A topological L-function for a threefold

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1 Introduction

In recent days, analogies between the number theory and the theory of three-folds are discussed by many mathematicians([1][6][7]). It will be Mazur who first pointed out analogies between primes and knots in the standard three dimensional sphere. Morishita([7]) has investigated a similarity between the absolute Galois of $\mathbb{Q}$ and a link group. (A link group is defined to be the fundamental group of a complement of a link in the standard three sphere.) Moreover he has interpreted various symbols (eg. Hilbert, Rédei) from a topological point of view. For an example, he has shown one may consider the Hilbert symbol of two primes as their “linking number”.

In this report, we will study a similarity between the number theory and the theory of topological threefold from a viewpoint of a representation theory. Namely an L-function associated to a topological threefold will be discussed. Since our definition of an L-function will be based on one of a local system on a curve defined over a finite field (i.e. the Hasse-Weil’s congruent L-function), we will recall the definition the L-function in the arithmetic case.

2 A brief review of the Hasse-Weil’s congruent L-function

In what follows, for an object $Z$ over a finite field $\mathbb{F}_q$, its base extension to $\overline{\mathbb{F}}_q$ will be denoted by $\overline{Z}$. We fix a rational prime $l$ which is prime to $q$.

Let $C$ be a smooth curve over a finite field $\mathbb{F}_q$ and let $C \rightarrow C^*$ be its compactification. Suppose we are given a $\mathbb{Q}_l$-smooth sheaf $\mathcal{F}$ on $C$. Then the $q$-th Frobenius $\phi_q$ acts on $H^1(C^*, \overline{j_*\mathcal{F}})$ and the Hasse-Weil L-function is defined to be

$$L(C, \mathcal{F}, T) = \det[1 - \phi_q^* T H^1(C^*, \overline{j_*\mathcal{F}})].$$

It has

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• an functional equation,

• an Euler product.

Suppose \( \mathcal{F} \) is deduced from an abelian fibration. Namely let \( A \to C \) be an abelian fibration whose moduli is not a constant. We set \( \mathcal{F} = R^1 f_* \mathcal{Q}_l \) and

\[
L(A, s) = L(C, \mathcal{F}, q^{-s}).
\]

Then \( L(A, s) \) is an entire function and Artin and Tate ([11]) have given a detailed conjecture for a special value of \( L \)-function, which is a geometric analogue of the Birch and Swinnerton-Dyer conjecture. Their conjecture predicts that the order of \( L(A, s) \) at \( s = 1 \) should be equal to the rank of the Mordell-Weil group of the fibration. They have shown this is equivalent to the finiteness of \( l \)-primary part of the Brauer group of \( A \).

3 A definition of an \( L \)-function of a topological threefold

3.1 The definition

Let \( X \) be the complement of a knot \( K \) in the standard three dimensional sphere. By the Alexander duality, we know \( H_1(X, \mathbb{Z}) \simeq \mathbb{Z} \) and therefore it admits a infinite cyclic covering

\[
Y \xrightarrow{\pi} X.
\]

Let \( S \) be a minimal Seifert surface of \( K \). Then its inverse image \( \pi^{-1}(S) \) is a disjoint union \( \sqcup_{n \in \mathbb{Z}} S_n \) of copies of \( S \) indexed by integers. We assume that the genus of \( S \) is greater than or equal to two and that the fundamental groups of \( S_0 = S \) and \( Y \) are isomorphic. Let \( \mathcal{L}_X \) be a polarized local system on \( X \) and let \( \mathcal{L}_S \) be its restriction to \( S \). The deck transformation of the covering may be considered as a diffeomorphism of \( S \) and it is easy to see it lifts to an isomorphism \( \phi \) of the local system \( \mathcal{L}_S \).

Let us compare our situation to the arithmetic one. The covering \( Y \xrightarrow{\pi} X \) corresponds to \( \overline{C} \to C \) and the local system \( \mathcal{L}_X \) is an analogy of \( \mathcal{F} \). Let \( \rho_X \) be the representation of \( \pi_1(X) \) associated to \( \mathcal{L}_X \). Since \( \pi^* \mathcal{L}_X \) is the local system for the restriction of \( \rho_X \) to \( \pi_1(Y) \simeq \pi_1(S) \), we may identify it with \( \mathcal{L}_S \). Hence \( \mathcal{L}_S \) is an analogy of \( \tilde{F} \) and \( \tilde{\phi} \) corresponds to the Frobenius.

