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SATO-TATE TWISTS & ARITHMETIC GROTHENDIECK DUALITY FOR MIXED CHARACTERISTICS LOCAL RINGS*

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CONTENTS

1. BRIEF BACKGROUND HISTORY

In this report, I will explain a certain nice p^m-torsion object "$\mathbb{Z}/p^m(r)_{\mathfrak{X}}$" $\in D^b(\mathfrak{X})$ which we call Sato-Tate twist, where $\mathfrak{X}$ is a regular scheme flat over Dedekind ring R having semi-stable reduction at primes in R lying over p. This, do we expect to play the same roles in the theory of p-torsion etale cohomology group for $\mathfrak{X}$ as Tate twist $\mu_{l^m}^{\otimes r}$ does in l-adic theory (l is invertible in $\mathfrak{X}$). It is P. Schneider who firstly gave the definition of $\mathbb{Z}/p^m(r)_{\mathfrak{X}}$ for the regular model $\mathfrak{X}$ of smooth projective variety X over local field having good reduction and afterwards it was generalized to semi-stable cases by Kanetomo Sato. The prototype of the theory is found in Bloch-Kato paper [BK1], "p-adic étale cohomology" in IHES. Schneider, however,

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* *The title* was "p-adic Hodge theory and Sato duality for mixed characteristic local rings of type $(0,p)$" when I talked in RIMS conference, to which I think the present title is easier and more preferrable.

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did not pursue any important properties of $Z/p^m(r)_{\mathfrak{X}}$ such as (P): Purity or (PD): Poincare Duality, although he gave nice attempts for (C): Cycle class maps for $\mathfrak{X}$. Then Sato in [Sat1] generalizing Schneider's definition to semi-stable cases completely proved properties (P) & (PD). This is actually big progress in the history of motivic cohomology from the viewpoint of constructing nice Tate twists for mixed characteristics schemes which are equipped with reasonable and desirable properties. He also made a nice application of his theory in [Sat3] to rewrite "Tamagawa Number Conjecture" by Bloch-Kato in [BK2] for certain motives and especially for arithmetic surfaces, he reinterprets through the p-adic cohomology with Sato-Tate twists coefficients the beautiful conjectural formula by Kato on values of L-functions of them stated in his Hasse principle paper [Ka2].

Then, I took his results to apply them to the proof of class field theory for complete regular local rings in mixed characteristics in [Ma4]. More correctly, I several years ago tried to prove class field theory for the fractional field of $\mathbb{Z}_p[[X_1, X_2]]$ which corresponds to mixed characteristics version of my Thesis in [Ma1], where I treated class field theory for the power series ring $\mathbb{F}_p[[X_1, X_2, X_3]]$. But what I encountered there was the terrible difficulty of dealing with or calculating the local cohomology $H^i_{\mathfrak{m}}(\mathrm{Spec}\ \mathbb{Z}_p[[X_1, X_2]], \mathbb{Z}/p)$ for $i > 0$. That is, I was very thirsty for some good formalism to calculate such local cohomologies, where in the case of $\mathbb{F}_p[[X_1, X_2, X_3]]$ we have the perfect duality with logarithmic Hodge-Witt sheaves which comes from Grothendieck duality for geometric local rings defined over fields. So, I was obliged to face with the severe situation that in mixed characteristics, I had no candidate which replaces with logarithmic Hodge-Witt sheaves in geometric cases. But afterwards, I studied Tsuji's C_{st} paper [Tsu] in Inventionnes and although with a short knowledge of syntomic complex by Kato, I imagined the very vague form of Sato-Tate twists, which I defined as the patching of vanishing cycles and syntomic complex on the special fibre, like an etale sheaf $\mathcal{F}$ on a scheme X with open immersion $U \hookrightarrow X$ and $Z = X \setminus U$ is defined by giving sheaves $\mathcal{F}_1, \mathcal{F}_2$ on U, Z, respectively together with the patching isomorphism on Z. But this definition by me is *not* correct, for it collapses when p is much bigger than the dimension of X. Moreover, it never tells the precise form of the original object which should be defined on the model, not on the special fibre. Then strangely at the same time, Shiho in the conversation at the computer room in Tohoku university suggested me that Sato was in the course of establishing such nice objects for general arithmetic schemes. I was very happy for this and asked Sato of his study, then he immediately showed me his object and the conjectural duality theorem, which is nothing but the duality that I was seeking for and longing for and dreaming of! I call this beautiful duality "Arithmetic Grothendieck Duality". But at that time, we concluded that it would be quite hard to prove, although in the 2-dimensional case, Shuji Saito in [Sa] did equivalent calculations. I actually proved this 2 dimensional case years ago shortly after our discussion in Sato's house in Nagoya independently with Saito, and felt sure of the holding of the general duality. Afterwards, I was busy with studying Ribet's paper on Galois representations, so I rather abandoned to prove it together with my feeling that something new will be necessary for the proof. But in Lille, I got the message from Sato that he proved duality completely which made me astonished. I soon after, imagined the proof, which was far from perfect, but at any rate Sato's success obliged me to prove it also by myself. But I struggled for much time, and it was only the last November that I found the complete proof of arithmetic Grothendieck duality for local rings, but in semi-stable cases, my understanding of the definition of Sato-Tate twists was completely wrong! I remedied my misunderstanding by getting correct definitions by Sato, and now I completed the proof. But in good reduction cases, the proof that I rediscovered is completely the same one as Sato did for varieties over local fields, and further I did a bit more than him in points that I calculated all wild and fierce ramifications along the special fibre which is Theorem B(see Section 3). The basic spirits is of course Kato's calculations of Milnor K-groups and of p-adic vanishing cycles. But one must pay good attention in working in derived categories. But at any rate, once we have the arithmetic Grothendieck duality, we can deduce from it various arithmetic applications to complete regular local rings in mixed characteristics such as class field theory, Hasse principle, vanishings or explicit representations by Milnor K-groups of many local cohomologies..... Hopefully this report will be an easy introductory guide to Sato-Tate twists.

