$ corresponds to the homomorphism

$$\begin{array}{ccc}
\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{\phi_{\ell}} & GL_2(H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \\
\cup & & \cup \\
\text{Gal}(\overline{\mathbb{Q}}/F') & \longrightarrow & (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \oplus (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \\
\sigma & \mapsto & (\delta'(\sigma), \delta'(\rho\sigma\rho)).
\end{array}$$

By Proposition 2.4 and Proposition 3.3, we have that one of the following two statements holds:

(a) $\chi^{(1)}_{\ell}(\sigma) = \gamma_1(\theta(\sigma))^{-1}\delta'(\sigma)$, $\chi^{(2)}_{\ell}(\sigma) = \gamma_2(\theta(\sigma))^{-1}\delta'(\rho\sigma\rho)$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$;

(b) $\chi^{(1)}_{\ell}(\sigma) = \gamma_1(\theta(\sigma))^{-1}\delta'((\rho\sigma\rho)(\sigma))$, $\chi^{(2)}_{\ell}(\sigma) = \gamma_2(\theta(\sigma))^{-1}\delta'(\sigma)$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$.

We will prove that (b) is impossible. For this we assume that (b) holds. For $k = 1, 2$, we define $\overline{\chi^{(k)}_{\ell}}$ similarly with $\tilde{\chi}_{\ell}$. If $\sigma_{|F_{ab}} = [x, F']$ ($[x, F']$ denotes the image of $x \in F_A^\times$ by the reciprocity law of $F'$), then we have $\rho\sigma\rho_{|K_{ab}} = [\rho(N_{F'/K}(x)), K]$ by the class field theory. Therefore, for any $x \in F_A^\times$, we have

$$\begin{align*}
\overline{\chi^{(1)}_{\ell}}(x) &= \gamma_1(\tilde{\theta}(x)^{-1})\delta'(\rho(N_{F'/K}(x))) \\
&= \gamma_1((\rho\sigma\rho)(x))^{-1}\gamma_1((N_{F'/K}(x))_{\ell})\alpha_{C/K}(\rho(N_{F'/K}(x)))\gamma_1((\rho\sigma\rho)(x))^{-1}.
\end{align*}$$

Since $\gamma_1 \circ \rho = \gamma_2$, this is rephrased to that

$$\begin{align*}
\frac{\gamma_1((N_{F'/K}(x))_{\ell})}{\gamma_2((N_{F'/K}(x))_{\ell})} &= \frac{\overline{\chi^{(1)}_{\ell}}(x)\alpha_{E/F'}(x)}{\alpha_{C/K}(\rho(N_{F'/K}(x)))}.
\end{align*}$$

We can take a transcendental element $\pi$ of $\mathbb{Q}_{\ell}$ over $\mathbb{Q}$ and put $x_0 := 1 \otimes 1 + \sqrt{-D} \otimes \pi \in (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \subseteq (F' \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \subseteq F_A^\times$. Now we suppose that $\ell$ splits completely in $H$. Since $K \subseteq H$, we can view $K \subseteq \mathbb{Q}_\ell$. By the isomorphism

$$\left( \prod_{j: H^\times \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell} j \otimes 1 \oplus \prod_{r: H^\times \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell} r \otimes 1 \right) : H \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell \sim (\mathbb{Q}_\ell \oplus \cdots \oplus \mathbb{Q}_\ell) \oplus (\mathbb{Q}_\ell \oplus \cdots \oplus \mathbb{Q}_\ell),$$

the element

$$\frac{\gamma_1((N_{F'/K}(x_0))_{\ell})}{\gamma_2((N_{F'/K}(x_0))_{\ell})} = \frac{\gamma_1(x_0^d)}{\gamma_2(x_0^d)},$$

where $d$ is the degree of $F'/\mathbb{Q}$. Since $x_0 \in (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$, we have

$$\frac{\gamma_1((N_{F'/K}(x_0))_{\ell})}{\gamma_2((N_{F'/K}(x_0))_{\ell})} = \frac{\gamma_1(x_0^d)}{\gamma_2(x_0^d)}.$$
is mapped to
\[
\left(\frac{1 + \pi \sqrt{-D}}{1 - \pi \sqrt{-D}}, \ldots, \frac{1 + \pi \sqrt{-D}}{1 - \pi \sqrt{-D}}, \frac{1 - \pi \sqrt{-D}}{1 + \pi \sqrt{-D}}, \ldots, \frac{1 - \pi \sqrt{-D}}{1 + \pi \sqrt{-D}}\right),
\]
where \(d = [F' : K]\). Putting
\[
\xi := \frac{\chi_{\ell}^{(1)}(x_0)\alpha_{E/F'}(x_0)}{\alpha_{c/K}(\rho(N_{F'/K}(x_0)))} \in H^x \subseteq (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^x
\]
and taking \(j : H \hookrightarrow \mathbb{Q}_\ell\) with \(j(\sqrt{-D}) = \sqrt{-D}\), we have that
\[
\left(\frac{1 + \pi \sqrt{-D}}{1 - \pi \sqrt{-D}}\right)^d = j(\xi) \quad \text{in} \quad \mathfrak{G}.
\]
We note that \(j(\xi)\) is algebraic over \(\mathbb{Q}\). So we have that
\[
\pi = \frac{\sqrt{j(\xi)} - 1}{\sqrt{-D}(1 + \sqrt{j(\xi)})} \in \overline{\mathbb{Q}}.
\]
This is a contradiction. Hence we have proved that (a) holds.