According to the observation above, we will make the following set up.

Let \( \tilde{X} \) be a compact smooth threefold which may have smooth boundaries. Suppose it has an infinite cyclic covering

\[
\tilde{Y} \xrightarrow{\tilde{\pi}} \tilde{X}
\]

which satisfies the following properties.
1. There is a smoothly embedded connected surface

\[ \mathcal{S} \hookrightarrow \mathcal{X} \]

whose genus is greater than or equal to 2 and the boundaries are contained in \( \partial \mathcal{X} \) via \( i \).

2. Let \( \mathcal{T} \) be the inverse image of \( \mathcal{S} \) by \( \pi \), which is a disjoint union of copies of \( \mathcal{S} \) indexed by integers:

\[ \mathcal{T} = \bigsqcup_{n \in \mathbb{Z}} \mathcal{S}_n, \quad \mathcal{S} \simeq \mathcal{S}_n. \]

Then the map \( i_0 \) induces an isomorphism

\[ \pi_1(\mathcal{S}, s_0) \simeq \pi_1(\mathcal{Y}, i_0(s_0)). \]

We will refer such an infinite cyclic covering to be of a surface type. The following notations will be used.

**Notations 3.1.**

1. \( X \) (resp. \( S, Y \)) is the interior of \( \overline{X} \) (resp. \( \overline{S}, \overline{Y} \)).

2. \( \Pi_S \) (resp. \( \Pi_Y, \Pi_X \)) is the fundamental group of \( S \) (resp. \( Y, X \)) with respect to the base point \( s_0 \) (resp. \( i_0(s_0), \pi(i_0(s_0)) \)).

3. \( \Pi_{Y/X} \) is the covering transformation group of \( Y \to X \).

Let \( \Phi \) be a deck transformation generating \( \Pi_{Y/X} \). Identifying \( S \) with \( S_0 \) (resp. \( S_1 \)) via \( i_0 \) (resp. \( i_1 \)), \( \Phi \) induces a diffeomorphism \( \phi \) on \( S \) by restriction. Since the genus of \( S \) is greater than or equal to 2, it is diffeomorphic to a quotient of the Poincaré upper half plane \( \mathbb{H}^2 \) by a discrete subgroup \( \Gamma \) of \( PSL_2(\mathbb{R}) \). Adding cusps \( \Sigma \) to the quotient, we get compactification \( S^* \). We will sometimes identify \( S \) with \( S^* \setminus \Sigma \).

**Remark 3.1.** Note that there is an exact sequence

\[ 1 \to \Pi_S \to \Pi_X \to \mathbb{Z} \to 1. \]

This is a geometric counterpart of the following situation in arithmetic geometry. Let \( C \) be a smooth curve defined over \( \mathbb{F}_q \) and let \( \overline{C} \) be its base extension to \( \overline{\mathbb{F}}_q \). Then their fundamental groups fit in the exact sequence

\[ 1 \to \pi_1(\overline{C}) \to \pi_1(C) \to \hat{\mathbb{Z}} \to 1. \]

Let \( F \) be a field of characteristic 0 and let \( L \) be a vector space over \( F \) of dimension \( 2g \) with a skew-symmetric nondegenerate pairing \( \alpha \). Suppose we are given a representation

\[ \Pi_X \xrightarrow{\rho_X} Aut(L, \alpha) \]

such that

\[ L^{\Pi_S} = 0, \quad (1) \]
Let $\rho_S$ be the restriction of $\rho_X$ to $\Pi_S$ and the local system associated to $\rho_X$ (resp. $\rho_S$) will be denoted by $\mathcal{L}_X$ (resp. $\mathcal{L}_S$). Then the diffeomorphism $\phi$ induces an isomorphism of a polarized local system:

\[
\begin{array}{ccc}
\mathcal{L}_S & \overset{\hat{\phi}}{\sim} & \mathcal{L}_S \\
S & \overset{\phi}{\sim} & S
\end{array}
\]