2. The Definition of Sato-Tate Twist $\mathbb{Z}/p^m(r)_X$ & Its General Formalism

Let $\mathfrak{X}$ be a regular scheme flat and semi-stable scheme over the integer ring $\mathcal{O}_k$ of k, where $[k\colon \mathbb{Q}_p] < \infty$ and $F\colon = \mathcal{O}_k/\pi_k$ with its uniformizer π_k. We begin to give the definiton of Sato-Tate twist $\mathbb{Z}/p^m(r)_{\mathfrak{X}} \in D^b(\mathfrak{X})$ in the below. Firstly we briefly recall important preparations.

Proposition/Definiton 2.1 (Sato). *Let $\mathfrak{X}$ be as above and Y be its special fibre which is a normal crossing variety over a finite field F. Let Y^0, Y^1 be sets of generic points or codimension 1 points of Y. Then there exists a canoinical boundary map*

$$\bigoplus_{y\in Y^0} W_m\Omega^r_{y,\log} \xrightarrow{\partial} \bigoplus_{y'\in Y^1} W_m\Omega^{r-1}_{y',\log},$$

by which we define

$$\nu^r_{m,Y}\colon = \mathrm{Ker}(\partial\colon \bigoplus_{y\in Y^0} W_m\Omega^r_{y,\log} \longrightarrow \bigoplus_{y'\in Y^1} W_m\Omega^{r-1}_{y',\log}). \qquad (2.1)$$

Also the natural map $W_m\mathcal{O}_Y^* \to W_m\Omega^1_{Y,\log}; \underline{a_1} \mapsto \frac{da_1}{\underline{a_1}}$ *induces another sheaf*

$$\lambda^r_{m,Y} := \operatorname{Image}\left(((W_m\mathcal{O})^*_Y)^{\otimes r} \to W_m\Omega^r_{Y,\log}\right); \underline{a_1} \otimes \ldots \otimes \underline{a_r} \mapsto \frac{da_1}{\underline{a_1}} \wedge \ldots \wedge \frac{da_r}{\underline{a_r}}.$$

If Y *is smooth, it holds that* $\nu^r_{m,Y} = \lambda^r_{m,Y} = W_m\Omega^r_{Y,\log}$, *which is the usual logarithmic Hodge-Witt sheaf of* Y.

Further, we have the following inclusion relations between Hyodo's logarithmic Hodge-Witt sheaves $W_m\omega^r_{Y,\log}$ *for* Y *defined by logarithmic structure studied by Kato in* [Ka5]:

$$\lambda^r_{m,Y} \subset W_m\omega^r_{Y,\log} \subset \nu^r_{m,Y}.$$

For the proof, we refer to Sato's paper [Sat2]. The beautiful perfect duality is the following:

Theorem 2.2 (Hyodo-Sato). *Let* Y *be a normal crossing variety of dimension* N *over a finite field* F. *Then there exist canoinical perfect dualities*:

$$H^i(Y, W_m\omega^r_Y) \times H^{N-i}(Y, W_m\omega^{N-r}_Y) \to H^N(Y, W_m\omega^N_Y) \cong \mathbb{Z}/p^m$$
$$H^i(Y, W_m\omega^r_{Y,\log}) \times H^{N+1-i}(Y, W_m\omega^{N-r}_{Y,\log}) \to H^{N+1}(Y, W_m\omega^N_{Y,\log}) \cong \mathbb{Z}/p^m$$
$$H^i(Y, \nu^r_{m,Y}) \times H^{N+1-i}(Y, \lambda^{N-r}_{m,Y}) \to H^{N+1}(Y, \nu^N_{m,Y}) \cong \mathbb{Z}/p^m,$$

where all cohomology groups in these pairings are finite.

