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A topological L-function for a threefold

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1 Introduction

In recent days, analogies between the number theory and the theory of threefolds are discussed by many mathematicians([1][6][7]). It will be Mazur who first pointed out analogies between primes and knots in the standard three dimensional sphere. Morishita([7]) has investigeted a similarity between the absolute Galois of $\mathbf{Q}$ and a link group. (*A link group* is defined to be the fundamental group of a complement of a link in the standard three sphere.) Moreover he has interpreted various symbols (eg. Hilbert, Rédei) from a topological point of view. For an example, he has shown one may consider the Hilbert symbol of two primes as their "linking number".

In this report, we will study a similarity between the number theory and the theory of topological threefold from a viewpoint of a representation theory. Namely an L-function assosiated to a topological threefold will be discussed. Since our definition of an L-function will be based on one of a local system on a curve defined over a finite field (i.e. the Hasse-Weil's congruent L-function), we will recall the definition the L-function in the arithmetic case.

2 A brief review of the Hasse-Weil's congruent L-function

In what follows, for an object Z over a finite field $\mathbf{F}_q$, its base extension to $\bar{\mathbf{F}}_q$ will be denoted by $\bar{Z}$. We fix a rational prime l which is prime to q.

Let C be a smooth curve over a finite field $\mathbf{F}_q$ and let $C \overset{j}{\hookrightarrow} C^*$ be its compactification. Suppose we are given a $\mathbf{Q}_l$-smooth sheaf $\mathcal{F}$ on C. Then the q-th Frobenius ϕ_q acts on $H^1(\overline{C^*}, \overline{j_*\mathcal{F}})$ and the Hasse-Weil L-function is defined to be

$$L(C, \mathcal{F}, T) = \det[1 - \phi_q^* T | H^1(\overline{C^*}, \overline{j_*\mathcal{F}})].$$

It has

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- an functional equation,
- an Euler product.

Suppose $\mathcal{F}$ is deduced from an abelian fibration. Namely let $A \xrightarrow{f} C$ be an abelian fibration whose moduli is not a constant. We set $\mathcal{F} = R^1 f_* \mathbf{Q}_l$ and

$$L(A, s) = L(C, \mathcal{F}, q^{-s}).$$

Then $L(A, s)$ is an entire function and Artin and Tate ([11]) have given a detailed conjecture for a special value of L-function, which is a geometric analogue of the Birch and Swinnerton-Dyer conjecture. Their conjecture predicts that the order of $L(A, s)$ at $s = 1$ should be equal to the rank of the Mordell-Weil group of the fibration. They have shown this is equivalent to the finiteness of l-primary part of the Brauer group of A.

3 A definition of an L-function of a topological threefold

3.1 The definition

Let X be the complement of a knot K in the standard three dimensional sphere. By the Alexander duality, we know $H_1(X, \mathbf{Z}) \simeq \mathbf{Z}$ and therefore it admits a infinite cyclic covering

$$Y \xrightarrow{\pi} X.$$

Let S be a minimal Seifert surface of K. Then its inverse image $\pi^{-1}(S)$ is a disjoint union $\sqcup_{n \in \mathbf{Z}} S_n$ of copies of S indexed by integers. We assume that the genus of S is greater than or equal to two and that the fundamental groups of $S_0 = S$ and Y are isomorphic. Let $\mathcal{L}_X$ be a polarized local system on X and let $\mathcal{L}_S$ be its restriction to S. The deck transformation of the covering may be considered as a diffeomorphism of S and it is easy to see it lifts to an isomorphism $\hat{\phi}$ of the local system $\mathcal{L}_S$.

Let us compare our situation to the arithmetic one. The covering $Y \xrightarrow{\pi} X$ corresponds to $\bar{C} \to C$ and the local system $\mathcal{L}_X$ is an analogy of $\mathcal{F}$. Let ρ_X be the representation of $\pi_1(X)$ associated to $\mathcal{L}_X$. Since $\pi^* \mathcal{L}_X$ is the local system for the restriction of ρ_X to $\pi_1(Y) \simeq \pi_1(S)$, we may identify it with $\mathcal{L}_S$. Hence $\mathcal{L}_S$ is an analogy of $\bar{\mathcal{F}}$ and $\hat{\phi}$ corresponds to the Frobenius.