Fig. 2.2

Let $j$ be the open immersion of $S$ into $S^*$ and let $i$ be the inclusion of $\Sigma$ into $S^*$. Then $\hat{\phi}$ acts on $H^1(S^*, j_* \mathcal{L}_S)$, which is a geometric analogue of the Frobenius action. For a point $P$ in $\Sigma$, let $\Delta_P$ be a small disc centered at $P$ and we set $\Delta_P^* = \Delta_P \setminus \{P\}$. The parabolic cohomology $H^1_P$ is defined to be

\[H^1_P(S, \mathcal{L}_S) = \text{Ker}[H^1(S, \mathcal{L}_S) \to \oplus_{P \in \Sigma} H^1(\Delta_P^*, \mathcal{C}s)].\]

One can easily see that $H^1_P(S, \mathcal{L}_S)$ admits an action of $\hat{\phi}$ and it is isomorphic to $H^1(S^*, j_* \mathcal{L}_S)$ as a $F[\hat{\phi}]$-module. Also the nondegenerate skew-symmetric pairing $\alpha$ and the Poincaré duality induce a perfect pairing on $H^1_P(S, \mathcal{L}_S)$, which is invariant under the action of $\hat{\phi}$. Hence $H^1_P(S, \mathcal{L}_S)$ is a semisimple $F[\hat{\phi}]$-module and it is isomorphic to its dual as a $F[\hat{\phi}]$-module.

Now we define the topological $L$-function $L(X, \mathcal{L}_X)$ for the local system $\mathcal{L}_X$ to be

\[L(X, \mathcal{L}_X) = \det[1 - \hat{\phi}^* T| H^1_P(S, \mathcal{L}_S)].\]

Here $T$ is an indeterminate.

Let $M_\phi(S)$ be the mapping torus of $\phi$ and let $M_\phi(\mathcal{L}_S)$ be the local system on $X$ which is obtained by the same way as "mapping torus" from the isomorphism $\hat{\phi}^* \mathcal{L}_S \cong \mathcal{L}_S$. Note that by the definition we have

\[L(X, \mathcal{L}_X) = L(M_\phi(S), M_\phi(\mathcal{L}_S)).\]

### 3.2 Examples

Let $K$ be a knot embedded in the standard three dimensional sphere $S^3$ and let $N_K$ be its tubular neighborhood. Let $\bar{X}$ be the closure of the complement of
$N_K$ in $S^3$. Then $H_1(\bar{X}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ by the Alexander duality and $\bar{X}$ admits an infinite cyclic covering

$$\bar{Y} \rightarrow \bar{X}.$$

Let $X$ (resp. $Y$) be the interior of $\bar{X}$ (resp. $\bar{Y}$) (cf. Notations 3.1). Then the map induces an exact sequence

$$1 \rightarrow \Pi_Y \rightarrow \Pi_X \rightarrow \mathbb{Z} \rightarrow 1. \quad (2)$$

Let $\hat{S}$ be a minimal Seifert surface of $K$ and we set

$$\bar{S} = \hat{S} \cap \bar{X}.$$ 

It is known if $\Pi_Y$ is finitely generated, $S \rightarrow Y$ induces an isomorphism ([5])

$$\Pi_S \simeq \Pi_Y.$$ 

Moreover Murasugi has shown if the absolute value of the Alexander polynomial $\Delta_K(t)$ of $K$ at $t = 0$ is equal to 1, then $\Pi_Y$ is finitely generated.

**Fact 3.1.** ([4/IV. Proposition 5]) Suppose every closed incompressible surface in $X$ is boundary parallel. Then either

1. $X$ is Seifert fibred,

or

2. $X$ is hyperbolic. Namely there is the maximal order $O_F$ of an algebraic number field $F$ and a torsion free subgroup $\Gamma \subset PSL_2(O_F)$ such that $X$ is diffeomorphic to $\Gamma \backslash H^3$. Here after fixing an embedding $F \rightarrow C$, $\Gamma$ is regarded to be a subgroup of $PSL_2(C)$.

Now we assume that the infinite cyclic covering satisfies the following conditions.