Next for a complete normal crossing local ring A *of dimension* N *over* F, *such as* $A = F[[X_1, \ldots, X_{N+1}]]/X_1 \cdots X_i\,(1 \le i \le N+1)$, *we have the following perfect dualities*:

$$H^i_{\mathfrak{m}_A}(A, W_m\omega^r_A) \times H^{N-i}(A, W_m\omega^{N-r}_A) \to H^N_{\mathfrak{m}_A}(A, W_m\omega^N_A) \cong I_A$$
$$H^i_{\mathfrak{m}_A}(A, W_m\omega^r_{A,\log}) \times H^{N+1-i}(A, W_m\omega^{N-r}_{A,\log}) \to H^{N+1}_{\mathfrak{m}_A}(A, W_m\omega^N_{A,\log}) \cong \mathbb{Z}/p^m$$
$$H^i_{\mathfrak{m}_A}(A, \nu^r_{m,A}) \times H^{N+1-i}(A, \lambda^{N-r}_{m,A}) \to H^{N+1}_{\mathfrak{m}_A}(A, \nu^N_{m,A}) \cong \mathbb{Z}/p^m,$$

where I_A is the injective hull for A in [Ha] *and by abuse* $H^{N+1}_{\mathfrak{m}_A}(A, W_m\omega^N_{A,\log})\colon = H^{N+1}_{\mathfrak{m}_A}(\operatorname{Spec} A, W_m\omega^N_{\operatorname{Spec} A,\log})$ *etc. Also in above pairings, we put the discrete topology on each left hand side and* $\mathfrak{m}_A$*-adic topology on each right hand side.*

We refer for the proof also to Hyodo's paper [Hyo1], [Hyo1] and [Sat2]. The following result by Bloch-Hyodo-Kato is also important in later arguments:

Theorem 2.3 (Bloch-Hyodo-Kato, Sato). *Let $\mathfrak{X}$ be a regular proper flat scheme over* $\operatorname{Spec}\mathcal{O}_k$ *having semi-stable reduction. Consider the diagram:* $X \overset{j}{\hookrightarrow} \mathfrak{X} \overset{i}{\hookleftarrow} Y$*, where X, Y denote the generic and special fibres over* $\operatorname{Spec}\mathcal{O}_k$*, respectively and put* $M^r_{m,X}\colon = i^*\mathbb{R}^r j_*\mu_{p^m}^{\otimes r}$*, which is the p-adic vanishing cycles by Kato. Then, there exist Kato-filtrations* $U^iM^r_{m,X} \supset V^iM^r_{m,X} \supset U^{i+1}M^r_{m,X}$ *with* $U^0M^r_{m,X} = M^r_{m,X}$*. Sato also defined a certain filtration* $M^r_{m,X} \supset FM^r_{m,X} \supset U^1M^r_{m,X}$ *in* [Sat1]*. Then each graded quotient is calculated as follows:*

$$Gr^0_0\colon = U^0M^r_{1,X}/V^0M^r_{1,X} \cong \omega^r_{Y,\log}, \quad Gr^0_1\colon = V^0M^r_{1,X}/U^1M^r_{1,X} \cong \omega^{r-1}_{Y,\log},$$
$$M^r_{m,X}/FM^r_{m,X} \cong \nu^{r-1}_{m,Y}, \quad FM^r_{m,X}/U^1M^r_{m,X} \cong \lambda^r_{m,Y}.$$

For $p \nmid i > 0$, grs of Kato-filtrations are as follows:

$$Gr^i_0 := U^iM^r_{1,X}/V^iM^r_{1,X} \cong \omega^r_Y/B\omega^r_Y,\ Gr^i_1 := V^iM^r_{1,X}/U^{i+1}M^r_{1,X} \cong \omega^r_Y/Z\omega^r_Y$$

and also for $p \mid i$, we have

$$Gr^i_0 := U^iM^r_{1,X}/V^iM^r_{1,X} \cong \omega^r_Y/Z\omega^r_Y,\ Gr^i_1 := V^iM^r_{1,X}/U^{i+1}M^r_{1,X} \cong \omega^r_Y/Z\omega^r_Y,$$

where Z denotes d-closed form and $B = d\omega$ denotes the perfect forms.