According to the observation above, we will make the following set up.

Let $\bar{X}$ be a compact smooth threefold which may have smooth boundaries. Suppose it has an infinite cyclic covering

$$\bar{Y} \xrightarrow{\pi} \bar{X}$$

which satisfies the following properties.

1. There is a smoothly embedded connected surface

$$\bar{S} \overset{i}{\hookrightarrow} \bar{X}$$

whose genus is greater than or equal to 2 and the boundaries are contained in $\partial\bar{X}$ via i.

2. Let $\bar{T}$ be the inverse image of $\bar{S}$ by π, which is a disjoint union of copies of $\bar{S}$ indexed by integers:

$$\bar{T} = \sqcup_{n\in\mathbf{Z}}\bar{S}_n, \quad \bar{S} \overset{i_n}{\simeq} \bar{S}_n.$$

Then the map i_0 induces an isomorphism

$$\pi_1(\bar{S}, s_0) \overset{\pi_1(i_0)}{\simeq} \pi_1(\bar{Y}, i_0(s_0)).$$

We will refer such an infinite cyclic covering to be *of a surface type.* The following notations will be used.

Notations 3.1. *1. X (resp. S, Y) is the interior of $\bar{X}$ (resp. $\bar{S}$, $\bar{Y}$).*

2. Π_S (resp. Π_Y, Π_X) is the fundamental group of S (resp. Y, X) with respect to the base point s_0 (resp. $i_0(s_0)$, $\pi(i_0(s_0))$).

3. $\Pi_{Y/X}$ is the covering transformation group of $Y \to X$.

Let Φ be a deck transformation generating $\Pi_{Y/X}$. Identifing S with S_0 (resp. S_1) via i_0 (resp. i_1), Φ induces a diffeomorphism ϕ on S by restriction. Since the genus of S is greater than or equal to 2, it is diffeomorphic to a quotient of the Poincaré upper half plane $\mathbf{H}^2$ by a discrete subgroup Γ of $PSL_2(\mathbf{R})$. Adding cusps Σ to the quotient, we get compactification S^*. We will sometimes identify S with $S^* \setminus \Sigma$.

Remark 3.1. *Note that there is an exact sequence*

$$1 \to \Pi_S \to \Pi_X \to \mathbf{Z} \to 1.$$

This is a geometric counterpart of the following situation in arithmetic geometry. Let C be a smooth curve defined over $\mathbf{F}_q$ and let $\bar{C}$ be its base extension to $\overline{\mathbf{F}}_q$. Then their fundamental groups fit in the exact seqence

$$1 \to \pi_1(\bar{C}) \to \pi_1(C) \to \hat{\mathbf{Z}} \to 1.$$

Let F be a field of characteristic 0 and let L be a vector space over F of dimension $2g$ with a skew-symmetric nondegenerate pairing α. Suppose we are given a representation

$$\Pi_X \overset{\rho_X}{\to} Aut(L, \alpha)$$

such that

$$L^{\Pi_S} = 0, \tag{1}$$

Let ρ_S be the restiction of ρ_X to Π_S and the local system associated to ρ_X (resp. ρ_S) will be denoted by $\mathcal{L}_X$ (resp. $\mathcal{L}_S$). Then the diffeomorphism ϕ induces an isomorphism of a polalized local system:

$$\begin{array}{ccc} \mathcal{L}_S & \xrightarrow[\simeq]{\hat{\phi}} & \mathcal{L}_S \\ \downarrow & & \downarrow \\ S & \xrightarrow[\simeq]{\phi} & S \end{array}$$

Fig. 2.2

Let j be the open immersion of S into S^* and let i be the inclusion of Σ into S^*. Then $\hat{\phi}$ acts on $H^1(S^*, j_*\mathcal{L}_S)$, which is a geometric analogue of the Frobenius action. For a point P in Σ, let Δ_P be a small disc centered at P and we set $\Delta_P^* = \Delta_P \setminus \{P\}$. The *parabolic cohomology* H_P^1 is defined to be

$$H_P^1(S, \mathcal{L}_S) = \mathrm{Ker}[H^1(S, \mathcal{L}_S) \to \oplus_{P\in\Sigma} H^1(\Delta_P^*, \mathcal{L}_S)].$$