**Condition 3.1.** 1. $\Pi_Y$ is finitely generated.

2. Either

(a) $X$ is Seifert fibred,

or

(b) there is the maximal order $O_F$ of an algebraic number field $F$ and a torsion free subgroup $\Gamma \subset SL_2(O_F)$ which freely acts on $H^3$ so that $X$ is diffeomorphic to $\Gamma \backslash H^3$. As before after fixing an embedding $F \rightarrow C$, $\Gamma$ is regarded to be a subgroup of $SL_2(C)$.

**Remark 3.2.** Professor Fujii kindly informed us that if $X$ is hyperbolic, then the Condition 3.1. 2 (b) is always satisfied.
Suppose $X$ satisfies 1 and 2(b) of Condition 3.1. Then we have the canonical representation
\[ \Pi_X \simeq \Gamma \rho_X^* SL_2(\mathbb{C}). \]
We set
\[ L = \mathbb{C}^{\oplus 2}, \]
and let $\alpha$ be the standard symplectic form on $L$. Namely for elements $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ of $L$, $\alpha(x,y)$ is defined as
\[ \alpha(x,y) = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}. \]
This invariant under the action of $\Pi_X$. The result of Neuwirth([5]) implies
\[ \Pi_S \simeq \Pi_Y \pi_1(i_{\mathbb{O}}), \]
and it is easy to see
\[ L^{\Pi_S} = 0. \]
Hence the conditions in §3.1 are satisfied.

Next suppose that $X$ satisfies 1 and 2(a) of Condition 3.1. Then under a mild condition, one can check its fundamental group has a linear representation
\[ \Pi_X \rho_X \simeq SL_2(\mathbb{C}) \]
such that
\[ L^{\Pi_S} = 0. \]
Details will be found in [9].

**Remark 3.3.** Even if we take the trivial representation, we can define an $L$-function for a knot complement. Note that this is nothing but the Alexander polynomial, which corresponds to the congruent zeta function of a curve. But contrary to the arithmetic case, as we have seen, we have a priori a two-dimensional irreducible linear representation of $\pi_1(X)$. This is one of the main reasons to consider the $L$-function.

4 Properties of a topological $L$-function

In the present section, we will list up basic properties of our topological $L$-function. Proofs of the statements will be found in [9].
4.1 A functional equation

Let $b(\mathcal{L}_S)$ be the dimension of $H^1_P(S, \mathcal{L}_S)$.

**Theorem 4.1.** *(The functional equation)*

$$L(X, \mathcal{L}_X)(T) = (-T)^{b(\mathcal{L}_S)}L(X, \mathcal{L}_X)(T^{-1}).$$

**Corollary 4.1.** Suppose $b(\mathcal{L}_S)$ is odd. Then $L(X, \mathcal{L}_X)(1)$ vanishes and in particular the dimension of $H^1_P(S, \mathcal{L}_S)\phi$ is positive.

4.2 A geometric analogue of Birch and Swinnerton-Dyer conjecture

In the present section, we will work with the holomorphic category.

Let $A^*$ be a smooth projective variety with a morphism

$$A^* \xrightarrow{\bar{\mu}} S^*$$

such that its restriction to $S$

$$A \xrightarrow{\mu} S$$

is a smooth fibration whose fibres are abelian varieties of dimension $g$. Moreover we assume $\bar{\mu}$ satisfies the following conditions.

**Condition 4.1.**

1. $A^*/S^*$ is the Neron model of $A/S$ and has a semistable reduction at each point $s \in \Sigma$.

2. $R^1\mu_* \mathbb{Q}$ is isomorphic to the local system $\mathcal{L}_S$.

3. $H^0(S^*, R^1\bar{\mu}_* \mathcal{O}_A^*) = 0$.