For the proof we refer to [Ka1], [BK1], [Hyo1], [Hyo2] and [Sat1], [Sat2]. Now, we will define our main games, which is Sato-Tate twists. Recall that for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b(\mathfrak{X})$ and $f\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$, $\operatorname{Cone}(\mathcal{F}^\bullet \to \mathcal{G}^\bullet)\colon = \mathcal{F}^i \oplus \mathcal{G}^{i-1}$

with $d^i(a,b) = (d(a), -f(a)+d(b))$. Also for $\mathcal{F}^\bullet$, $\tau_{\le r}\mathcal{F}^\bullet$ is defined as its degree i part is $\mathcal{F}^i$ for $i<r$, degree r part is $\mathrm{Ker}(d^r)$ and degree i part is 0 for $i>r$. Here is the definition:

Definition 1 (Sato-Tate twists). Let $\mathfrak{X}$ be a regular flat scheme over $\mathrm{Spec}\, R$ where R is the Dedekind ring in mixed characteristics having semi-stable reduction at each prime $\mathfrak{p}$ lying over p. Then we have the diagram $X := \mathfrak{X} \setminus Y \overset{j}{\hookrightarrow} \mathfrak{X} \overset{i}{\hookleftarrow} Y$, where Y is the union of all special fibres at prime $\mathfrak{p}$ lying over p (namely all irreducible components of Y are in characteristic p). Then the Sato-Tate twist $\mathbb{Z}/p^m(r)_{\mathfrak{X}}$ for $\mathfrak{X}$ is given as the following object:

$$\mathbb{Z}/p^m(r)_{\mathfrak{X}} := \mathrm{Cone}(\tau_{\le r}\mathbb{R}j_*\mu_{p^m}^{\otimes r} \overset{tame'}{\longrightarrow} i_*\nu_{m,Y}^{r-1}[-r]) \in D^b(\mathfrak{X}),$$

where we consider the single sheaf $i_*\nu_{m,Y}^{r-1}[-r]$ as the complex sitting in degree r and $\nu_{m,Y}^{r-1}$ is the modified logarithmic Hodge-Witt sheaf by Sato in Definition 2.1 and tame' denotes the map coming from $M_{m,X}^r/FM_{m,X}^r \cong \nu_{m,Y}^{r-1}$ in Theorem 2.3.

Remark 1. If $\mathfrak{X}$ is proper smooth over $\mathrm{Spec}\,\mathcal{O}_k$ which is the spec of an integer ring of a local field k s.t. $[k\colon \mathbb{Q}_p] < \infty$, the Sato-Tate twist $\mathbb{Z}/p^m(r)_{\mathfrak{X}}$ for $\mathfrak{X}$ becomes simply as

$$\mathbb{Z}/p^m(r)_{\mathfrak{X}} := \mathrm{Cone}(\tau_{\le r}\mathbb{R}j_*\mu_{p^m}^{\otimes r} \overset{tame}{\longrightarrow} i_*W_m\Omega_{Y,\log}^{r-1}[-r]) \in D^b(\mathfrak{X}),$$

where $W_m\Omega_{Y,\log}^{r-1}$ is the logarithmic Hodge-Witt sheaf of Y and tame denotes tame symbol in Milnor K-theory.

Here we see some important properties of them.

Theorem 2.4 (General Formalisms of $\mathbb{Z}/p^m(r)_{\mathfrak{X}}$, Sato, Kurihara). *Let $\mathfrak{X}$ be a regular flat scheme over* $\mathrm{Spec}\,\mathcal{O}_k$ *with* $[k\colon \mathbb{Q}_p] < \infty$ *and* $\mathcal{O}_k$ *is its valuation ring having semi-stable reduction. Let* $X \overset{j}{\hookrightarrow} \mathfrak{X} \overset{i}{\hookleftarrow} Y$ *be as in*

Definition 1. Then, the following 3 *properties hold:*

1. $j^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}} \cong \mu_{p^m}^{\otimes r}$
2. $\mathbb{R}^i i^! \mathbb{Z}/p^m(r)_{\mathfrak{X}} = 0 \quad (i < r+1)$
   $\mathbb{R}^{r+1} i^! \mathbb{Z}/p^m(r)_{\mathfrak{X}} \cong W_m \Omega^{r-1}_{Y,\log}$
   $\mathbb{R}^i i^! \mathbb{Z}/p^m(r)_{\mathfrak{X}} \cong i^{-1}\mathbb{R}^{i-1} j_* \mu_{p^m}^{\otimes r} \quad (i > r+1)$
3. $i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}} \cong S_{m,\mathfrak{X}}(r)$ if $\mathfrak{X}$ has good reduction and $r < p-1$,

where $S_{m,\mathfrak{X}}(r)$ *denotes Kato's syntomic complex.*

I learned from Sato that even if Y has semi-stable reduction, but not good reduction, then the isomorphism in 3. collapses even for $r < p-1$. These properties are often quite useful, for in Section 3 we heavily use these properties in proving main theorems. I must also mention that I need deep results by Kurihara for 3. in the above in the good reduction case, which was taught by Sato. In the next Section 3, we will see that Sato-Tate twists are quite nice in cohomological behaviours.