One can easily see that $H_P^1(S, \mathcal{L}_S)$ admits an action of $\hat{\phi}$ and it is isomorphic to $H^1(S^*, j_*\mathcal{L}_S)$ as a $F[\hat{\phi}]$-module. Also the nondegenetate skewsymmetric pairing α and the Poincaré duality induce a perfect pairing on $H_P^1(S, \mathcal{L}_S)$, which is invariant under the action of $\hat{\phi}$. Hence $H_P^1(S, \mathcal{L}_S)$ is a semisimple $F[\hat{\phi}]$-module and it is isomorphic to its dual as a $F[\hat{\phi}]$-module.

Now we define the *topological L-fuction* $L(X, \mathcal{L}_X)$ for the local system $\mathcal{L}_X$ to be

$$L(X, \mathcal{L}_X) = \det[1 - \hat{\phi}^* T | H_P^1(S, \mathcal{L}_S)].$$

Here T is an indeterminate.

Let $M_\phi(S)$ be the mapping torus of ϕ and let $M_\phi(\mathcal{L}_S)$ be the local system on X which is the obtained by the same way as "mapping torus" from the isomorphism $\mathcal{L}_S \overset{\hat{\phi}}{\simeq} \mathcal{L}_S$. Note that by the definition we have

$$L(X, \mathcal{L}_X) = L(M_\phi(S), M_{\hat{\phi}}(\mathcal{L}_S)).$$

3.2 Examples

Let K be a knot embedded in the standard three dimensional sphere S^3 and let N_K be its tubular neighborhood. Let $\bar{X}$ be the closure of the complement of

N_K in S^3. Then $H_1(\bar{X}, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ by the Alexander duality and $\bar{X}$ admits an infinite cyclic covering

$$\bar{Y} \stackrel{\bar{\mu}}{\to} \bar{X}.$$

Let X (resp. Y) be the interior of $\bar{X}$ (resp. $\bar{Y}$) (cf. **Notations 3.1**). Then the map induces an exact sequence

$$1 \to \Pi_Y \to \Pi_X \to \mathbf{Z} \to 1. \quad (2)$$

Let $\hat{S}$ be a minimal Seifert surface of K and we set

$$\bar{S} = \hat{S} \cap \bar{X}.$$

It is known if Π_Y is finitely generated, $S \stackrel{i_0}{\hookrightarrow} Y$ induces an isomorphism ([5])

$$\Pi_S \simeq \Pi_Y.$$

Moreover Murasugi has shown if the absolute value of the Alexander polynomial $\Delta_K(t)$ of K at $t = 0$ is equal to 1, then Π_Y is finitely generated.

Fact 3.1. *([4]*__IV. Proposition 5__*) Suppose every closed imcompressive surface in X is boundary parallel. Then either*

1. *X is Seifert fibred,*

   or

2. *X is hyperbolic. Namely there is the maximal order $\mathcal{O}_F$ of an algebraic number field F and a torsion free subgroup $\Gamma \subset PSL_2(\mathcal{O}_F)$ such that X is diffeomorphic to $\Gamma\backslash\mathbf{H}^3$. Here after fixing an embedding F ito $\mathbf{C}$, Γ is regarded to be a subgroup of $PSL_2(\mathbf{C})$.*

Now we assume that the infinite cyclic covering satisfies the following conditions.

Condition 3.1. *1. Π_Y is finitely generated.*

2. Either

   (a) X is Seifert fibred,

   or

   (b) there is the maximal order $\mathcal{O}_F$ of an algebraic number field F and a torsion free subgroup $\Gamma \subset SL_2(\mathcal{O}_F)$ which freely acts on $\mathbf{H}^3$ so that X is diffeomorphic to $\Gamma\backslash\mathbf{H}^3$. As before after fixing an embedding F ito $\mathbf{C}$, Γ is regarded to be a subgroup of $SL_2(\mathbf{C})$.