Suppose there is a commutative diagram

$$
\begin{array}{ccc}
A^* & \xrightarrow{\Phi} & A^* \\
\downarrow \bar{\mu} & \cong & \downarrow \bar{\mu} \\
S^* & \xrightarrow{\phi} & S^* \\
\end{array}
$$

Fig. 4.1
such that $\phi(\Sigma) = \Sigma$. Since $L_S = R^1 \mu_* Q$, this induces the diagram as Fig. 2.2. We define the Mordell-Weil group $MW_X(A)$ to be

$$MW_X(A) = A(S)^{\phi}$$

and its rank will be denoted by $r_X(A)$. Since the cycle map induces an imbedding

$$A(S) \otimes Q \hookrightarrow H^1_F(S, L_S),$$

$r_X(A)$ is less than or equal to the order of the topological $L$-function $L(X, L_X)$ at $T = 1$.

**Theorem 4.2.** Suppose $H^2(A^*, O_{A^*}) = 0$. Then $r_X(A)$ is equal to the order of the topological $L$-function $L(X, L_X)$ at $T = 1$.

We define the topological Brauer group $Br_{top}(A^*)$ to be

$$Br_{top}(A^*) = H^2(A^*, O_{A^*}^*).$$

Then the exponential sequence

$$0 \to Z \to O_{A^*} \to O_{A^*}^* \to 0$$

implies the exact sequence

$$H^2(A^*, Z) \to H^2(A^*, O_{A^*}) \to Br_{top}(A^*) \to H^3(A^*, Z).$$

Since $A^*$ is compact, both $H^2(A^*, Z)$ and $H^3(A^*, Z)$ are finitely generated abelian groups. Hence $Br_{top}(A^*)$ is finitely generated if and only if $H^2(A^*, O_{A^*})$ vanish since the latter is a complex vector space.

**Corollary 4.2.** Suppose $Br_{top}(A^*)$ is finitely generated. Then the rank of the Mordell-Weil group $r_X(A)$ is equal to the order of the topological $L$-function $L(X, L_X)$ at $T = 1$.

Note that the corollary above is a geometric analogue of the theorem of Artin and Tate. ([10][11])

### 4.3 An Euler product and an Euler system

Suppose the map $\phi$ in Fig. 2.2 satisfies the following condition.

**Condition 4.2.** There exists a diffeomorphism $\phi_0$ of $S$ such that

1. $\phi_0$ is homotopic to $\phi$,

and

2. every fixed point of $\phi_0^n$ is non-degenerate and is isolated for any positive integer $n$. 
Because of Condition 4.2(1), Fig. 2.2 may be replaced by:

\[ \phi_0 \]
\[ \mathcal{L}_S \rightarrow \mathcal{L}_S \]
\[ \phi_0 \]
\[ \mathcal{L}_S \rightarrow \mathcal{L}_S \]
\[ S \rightarrow S \]

Fig. 7.1

We prepare some notations. Let us fix a positive integer \( n \). The set of fixed points of \( \phi_0^n \) will be denoted by \( S^{\phi_0^n} \). We define \( \Phi_0(n) \) to be the orbit space of the action of \( \phi_0 \) on

\[ \{ s \in S | \phi_0^n(s) = s \quad \text{and} \quad \phi_0^m(s) \neq s \quad \text{for} \quad 1 \leq m \leq n - 1 \} \]

and we set

\[ \Phi_0 = \bigcup_{n=1}^{\infty} \Phi_0(n). \]

For an element \( \gamma \) of \( \Phi_0(n) \), we call the integer \( n \) its \textit{length} and we will denote it by \( l(\gamma) \). Let \( x \in S^{\phi_0^{l(\gamma)}} \) be a representative of \( \gamma \in \Phi_0 \). Then \( \phi_0^{l(\gamma)} \) defines an automorphism of the fibre of \( \mathcal{L}_S \otimes \mathbb{Q} \) at \( x \) and the polynomial

\[ \det[1 - \hat{\phi}_0^{l(\gamma)} T | (\mathcal{L}_S \otimes \mathbb{Q})_x] \]

is independent of the choice of \( x \), which will be written as \( P_{\gamma}(T) \).

Let \( V^P \) is the invariant subspace of \( L \otimes \mathbb{Q} \) under the action of \( \pi_1(\Delta_P^\ast) \). It is easy to see \( \bigoplus_{P \in \Sigma} V^P \) has an action of \( \hat{\phi} \). Now the Grothendieck-Lefshetz trace formula implies the following theorem.