3. Main Theorems and The Sketch of Proofs

Before stating our main results, we will review Sato's beautiful arithmetic dualities for arithmetic schemes over integer ring $\mathcal{O}_k$ of local field k ($[k\colon \mathbb{Q}_p] < \infty$).

Theorem(Arithmetic Duality; Sato(2002)). *Let k be a local field s.t. $[k\colon \mathbb{Q}_p] < \infty$ and let $\mathfrak{X}$ be the regular scheme of Krull dimension N proper flat over the integer ring $\mathcal{O}_k$ of k having semi-stable reduction. Applying Definition* 1 *for $\mathfrak{X}$, where X, Y there replace with generic and special fibres of $\mathfrak{X}$, respectively, we obtain Sato-Tate twist $\mathbb{Z}/p^m(r)_{\mathfrak{X}}$ for $r \geq 0$. Then the canonical trace isomorphism*

$$\mathrm{Trace}\colon \mathbb{H}^{2N+1}_Y(\mathfrak{X}, \mathbb{Z}/p^m(N)_{\mathfrak{X}}) \cong \mathbb{Z}/p^m$$

exists and moreover, it holds the following perfect pairing:

$$\mathbb{H}^i_Y(\mathfrak{X}, \mathbb{Z}/p^m(r)_{\mathfrak{X}}) \times \mathbb{H}^{2N+1-i}(\mathfrak{X}, \mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) \overset{\cup}{\to} \mathbb{H}^{2N+1}_Y(\mathfrak{X}, \mathbb{Z}/p^m(N)_{\mathfrak{X}}) \overset{trace}{\cong} \mathbb{Z}/p^m$$

between finite groups.

This is big success actually, especially in that it proves that Sato-Tate twists are actually nice objects so that they give Poincare-duality even for "model" $\mathfrak{X}$ of varieties over local fields. It is widely known that to do some calculations in the model level is often quite difficult and to have perfect duality with finite coefficients is very frequently impossible. For example, the success of the famous p-adic Hodge theory by Kato-Hyodo-Kurihara-Tsuji comes from deep calculations of cohomologies of syntomic complexes on "models", which is the core of the proof of C_{st} conjecture. The above duality by Sato assures us at least in the cohomological viewpoint that Sato-Tate twists in mixed characteristics work satisfactorily and perfectly comparably to usual Tate-twists in l-adic theory. Also it is important that Sato still needs deep calculations of p-adic vanishing cycles by Kato in his proof of the above theorem.

Now, it is my turn. The important fact is that Sato's arithmetic dualities are actually possible and inheritable also to local rings in mixed characteristics. Now, we shall state our main theorems for local rings:

Theorem A(Arithmetic Grothendieck Duality; P. Matsumi). *Let k be a local field with $[k\colon \mathbb{Q}_p] < \infty$ and $\mathcal{O}_k$ be its integer ring. Let A be the complete regular local ring over $\mathcal{O}_k$ of Krull dimension N having semi-stable reduction(A is, for example, $\mathcal{O}_k[[X_1, \dots, X_{N-1}]]$ or $\mathcal{O}_k[[X_1, \dots, X_N]]/(X_1 \cdots X_i - \pi_k)$). We apply Definition 1 for $\mathfrak{X} := \operatorname{Spec} A$, where X, Y there replace as $X := \operatorname{Spec} A[\frac{1}{\pi_k}]$, $Y := \operatorname{Spec} A/\pi_k$,* *respectively. Then, the canonical trace isomorphism*

$$\text{Trace}\colon \mathbb{H}^{2N+1}_{\mathfrak{m}_A}(\mathfrak{X}, \mathbb{Z}/p^m(N)_{\mathfrak{X}}) \cong \mathbb{Z}/p^m$$

exists and moreover, it holds the following perfect pairing:

$$\mathbb{H}^{i}_{\mathfrak{m}_A}(\mathfrak{X}, \mathbb{Z}/p^m(r)_{\mathfrak{X}}) \times \mathbb{H}^{2N+1-i}(\mathfrak{X}, \mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) \xrightarrow{\cup} \mathbb{H}^{2N+1}_{\mathfrak{m}_A}(\mathfrak{X}, \mathbb{Z}/p^m(N)_{\mathfrak{X}}) \overset{trace}{\cong} \mathbb{Z}/p^m,$$

where we put the discrete topology on the L.H.S. and $\mathfrak{m}_A$-adic topology on the R.H.S.