We set
\[
\iota''' : H \longrightarrow \text{End}_K^0(\rho C), \quad a \mapsto \theta(\rho(a))|_{\rho C}.
\]
Then \((\rho C, \iota''')\) is an abelian variety with complex multiplication defined over \(K\) which has the same CM-type with \((C, \iota'')\). As case of \((C, \iota'')\), we have \(\alpha_{c/K} : K^x \longrightarrow H^x\).
Since \(\alpha_{c/K} = \rho \circ \alpha_{c/K} \circ \rho\), it holds that
\[
(a) \iff \chi_{\ell}^{(1)}(x) = \alpha_{E/F'}(x)^{-1}\alpha_{c/K}(N_{F'/K}(x)), \quad \overline{\chi_{\ell}^{(2)}}(x) = \rho(\alpha_{E/F'}(x)^{-1}\alpha_{c/K}(N_{F'/K}(x)))
\]
for any \(x \in F_{\mathbb{A}}^x\).

Therefore we have
\[
\text{Hom}_{F'}(E, J_f) \neq \{0\} \iff \chi^{(1)} = 1 \quad \text{or} \quad \chi^{(2)} = 1 \iff \chi_{\ell}^{(1)} = 1 \quad \text{or} \quad \chi_{\ell}^{(2)} = 1
\]
\[
\iff \alpha_{c/K} \circ N_{F'/K} = \alpha_{E/F'} \quad \text{or} \quad \alpha_{c/K} \circ N_{F'/K} = \alpha_{E/F'}.
\]
Set \(\lambda' : \rho \circ \lambda \circ \rho : I_K(\rho(m)) \longrightarrow \mathbb{C}^x\). As Case 1, we can construct a Grössen-character \(\lambda\) (resp. \(\overline{\lambda}\)) of \(K^x\) from \(\lambda\) (resp. \(\lambda'\)). Then we have
\[
\alpha_{c/K} \circ N_{F'/K} = \alpha_{E/F'} \quad \text{or} \quad \alpha_{c/K} \circ N_{F'/K} = \alpha_{E/F'} \iff \overline{\lambda} \circ N_{F'/K} = \beta_{E/F'} \quad \text{or} \quad \overline{\lambda} \circ N_{F'/K} = \beta_{E/F'}.
\]
Hence we can take \(\lambda\) or \(\overline{\lambda}\) as \(\gamma\) in (2).

Finally we will prove that (2) implies (1). By Lemma 2.2, it is sufficient to show that there exists a normalized newform \(f = f_\lambda\) of weight two constructed from some \(\lambda : I_K(m) \longrightarrow \mathbb{C}^x\) such that \(\text{Hom}_{F'}(E, J_f) \neq \{0\}\).
Claim. Let $\gamma$ be as in (2) and $n_0$ be the conductor of $\gamma$. As defining $\alpha_{J_f/K}$ from $\alpha_{J_f/K}$ in Section 4, we can also define $\widetilde{\gamma}: I_K(n_0) \rightarrow \mathbb{C}^\times$ from $\gamma$. Then it holds that for any $x \in K^\times$ s.t. $x \equiv 1 \mod n_0$,
\[
\widetilde{\gamma}(x) = x.
\]

By Claim, from $\widetilde{\gamma}$ we can construct a normalized newform $f = f_{\widetilde{\gamma}}$ of weight two. Then the arguments in the proof of the statement: (1) $\Rightarrow$ (2) imply that
\[
\gamma \circ N_{F'/K} = \beta_{E/F'} \iff \begin{cases} 
\alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} & \text{(if } K \nsubseteq H) \\
\alpha_{c/K} \circ N_{F'/K} = \alpha_{E/F'} & \text{(if } K \subseteq H) 
\end{cases}
\implies \text{Hom}_{F'}(E, J_f) \neq \{0\}.
\]
So we have proved that (2) $\Rightarrow$ (1).

Theorem 5.2. Let $E/F$, $K$, $F'$, and $\beta_{E/F'}$ be as in Theorem 5.1. Assume that the condition (2) in Theorem 5.1 holds. Let $m$ be the conductor of $\gamma$ and set
\[
f(z) = f_{\widetilde{\gamma}}(z) := \sum_{a \in I_K(m)} \widetilde{\gamma}(a)q^{N(a)} = \sum_{m \geq 1} a_m q^m \quad (q = e^{2\pi i z}).
\]

Put $H := \mathbb{Q}(a_m | m \geq 1)$. Then we have the followings:
1. For any normalized newform $g$ of weight two, $\text{Hom}_F(E, J_g) \neq \{0\}$ if and only if there exists some $\gamma$ as above such that $g = f_{\widetilde{\gamma}}$.
2. Case 1: $K \nsubseteq H$. Then we have
\[
J_f \sim_F E \times \cdots \times E \quad (n = \dim J_f = [H: \mathbb{Q}]).
\]

Case 2: $K \subseteq H$.
(a) If $\gamma = \rho \circ \gamma \circ \rho$ on $P := K^\times N_{F'/K}(F_{\Lambda}^\times)$, then we have
\[
J_f \sim_F E \times \cdots \times E.
\]
(b) If $\gamma \neq \rho \circ \gamma \circ \rho$ on $P$, then we have that $F = F'$ and there exists an abelian variety $A$ of dimension $\frac{n}{2}$ defined over $K$ such that
\[
J_f \sim_F E \times \cdots \times E \times A_{/F}, \quad \text{Hom}_F(E, A_{/F}) = \{0\}.
\]
References