\textbf{Theorem 4.3.} \textit{(Euler product formula)} Suppose the map \( \phi \) in Fig. 2.2 satisfies the Condition 4.2. Then

\[ L(X, L_X) = (\det[1 - \hat{\phi}^* T | \bigoplus_{P \in \Sigma} V^P])^{-1} \prod_{\gamma \in \Phi_0} P_{\gamma}(T^{n(\gamma)})^{-1}. \]

Our L-function has a \textit{Euler system}, which has been considered by Kolyvagin in the Iwasawa theory of an elliptic curve ([8]). Let \( \phi_0 \) be a diffeomorphism.
of $S$ satisfying the Condition 4.2 and let us fix a generator $t$ of $\Pi_{Y/X} \simeq \mathbb{Z}$. Then $Q[\Pi_{Y/X}]$ may be identified with $P = Q[t, t^{-1}]$ and defining the action of $t$ by $(\phi_{0}^{*})^{-1}$, the compact supported cohomology group $H_{c}^{1}(S, L_{S} \otimes Q)$ may be regarded as a $P$-module. In general, the Fitting ideal of a finitely generated $P$-module $M$ will be denoted by $\text{Fitt}_{P}(M)$. The following lemma directly follows from the definition of our $L$ function.

**Lemma 4.1.**

$$\text{Fitt}_{P}(H_{c}^{1}(S, L_{S} \otimes Q)) = (L_{c}(X, L_{X})),$$

where $L_{c}(X, L_{X})$ is defined to be

$$L_{c}(X, L_{X}) = \det[1 - \delta^{*} t|H_{c}^{1}(S, L_{S} \otimes Q)].$$

For $\gamma \in \Phi_{0}(n)$, let $O_{\gamma} \subset S$ be the corresponding orbit of $\phi_{0}$ and let $S_{\gamma}$ be its complement. The corestriction map

$$H_{c}^{1}(S_{\gamma}, L_{S} \otimes Q) \xrightarrow{\text{Cor}} H_{c}^{1}(S, L_{S} \otimes Q)$$

is defined to be the Poincaré dual of the restriction map

$$H^{1}(S, L_{S} \otimes Q) \xrightarrow{\text{Res}} H^{1}(S_{\gamma}, L_{S} \otimes Q).$$

Observe that both of them are homomorphism of $P$-modules. The Thom-Gysin exact sequence implies

$$0 \rightarrow H^{1}(S, L_{S} \otimes Q) \xrightarrow{\text{Res}} H^{1}(S_{\gamma}, L_{S} \otimes Q) \rightarrow \oplus_{x \in O_{\gamma}}(L_{S} \otimes Q)_{x} \rightarrow 0,$$

and let

$$0 \rightarrow \oplus_{x \in O_{\gamma}}(L_{S} \otimes Q)_{x} \rightarrow H_{1}^{1}(S_{\gamma}, L_{S} \otimes Q) \xrightarrow{\text{Cor}} H_{c}^{1}(S, L_{S} \otimes Q) \rightarrow 0 \quad (3)$$

be its dual sequence. The following lemma follows from the observation:

$$\text{Fitt}_{P}((\oplus_{x \in O_{\gamma}}(L_{S} \otimes Q)_{x}) = (P_{\gamma}(t^{l(\gamma)})).$$

**Lemma 4.2.**

$$\text{Fitt}_{P}(H_{c}^{1}(S_{\gamma}, L_{S} \otimes Q)) = (L_{c}(X, L_{X}) \cdot P_{\gamma}(t^{l(\gamma)})).$$

In general for an $N$-tuples of distinct elements $\{\gamma_{1}, \cdots, \gamma_{N}\}$ of $\Phi_{0}$, we set

$$S_{\gamma} = S \setminus (O_{\gamma_{1}} \cup \cdots \cup O_{\gamma_{N}}). \quad (4)$$

The induction on $N$ shows the following proposition.