Theorem B(Poincare Duality; P. Matsumi). *Let A be as above and $\mathfrak{X} := \operatorname{Spec} A$. For the generic fibre $X = \operatorname{Spec} A[\frac{1}{\pi_k}]$ of $\mathfrak{X}$, we set $X \overset{j}{\hookrightarrow} \mathfrak{X}$. Then, we have the perfect pairing:*

$$\mathbb{H}^{i}_{\mathfrak{m}_A}(\mathfrak{X}, j_!j^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}) \times \mathrm{H}^{2N+1-i}_{\mathrm{et}}(X, \mu_{p^m}^{\otimes(N-r)}) \xrightarrow{\cup} \mathbb{H}^{2N+1}_{\mathfrak{m}_A}(\mathfrak{X}, j_!j^{-1}\mathbb{Z}/p^m(N)_{\mathfrak{X}})$$
$$\to \mathbb{H}^{2N+1}_{\mathfrak{m}_A}(\mathfrak{X}, \mathbb{Z}/p^m(N)_{\mathfrak{X}}) \overset{trace}{\cong} \mathbb{Z}/p^m,$$

where we put natural topologies on both hands coming from Theorem A.

These are main results of me in the last year for mixed characteristics local rings, which spiritually comes from the first success of Kanetomo Sato in [Sat1] in proving his arithmetic dualities for arithmetic schemes mentioned above. We will sketch proofs of Theorems A, B.

Sketch of proofs. For Theorem A, we use the spectral sequence

$$\mathbb{E}_1^{s,t} := \mathbb{H}^{t}_{\mathfrak{m}_A}(\mathfrak{X}, \mathcal{H}^s(\mathbb{Z}/p^m(r)_{\mathfrak{X}})) \Rightarrow \mathbb{H}^{s+t}_{\mathfrak{m}_A}(\mathfrak{X}, \mathbb{Z}/p^m(r)_{\mathfrak{X}}).$$

The key device to calculate each $\mathbb{E}_1^{s,t}$-term is to use the following important isomorphisms with Kato's p-adic vanishing cycles:

$$\begin{aligned} \mathcal{H}^s(\mathbb{Z}/p^m(r)_{\mathfrak{X}}) &\cong \mathbb{R}^s j_* \mu_{p^m}^{\otimes r} \quad \text{for} \quad s < r, \\ \mathcal{H}^s(\mathbb{Z}/p^m(r)_{\mathfrak{X}}) &= 0 \qquad\qquad\ \ \text{for} \quad s > r, \\ \mathcal{H}^r(\mathbb{Z}/p^m(r)_{\mathfrak{X}}) &\cong U^0\mathbb{R}^r j_* \mu_{p^m}^{\otimes r}, \end{aligned}$$

where $U^0\mathbb{R}^r j_* \mu_{p^m}^{\otimes r}$ is in Theorem 2.3. Once we have these isomorphisms we can use Kato's calculations of p-adic vanishing cycles which relate them with differential forms of the special fibre Y. We see this principle in the typical proof of Theorem B, for which I will give more details from now. That is, we will prove Theorem B assuming Theorem A. For this, we will use localization sequences in the derived category. Namely, from the two distinguished triangles

$$j_! j^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}} \to \mathbb{Z}/p^m(r)_{\mathfrak{X}} \to i_* i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}} \to j_! j^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}[1],$$

$$\begin{aligned} i_*\mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}} \to \mathbb{Z}/p^m(N-r)_{\mathfrak{X}} \to \mathbb{R}j_*j^{-1}\mathbb{Z}/p^m(N-r)_{\mathfrak{X}} \to \\ i_*\mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}}[1], \end{aligned}$$

we deduce two long exact sequences

$$\mathbb{H}^{i-1}_{\mathfrak{m}_A}(\mathfrak{X}, i_*i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}) \to \mathbb{H}^{i}_{\mathfrak{m}_A}(\mathfrak{X}, j_!j^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}) \to \mathbb{H}^{i}_{\mathfrak{m}_A}(\mathfrak{X}, \mathbb{Z}/p^m(r)_{\mathfrak{X}}) \to,$$

$$\begin{aligned} \mathbb{H}^{2N+1-i}(\mathfrak{X},\, \mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) \to \mathrm{H}^{2N+1-i}_{\mathrm{et}}(X,\, \mu_{p^m}^{\otimes(N-r)}) \to \\ \mathbb{H}^{2N+2-i}(Y,\, \mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) \to . \end{aligned}$$

From these, we see that it suffices to establish the perfectness of the pairing

$$\mathbb{H}^{i-1}_{\mathfrak{m}_Y}(Y, i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}) \times \mathbb{H}^{2N+2-i}(Y,\, \mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) \to \mathbb{Z}/p^m. \tag{3.1}$$