Remark 3.2. *Professor Fujii kindly informed us that if X is hypebolic, then the* **Condition 3.1. 2 (b)** *is always satisfied.*

Suppose X satisfies 1 and 2(b) of **Condition 3.1**. Then we have the canonical representation

$$\Pi_X \simeq \Gamma \overset{\rho_X}{\hookrightarrow} SL_2(\mathbf{C}).$$

We set

$$L = \mathbf{C}^{\oplus 2},$$

and let α be the standard symplectic form on L. Namely for elements $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ of L, $\alpha(x,y)$ is defined as

$$\alpha(x,y) = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

This invariant under the action of Π_X. The result of Neuwirth([5]) implies

$$\Pi_S \overset{\pi_1(i_0)}{\simeq} \Pi_Y,$$

and it is easy to see

$$L^{\Pi_S} = 0.$$

Hence the conditions in §3.1 are satisfied.

Next suppose that X satisfies 1 and 2(a) of **Condition 3.1**. Then under a mild condition, one can check its fundamental group has a linear representation

$$\Pi_X \overset{\rho_X}{\to} SL_2(\mathbf{C})$$

such that

$$L^{\Pi_S} = 0.$$

Details will be found in [9].

Remark 3.3. *Even if we take the trivial representation, we can define an L-function for a knot complement. Note that this is nothing but the Alexander polynomial, which corresponds to the congruent zeta function of a curve. But contraty to the arithmetic case, as we have seen, we have* a priori *a two dimensional irreducible linear representation of $\pi_1(X)$. This is one of the main reasons to consider the L-function.*

4 Properties of a topological L-function

In the present section, we will list up basic properties of our topological L-function. Proofs of the statements will be found in [9].

4.1 A functional equation

Let $b(\mathcal{L}_S)$ be the dimension of $H^1_P(S, \mathcal{L}_S)$.

Theorem 4.1. *(The functional equation)*

$$L(X, \mathcal{L}_X)(T) = (-T)^{b(\mathcal{L}_S)} L(X, \mathcal{L}_X)(T^{-1}).$$

Corollary 4.1. *Suppose $b(\mathcal{L}_S)$ is odd. Then $L(X, \mathcal{L}_X)(1)$ vanishes and in particular the dimension of $H^1_P(S, \mathcal{L}_S)^{\hat{\phi}^*}$ is positive.*

4.2 A geometric analogue of Birch and Swinnerton-Dyer conjecture

In the present section, we will work with the holomorphic category.

Let A^* be a smooth projective variety with a morphism

$$A^* \stackrel{\bar{\mu}}{\to} S^*$$

such that its resriction to S

$$A \stackrel{\mu}{\to} S$$

is a smooth fibration whose fibres are abelian varieties of dimension g. Moreover we assume $\bar{\mu}$ satisfies the following conditions.

Condition 4.1. *1. A^*/S^* is the Neron model of A/S and has a semistable reduction at each point $s \in \Sigma$.*

2. $R^1\mu_*\mathbf{Q}$ is isomorphic to the local system $\mathcal{L}_S$.

3.

$$H^0(S^*, R^1\bar{\mu}_*\mathcal{O}_{A^*}) = 0.$$

Suppose there is a commutative diagram

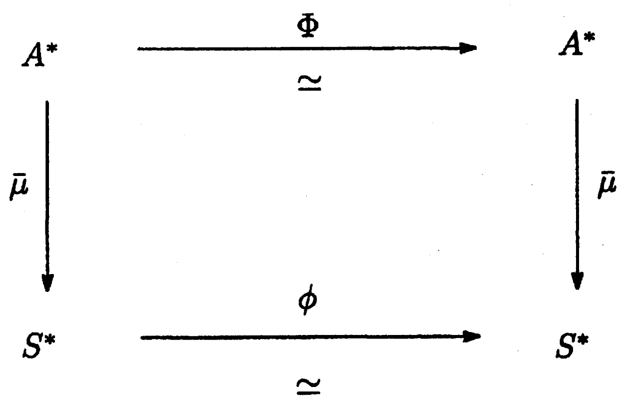

Fig. 4.1

such that $\phi(\Sigma) = \Sigma$. Since $\mathcal{L}_S = R^1\mu_*\mathbf{Q}$, this induces the diagram as **Fig. 2.2**. We define *the Mordell-Weil group* $MW_X(A)$ to be