**Proposition 4.1.**

$$\text{Fitt}_{P}(H_{c}^{1}(S_{\gamma}, L_{S} \otimes Q)) = (L_{c}(X, L_{X}) \cdot \prod_{i=1}^{N} P_{\gamma_{i}}(t^{l(\gamma_{i})})).$$
Definition 4.1. (An Euler system of a topological L-function) Let \( \gamma \) be the empty set or an \( N \)-tuples of distinct elements of \( \Phi_0 \). Suppose a finitely generated \( P \)-modules \( V_\gamma \) is given for such \( \gamma \). If \( \{V_\gamma\}_\gamma \) satisfy the following conditions, they will be referred as Euler system of the topological L-function.

1. 
   \[
   \text{Fitt}_P(V_\phi) = (L_c(X, \mathcal{L}_X)).
   \]

2. Suppose 
   \[
   \gamma' = \gamma \cup \{\gamma_{N+1}\}, \quad \gamma_{N+1} \notin \gamma.
   \]
   Then there is a surjection as \( P \)-modules
   \[
   V_{\gamma'} \to V_\gamma
   \]
   and their Fitting ideals satisfy the relation
   \[
   \text{Fitt}_P(V_{\gamma'}) = \text{Fitt}_P(V_\gamma) \cdot (P_{\gamma N+1}(t^{l(\gamma_{N+1})})).
   \]

We set 
\[
V_\phi = H_c^1(S, \mathcal{L}_S \otimes \mathbb{Q})
\]
and for an \( N \)-tuples of distinct elements \( \gamma \) of \( \Phi_0 \) we define 
\[
V_\gamma = H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbb{Q}).
\]

Then \( \{V_\gamma\}_\gamma \) is an Euler system by Proposition 4.1.

Next we will show how Kolyvagin's Euler system appears in our geometric situation. We assume any two of \( \{P_\gamma(t^{l(\gamma)})\}_{\gamma \in \Phi_0} \) are relatively prime. Let \( \gamma \) and \( \gamma' \) be as 2. of Definition 4.1. The same arguments of to obtain (3) shows
\[
0 \to \oplus_{x \in O_{\gamma_{N+1}}} (\mathcal{L}_S \otimes \mathbb{Q})_x \to H_c^1(S_{\gamma'}, \mathcal{L}_S \otimes \mathbb{Q}) \xrightarrow{Cor} H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbb{Q}) \to 0.
\]

Note that \( P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})}) \) annihilates \( \oplus_{x \in O_{\gamma_{N+1}}} (\mathcal{L}_S \otimes \mathbb{Q})_x \) and by the assumption its multiplication on \( H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbb{Q}) \) is an isomorphism. These observations imply the following lemma.

Lemma 4.3. Let us fix \( x_\gamma \in H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbb{Q}) \). If we take \( y_{\gamma'} \in H_c^1(S_{\gamma'}, \mathcal{L}_S \otimes \mathbb{Q}) \) so that 
\[
\text{Cor}(y_{\gamma'}) = x_\gamma.
\]
Then we have 
\[
\text{Cor}(P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})y_{\gamma'}) = P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})x_\gamma.
\]

Moreover \( P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})y_{\gamma'} \) is independent of the choice of \( y_{\gamma'} \).
Now we fix a non-zero element $c_\phi$ of $H_2^1(S, \mathcal{L}_S \otimes \mathbb{Q})$. For an $N$-tuples of distinct elements $\gamma$ of $\Phi_0$ we will inductively define an element $c_\gamma$ of $H_2^1(S_\gamma, \mathcal{L}_S \otimes \mathbb{Q})$. Let $\gamma' = \gamma \cup \{\gamma_{N+1}\}$ be as before. We take any $d_\gamma \in H_2^1(S_\gamma, \mathcal{L}_S \otimes \mathbb{Q})$ to be

$$Cor(d_\gamma) = c_\gamma,$$

and we set

$$c_{\gamma'} = P_{\gamma_{N+1}}(t^{d(\gamma_{N+1})})d_\gamma.$$

Then the system $\{c_\gamma\}_\gamma$ is well-defined by Lemma 4.3 and they satisfy

$$Cor(c_{\gamma'}) = P_{\gamma_{N+1}}(t^{d(\gamma_{N+1})})c_\gamma,$$

which is the same relation as Kolyvagin's Euler system ([8]). One may realize an Euler system is an another appearence of Euler product.