For this, we again consider the spectral sequence

$$\mathbb{E}_1^{s,t} := \mathbb{H}^t_{\mathrm{m}_Y}(Y, \mathcal{H}^s(i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}})) \Rightarrow \mathbb{H}^{s+t}_{\mathrm{m}_Y}(Y, i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}})$$
$$\mathbb{E}_1^{s,t} := \mathbb{H}^t(Y, \mathcal{H}^s(\mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}})) \Rightarrow \mathbb{H}^{s+t}(Y, \mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}})$$

As seen above, we have

$$\begin{aligned} &\mathcal{H}^s(i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}) \cong i^{-1}\mathbb{R}^s j_*\mu_{p^m}^{\otimes r} && \text{for} \quad s<r \\ &\mathcal{H}^s(i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}) = 0 && \text{for} \quad s>r \\ &\mathcal{H}^r(i^{-1}\mathbb{Z}/p^m(r)_{\mathfrak{X}}) \cong i^{-1}U^0\mathbb{R}^r j_*\mu_{p^m}^{\otimes r}. \end{aligned}$$

and by Theorem 2.4, we have

$$\begin{aligned} &\mathcal{H}^s(\mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) = 0 && \text{for} \quad s<N-r+1 \\ &\mathcal{H}^{N-r+1}(\mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) \cong W_m\Omega^{N-r-1}_{Y,\log} \\ &\mathcal{H}^s(\mathbb{R}i^!\mathbb{Z}/p^m(N-r)_{\mathfrak{X}}) \cong i^{-1}\mathbb{R}^{s-1} j_*\mu_{p^m}^{\otimes N-r} && \text{for} \quad s>N-r+1 \end{aligned}$$

From these the aiming pairing (3.1) is rewritten, for example, as

$$\mathbb{H}^{i-1-s}_{\mathrm{m}_Y}(Y, i^{-1}\mathbb{R}^s j_*\mu_{p^m}^{\otimes r}) \times \mathbb{H}^{2N+2-i-s}(Y,\ i^{-1}\mathbb{R}^{s-1} j_*\mu_{p^m}^{\otimes N-r}) \to \mathbb{Z}/p^m.$$

Further herein we replace $s \mapsto r-s$ in L.H.S. and $s \mapsto (N-r+2)+s$ in R.H.S. obtaining

$$\mathbb{H}^{i-1-r+s}_{\mathrm{m}_Y}(Y, i^{-1}\mathbb{R}^{r-s} j_*\mu_{p^m}^{\otimes r}) \times \mathbb{H}^{N+r-i-s}(Y,\ i^{-1}\mathbb{R}^{N-r+s+1} j_*\mu_{p^m}^{\otimes N-r}) \to \mathbb{Z}/p^m.$$

After passing to $m=1$, gathering these pairings and cutting them piece by piece in a suitable way with the help of Bloch-Hyodo-Kato Theorem 2.3, the desired perfectness of (3.1) is reduced to the powerful duality Theorem 2.2 by Hyodo-Sato paying attention to the fact that $(i-1-r+s)+(N+r-i-s) = N-1$, $(r-s-1)+(N-r+s+1-1) = N-1$. The precise calculation is found in [Ma4].

4. Arithmetic Applications

In this section, we will deduce various interesting arithmetic applications to complete regular local rings in mixed characteristics from Theorems A, B in Section 3.

Theorem 4.1 (Class Field Theory). *Let A be a complete regular local ring in mixed characteristics of Krull dimension N with finite residue field having semi-stable reduction over its coefficient ring and denote by K its fractional field. The idele class group $C_K := \varprojlim_{\mathcal{I}} \mathrm{H}^N_{\mathfrak{m}_A}(\operatorname{Spec} A, \mathcal{K}^{\mathrm{M}}_N(\mathcal{O}_A, \mathcal{I}))$ is endowed with the inverse limit topology induced from the discrete topology on each $\mathrm{H}^N_{\mathfrak{m}_A}(\operatorname{Spec} A, \mathcal{K}^{\mathrm{M}}_N(\mathcal{O}_A, \mathcal{I}))$, where $\mathcal{I}$ runs over all ideal sheaves of $\mathcal{O}_A$, and $\mathcal{K}^{\mathrm{M}}_N(\mathcal{O}_A, \mathcal{I})$ is a certain Milnor K-theoretic sheaf in the Nisnevitch topology. Then, C_K satisfies the following dual reciprocity isomorphism:*

$$\rho^*_K : \mathrm{H}^1_{\mathrm{Gal}}(K, \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{Hom}_{\mathrm{c}}(C_K, \mathbb{Q}_p/\mathbb{Z}_p),$$

where p is the characteristic of the residue field of A and $\mathrm{Hom}_{\mathrm{c}}$ is the set of all continuous characters of finite order.