$$MW_X(A) = A(S)^\Phi$$

and its rank will be denoted by $r_X(A)$. Since the cycle map induces an imbedding

$$A(S) \otimes \mathbf{Q} \hookrightarrow H^1_P(S, \mathcal{L}_S),$$

$r_X(A)$ is less than or equal to the order of the topological L-function $L(X, \mathcal{L}_X)$ at $T = 1$.

Theorem 4.2. *Suppose $H^2(A^*, \mathcal{O}_{A^*}) = 0$. Then $r_X(A)$ is equal to the order of the topological L-function $L(X, \mathcal{L}_X)$ at $T = 1$.*

We define the *topological Brauer group $Br_{top}(A^*)$* to be

$$Br_{top}(A^*) = H^2(A^*, \mathcal{O}^\times_{A^*}).$$

Then the exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_{A^*} \to \mathcal{O}^\times_{A^*} \to 0$$

implies the exact sequence

$$H^2(A^*, \mathbf{Z}) \to H^2(A^*, \mathcal{O}_{A^*}) \to Br_{top}(A^*) \to H^3(A^*, \mathbf{Z}).$$

Since A^* is compact, both $H^2(A^*, \mathbf{Z})$ and $H^3(A^*, \mathbf{Z})$ are finitely generated abelian groups. Hence $Br_{top}(A^*)$ is finitely generated if and only if $H^2(A^*, \mathcal{O}_{A^*})$ vanish since the latter is a complex vector space.

Corollary 4.2. *Suppose $Br_{top}(A^*)$ is finitely generated. Then the rank of the Mordell-Weil group $r_X(A)$ is equal to the order of the topological L-function $L(X, \mathcal{L}_X)$ at $T = 1$.*

Note that the corollary above is a geometric analogue of the theorem of Artin and Tate. ([10][11])

4.3 An Euler product and an Euler system

Suppose the map ϕ in **Fig. 2.2** satisfies the following condition.

Condition 4.2. *There exists a diffeomorphism ϕ_0 of S such that*

1. *ϕ_0 is homotopic to ϕ,*

   and

2. *every fixed point of ϕ_0^n is non-degenerate and is isolated for any positive integer n.*

Because of **Condition 4.2(1)**, **Fig. 2.2** may be replaced by:

$$\begin{array}{ccc} \mathcal{L}_S & \xrightarrow[\simeq]{\hat{\phi}_0} & \mathcal{L}_S \\ \downarrow & & \downarrow \\ S & \xrightarrow[\simeq]{\phi_0} & S \end{array}$$

Fig. 7.1

We prepare some notations. Let us fix a positive integer n. The set of fixed points of ϕ_0^n will be denoted by $S^{\phi_0^n}$. We define $\Phi_0(n)$ to be the orbit space of the action of ϕ_0 on

$$\{s \in S | \phi_0^n(s) = s \quad and \quad \phi_0^m(s) \neq s \quad for \quad 1 \leq \forall m \leq n-1\}$$

and we set

$$\Phi_0 = \sqcup_{n=1}^{\infty} \Phi_0(n).$$

For an element γ of $\Phi_0(n)$, we call the integer n its *length* and we will denote it by $l(\gamma)$. Let $x \in S^{\phi_0^{l(\gamma)}}$ be a representative of $\gamma \in \Phi_0$. Then $\hat{\phi}_0^{l(\gamma)}$ defines an automorphism of the fibre of $\mathcal{L}_S \otimes \mathbf{Q}$ at x and the polynomial

$$\det[1 - \hat{\phi}_0{}^{l(\gamma)} T | (\mathcal{L}_S \otimes \mathbf{Q})_x]$$

is independent of the choice of x, which will be written as $P_\gamma(T)$.