4.4 The Franz-Reidemeister torsion and a special value

We will briefly recall the theory of densities and the Franz-Reidemeister torsion ([2] [3] [9]). Throughout the subsection, let $F$ be equal to $\mathbb{R}$ or $\mathbb{C}$. Let $V$ be a vector space over $F$ of dimension $r > 0$ and let $\{v_1, \cdots, v_r\}$ be its basis. We set

$$(\Lambda^r V)^x = \{a \cdot v_1 \wedge \cdots \wedge v_r | a \in F^x\}$$

and

$$|\Lambda^r V| = (\Lambda^r V)^x / \{\pm 1\}.$$ Then $|\Lambda^r V|$ is isomorphic to $F^x / \{\pm 1\}$ and will be mentioned as the space of densities on $V$. Let

$$(\Lambda^r V)^x \overset{\pi}{\rightarrow} |\Lambda^r V|$$

be the canonical projection and the image $\pi(f)$ of $f \in (\Lambda^r V)^x$ will be denoted by $|f|$. For the 0 dimensional vector space $0$, we define

$$\Lambda^0 0 = F, \quad (\Lambda^0 0)^x = F^x$$

and

$$|\Lambda^0 0| = F^x / \{\pm 1\}.$$ Moreover for $f \in \Lambda^0 0 = F^x$, its image in $|\Lambda^0 0| = F^x / \{\pm 1\}$ will be denoted by $|f|$. If $F$ is $\mathbb{R}$, the canonical projection

$$(\Lambda^0 0)^x = \mathbb{R}^x \overset{\pi}{\rightarrow} |\Lambda^0 0| \simeq \mathbb{R}_{>0}$$

is nothing but the map of taking absolute value. In the followings, we always assume the 0 dimension vector space 0 has the density 1 $\in |\Lambda^0 0| = F^x / \{\pm 1\}$. Also we always assume every complex is bounded and consists of finite dimensional vector spaces over $F$. 
Definition 4.2. If a complex
\[ C' = [C^0 \rightarrow \cdots \rightarrow C^n] \]
has a density on each \( C^i \) and \( H^i \), we say the complex \( C' \) is given a density.

Remark 4.1. When \( C^i = H^i \), we assume \( H^i \) is given the same density as \( C^i \).

For a complex with a density
\[ C' = [C^0 \rightarrow \cdots \rightarrow C^n], \]
we can associate an element \( \tau_{FR}(C') \) of \( F^x/\{\pm 1\} \), which is called as the Franz-Reidemeister torsion (the FR-torsion for simplicity). Let \(|C^i|\) (resp. \(|H^i|\)) be the density on \( C^i \) (resp. \( H^i \)). Then one may intuitively think of \( \tau_{FR}(C') \) as
\[ \tau_{FR}(C') = \prod_{i=1}^{n} \left( \frac{|C^i|}{|H^i|} \right)^{(-1)^i}. \]

Let us take a finite triangulation of \( S \) which is preserved by \( \phi \). Then by a pararell transformation of the symplectic form \( \alpha \), we obtain a complex with a density \( C_\phi' \) such that its cohomology groups are isomorphic to \( H^* (M_\phi(S), M_{\overline{\phi}}(\mathcal{L}_S)) \). Using the previous observation:
\[ L(X, \mathcal{L}_X) = L(M_\phi(S), M_{\overline{\phi}}(\mathcal{L}_S)), \]
we can show the following theorem.

Theorem 4.4. Suppose \( \hat{\phi}^* - 1 \) is isomorphic on \( H^1_P(S, \mathcal{L}_S) \). Then we have
\[ |L(X, \mathcal{L}_X)(1)| = \tau_{FR}(C_\phi). \]

Remark 4.2. In general, we can show the following statement:
Let \( r \) be the dimension of \( \ker [(\hat{\phi}^* - 1)|H^1_P] \). Then we have
\[ \lim_{T \to 1} |(T - 1)^{-r} L(X, \mathcal{L}_X)(T)| = R((H^1_P)) \cdot \tau_{FR}(C_\phi). \]

Here \( R((H^1_P)) \) is the regulator of the local system. Note that this is quite similar to the formula which is predicted by the Birch and Swinnerton-Dyer conjecture.

References


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