We omit details of the proof mentioning just the following simple diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{H}^1_{\mathrm{Gal}}(X, \mathbb{Z}/p) & \to & \mathrm{H}^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) & \longrightarrow & \bigoplus_{\mathfrak{p} \in X^{(1)}} \mathrm{H}^2_{\mathfrak{p}}(X, \mathbb{Z}/p) & \longrightarrow & 0 \\
& \Big\downarrow \cong \ \mathrm{Thm}.B & & \Big\downarrow \rho^*_K/p & & \Big\downarrow \cong \ Kato & & \Big\downarrow \\
0 \to & \mathrm{Hom}_{\mathrm{c}}(C_X, \mathbb{Z}/p) & \to & \mathrm{Hom}_{\mathrm{c}}(C_K, \mathbb{Z}/p) & \to & \bigoplus_{\mathfrak{p} \in X^{(1)}} \mathrm{Hom}_{\mathrm{c}}(F^0 C_{K_{\mathfrak{p}}}, \mathbb{Z}/p) & \to & 0,
\end{array}$$

where $X = \operatorname{Spec} A[\frac{1}{\pi_k}]$, $X^{(1)}$ denotes the set of all height one primes in X and the top row is exact, where the final 0 comes from the absolute purity $\mathrm{H}^3_{\mathfrak{m}}(X, \mathbb{Z}/p) \cong 0$ for each height 2 prime $\mathfrak{m}$ and the bottom row is exact at $\mathrm{Hom}_{\mathrm{c}}(C_X, \mathbb{Z}/p)$ and $\mathrm{Hom}_{\mathrm{c}}(C_K, \mathbb{Z}/p)$ and finally in the extremely left vertical isomorphism, we used Theorem B in Section 3 with $i = 2N, r = N$ together with the isomorphism $\mathbb{H}^{2N}_{\mathfrak{m}_A}(\mathfrak{X}, j_! j^{-1} \mathbb{Z}/p(N)_{\mathfrak{X}}) \cong C_X/p$. It is easy to deduce from this diagram the desired isomorphism of ρ_K^*/p. The precise calculation is found in [Ma4].

Corollary 4.2. *Let A, K be as above. Then for an arbitary finite abelain extension L/K such that the integral closure B of A in L is also semi-stable over its coefficient ring, under the Gersten-Quillen conjecture for A and B and the Bloch-Milnor-Kato conjecture for K and L, it holds the following reciprocity isomorphism:*

$$\rho_K\colon C_K/\mathrm{Norm}(C_L) \cong \mathrm{Gal}(L/K).$$

Roughly, the assumption of the Bloch-Milnor-Kato conjecture is used in l-parts in interpreting local cohomologies by Milnor K-groups or in connecting local cohomologies to the idele class group C_K and on the other hand the Gersten conjecture is necessary also in l-parts in comparing idele class group C_K and those of various complete valuation fields obtained by completion at each height one prime of A. These are explained in the section by Sato in [Ma3]

Another interesting application is the following:

Theorem 4.3 (Hasse Principle). *Let A be a 3-dimensional complete regular local ring in mixed characteristics with finite residue field having semi-stable reduction over its coefficient ring and denote by K its fractional field. Then for arbitrary integr $m > 1$, the following Kato-complex for A is exact:*

$$0 \to \mathrm{H}^4_{\mathrm{Gal}}(K, \mu_m^{\otimes 3}) \to \bigoplus_{\mathfrak{p}:\,\mathrm{ht}\,1} \mathrm{H}^3_{\mathrm{Gal}}(\kappa(\mathfrak{p}), \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{m}:\,\mathrm{ht}\,2} \mathrm{H}^2_{\mathrm{Gal}}(\kappa(\mathfrak{m}), \mu_m) \xrightarrow{addition} \mathbb{Z}/m \to 0,$$

where for $\mathrm{ch}(k)^N m' = m$, we replace $\mu_m^{\otimes i}$ with $W_N\Omega^i_{k,\log}[-i] \oplus \mu_{m'}^{\otimes i}$.

This is the mixed characteristics analogy of [Ma2]. The proof goes completely in the same way as in [Ma2] just replacing X and $\mu_{p^m}^{\otimes 3}$ there with $\mathfrak{X}\colon = \mathrm{Spec}\, A$ and Sato-Tate twist $\mathbb{Z}/p^m(3)_{\mathfrak{X}}$, respectively.

Apology. I must apology to all audience for the following. In the conference, I stated at the end of my talk that Takagi's class field theory for number fields could be deduced easily from global arithmetic duality by reducing all cases to the case of $\operatorname{Spec}\mathbb{Z}$. But this was completely rediculous and wrong statements. Professor Takeshi Saito immediately showed the sign of suggesting my mistakes in his face, but I devoted myself to rushing my stupid opinions. Here, I heartily apology to all audience there including professor Takeshi Saito for my poor understanding and terribly wrong statements.

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