Let V^P is the invariant subspace of $L \otimes \mathbf{Q}$ under the action of $\pi_1(\Delta_P^*)$. It is easy to see $\oplus_{P\in\Sigma} V^P$ has an action of $\hat{\phi}$. Now the Grothendieck-Lefshetz trace formula implies the folloing theorem.

Theorem 4.3. *(Euler product formula) Suppose the map ϕ in* **Fig. 2.2** *satisfies the* **Condition 4.2**. *Then*

$$L(X, \mathcal{L}_X) = (\det[1 - \hat{\phi}^* T | \oplus_{P\in\Sigma} V^P])^{-1} \prod_{\gamma \in \Phi_0} P_\gamma(T^{l(\gamma)})^{-1}.$$

Our L-function has a *Euler system*, which has been considered by Kolyvagin in the Iwasawa theory of an elliptic curve ([8]). Let ϕ_0 be a diffeomorphism of S satisfying the **Condition 4.2** and let us fix a generator t of $\Pi_{Y/X} \simeq \mathbf{Z}$. Then $\mathbf{Q}[\Pi_{Y/X}]$ may be identified with $P = \mathbf{Q}[t, t^{-1}]$ and defining the action of t by $(\hat{\phi}_0^*)^{-1}$, the compact supported cohomology group $H_c^1(S, \mathcal{L}_S \otimes \mathbf{Q})$ may be regarded as a P-module. In gereral, the Fitting ideal of a finitely generated P-module M will be denoted by $Fitt_P(M)$. The following lemma directly follows from the definition of our L function.

Lemma 4.1.

$$Fitt_P(H_c^1(S, \mathcal{L}_S \otimes \mathbf{Q})) = (L_c(X, \mathcal{L}_X)),$$

where $L_c(X, \mathcal{L}_X)$ *is defined to be*

$$L_c(X, \mathcal{L}_X) = \det[1 - \hat{\phi}^* t | H_c^1(S, \mathcal{L}_S \otimes \mathbf{Q})].$$

For $\gamma \in \Phi_0(n)$, let $O_\gamma \subset S$ be the corresponding orbit of ϕ_0 and let S_γ be its complement. The *corestriction map*

$$H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q}) \overset{Cor}{\rightarrow} H_c^1(S, \mathcal{L}_S \otimes \mathbf{Q})$$

is defined to be the Poincaré dual of the restiction map

$$H^1(S, \mathcal{L}_S \otimes \mathbf{Q}) \overset{Res}{\rightarrow} H^1(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q}).$$

Observe that both of them are homomorphism of P-modules. The Thom-Gysin exact sequence implies

$$0 \rightarrow H^1(S, \mathcal{L}_S \otimes \mathbf{Q}) \overset{Res}{\rightarrow} H^1(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q}) \rightarrow \oplus_{x \in O_\gamma} (\mathcal{L}_S \otimes \mathbf{Q})_x \rightarrow 0,$$

and let

$$0 \rightarrow \oplus_{x \in O_\gamma} (\mathcal{L}_S \otimes \mathbf{Q})_x \rightarrow H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q}) \overset{Cor}{\rightarrow} H_c^1(S, \mathcal{L}_S \otimes \mathbf{Q}) \rightarrow 0 \qquad (3)$$

be its dual sequence. The following lemma follows from the obeservation:

$$Fitt_P(\oplus_{x \in O_\gamma} (\mathcal{L}_S \otimes \mathbf{Q})_x) = (P_\gamma(t^{l(\gamma)})).$$

Lemma 4.2.

$$Fitt_P(H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q})) = (L_c(X, \mathcal{L}_X) \cdot P_\gamma(t^{l(\gamma)})).$$

In general for an N-tuples of distinct elements $\{\gamma_1, \cdots, \gamma_N\}$ of Φ_0, we set

$$S_\gamma = S \setminus (O_{\gamma_1} \cup \cdots \cup O_{\gamma_N}). \qquad (4)$$

The induction on N shows the following proposition.

Proposition 4.1.

$$Fitt_P(H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q})) = (L_c(X, \mathcal{L}_X) \cdot \prod_{i=1}^{N} P_{\gamma_i}(t^{l(\gamma_i)})).$$

Definition 4.1. *(An Euler system of a topological L-function) Let γ be the empty set or an N-tuples of distinct elements of Φ_0. Suppose a finitely generated P-modules V_γ is given for such γ. If $\{V_\gamma\}_\gamma$ satisfy the following conditions, they will be refered as* Euler system *of the topological L-function.*

1.
$$Fitt_P(V_\phi) = (L_c(X, \mathcal{L}_X)).$$

2. *Suppose*
$$\gamma' = \gamma \cup \{\gamma_{N+1}\}, \quad \gamma_{N+1} \notin \gamma.$$
Then there is a surjection as P-modules
$$V_{\gamma'} \to V_\gamma$$
and their Fitting ideals satisfy the relation
$$Fitt_P(V_{\gamma'}) = Fitt_P(V_\gamma) \cdot (P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})).$$

We set
$$V_\phi = H^1_c(S, \mathcal{L}_S \otimes \mathbf{Q})$$
and for an N-tuples of distinct elements γ of Φ_0 we define
$$V_\gamma = H^1_c(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q}).$$
Then $\{V_\gamma\}_\gamma$ is an Euler system by **Proposition 4.1.**

Next we will show how Kolyvagin's Euler system appears in our geometric situation. We assume any two of $\{P_\gamma(t^{l(\gamma)})\}_{\gamma\in\Phi_0}$ are relatively prime. Let γ and γ' be as 2. of **Definition 4.1.** The same arguments of to obtain (3) shows

$$0 \to \oplus_{x\in O_{\gamma_{N+1}}}(\mathcal{L}_S \otimes \mathbf{Q})_x \to H^1_c(S_{\gamma'}, \mathcal{L}_S \otimes \mathbf{Q}) \overset{Cor}{\to} H^1_c(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q}) \to 0.$$

Note that $P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})$ annihilates $\oplus_{x\in O_{\gamma_{N+1}}}(\mathcal{L}_S \otimes \mathbf{Q})_x$ and by the assumption its multiplication on $H^1_c(S_\gamma, \mathcal{L}_S\otimes\mathbf{Q})$ is an isomorphism. These observations imply the following lemma.

Lemma 4.3. *Let us fix $x_\gamma \in H^1_c(S_\gamma, \mathcal{L}_S\otimes\mathbf{Q})$. If we take $y_{\gamma'} \in H^1_c(S_{\gamma'}, \mathcal{L}_S\otimes\mathbf{Q})$ so that*
$$Cor(y_{\gamma'}) = x_\gamma.$$
Then we have
$$Cor(P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})y_{\gamma'}) = P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})x_\gamma.$$
Moreover $P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})y_{\gamma'}$ is independent of the choice of $y_{\gamma'}$.

Now we fix a non-zero element c_ϕ of $H_c^1(S, \mathcal{L}_S \otimes \mathbf{Q})$. For an N-tuples of distinct elements γ of Φ_0 we will inductively define an element c_γ of $H_c^1(S_\gamma, \mathcal{L}_S \otimes \mathbf{Q})$. Let $\gamma' = \gamma \cup \{\gamma_{N+1}\}$ be as before. We take any $d_{\gamma'} \in H_c^1(S_{\gamma'}, \mathcal{L}_S \otimes \mathbf{Q})$ to be

$$Cor(d_{\gamma'}) = c_\gamma,$$

and we set

$$c_{\gamma'} = P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})d_{\gamma'}.$$

Then the system $\{c_\gamma\}_\gamma$ is well-defined by **Lemma 4.3** and they satisfy

$$Cor(c_{\gamma'}) = P_{\gamma_{N+1}}(t^{l(\gamma_{N+1})})c_\gamma,$$

which is the same relation as Kolyvagin's Euler system ([8]). One may realize an Euler system is an another appearence of *Euler product.*

4.4 The Franz-Reidemeister torsion and a special value

We will briefly recall the theory of densities and the Franz-Reidemeister torsion ([2] [3] [9]). Throughout the subsection, let F be equal to $\mathbf{R}$ or $\mathbf{C}$. Let V be a vector space over F of dimension $r > 0$ and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be its basis. We set

$$(\wedge^r V)^\times = \{a \cdot \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r | a \in F^\times\}$$

and

$$|\wedge^r V| = (\wedge^r V)^\times / \{\pm 1\}.$$

Then $|\wedge^r V|$ is isomorphic to $F^\times/\{\pm 1\}$ and will be mentioned as *the space of densities* on V. Let

$$(\wedge^r V)^\times \xrightarrow{\pi} |\wedge^r V|$$

be the canonical projection and the image $\pi(f)$ of $f \in (\wedge^r V)^\times$ will be denoted by $|f|$. For the 0 dimensional vector space $\mathbf{0}$, we define

$$\wedge^0 \mathbf{0} = F, \quad (\wedge^0 \mathbf{0})^\times = F^\times$$

and

$$|\wedge^0 \mathbf{0}| = F^\times / \{\pm 1\}.$$

Moreover for $f \in \wedge^0 \mathbf{0} = F^\times$, its image in $|\wedge^0 \mathbf{0}| = F^\times/\{\pm 1\}$ will be denoted by $|f|$. If F is $\mathbf{R}$, the canonical projection

$$(\wedge^0 \mathbf{0})^\times = \mathbf{R}^\times \xrightarrow{\pi} |\wedge^0 \mathbf{0}| \simeq \mathbf{R}_{>0}$$

is nothing but the map of taking absolute value. In the followings, we always assume the 0 dimesional vector space $\mathbf{0}$ has the density $1 \in |\wedge^0 \mathbf{0}| = F^\times/\{\pm 1\}$. Also we always assume every complex is bounded and consists of finite dimensional vector spaces over F.

Definition 4.2. *If a complex*

$$C^{\cdot} = [C^0 \to \cdots \to C^n]$$

has a density on each C^i *and* H^i, *we say the complex* $C^{\cdot}$ is given a density.

Remark 4.1. *When* $C^i = H^i$, *we assume* H^i *is given the same density as* C^i.

For a complex with a density

$$C^{\cdot} = [C^0 \to \cdots \to C^n],$$

we can associate an element $\tau_{FR}(C^{\cdot})$ of $F^{\times}/\{\pm 1\}$, which is called as *the Franz-Reidemeister torsion* (*the FR-torsion* for simplicity). Let $|C^i|$ (resp. $|H^i|$) be the density on C^i (resp. H^i). Then one may intuitively think of $\tau_{FR}(C^{\cdot})$ as

$$\tau_{FR}(C^{\cdot}) = \prod_{i=1}^{n}(\frac{|C^i|}{|H^i|})^{(-1)^i}.$$

Let us take a finite triagulation of S which is preserved by ϕ. Then by a pararell transformation of the symplectic form α, we obtain a complex with a density $C^{\cdot}_{\phi}$ such that its cohomology groups are isomorphic to $H^{\cdot}(M_{\phi}(S), M_{\hat{\phi}}(\mathcal{L}_S))$. Using the previous observation:

$$L(X, \mathcal{L}_X) = L(M_{\phi}(S), M_{\hat{\phi}}(\mathcal{L}_S)),$$

we can show the following theorem.

Theorem 4.4. *Suppose* $\hat{\phi}^* - 1$ *is isomorphic on* $H^1_P(S, \mathcal{L}_S)$. *Then we have*

$$|L(X,\ \mathcal{L}_X)(1)| = \tau_{FR}(C^{\cdot}_{\phi}).$$

Remark 4.2. *In general, we can show the following statement:*
Let r *be the dimension of* $\mathrm{Ker}\,[\hat{\phi}^* - 1|H^1_P]$. *Then we have*

$$\lim_{T\to 1} |(T-1)^{-r} L(X,\ \mathcal{L}_X)(T)| = R((H^1_P)^{\cdot}) \cdot \tau_{FR}(C^{\cdot}_{\phi}).$$

Here $R((H^1_P)^{\cdot})$ *is the* regulator *of the local system. Note that this is quite similar to the formula which is predicted by the Birch and Swinnerton-Dyer conjecture.*

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