数理解析研究所講究録 1376 巻 2004 年 154-162

On the group structure of Kummer étale K-group

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1. Introduction

The aim of this note is to propose a generalisation of algebraic K-groups in logarithmic geometry and to describe its structure as an Abelian group by usual K-groups for a wide class of logarithmic schemes.

The note is organised as follows: In Section 2 we review some language used in logarithmic geometry. This section contains no originality. Then we define the Két K-group for general fs log schemes in Section 3 and state Main Theorem on its structure in Section 4. Lastly, we give the sketch of its proof in Section 5.

2. Log Scheme and Kummer Étale Site

In this section we review some notions about logarithmic schemes and Kummer étale sites. For details, see [Kat89], [Nak97] and [Ill02]. The readers familiar with terminology in logarithmic geometry are recommended to skip to the next section.

Let X be a scheme. A pre-log structure on X is a pair (M, α), where M is a sheaf of monoids on X_{et} and α is a homomorphism from M to $\mathscr{O}_X$.

Remark. In this note all monoids are assumed to be commutative ones with units and maps of monoids to preserve the units. When regarding the structure sheaf $\mathscr{O}_X$ on a scheme X as a sheaf of monoids, we always do by means of the multiplication.

A pre-log structure (M, α) is called a log structure if α induces an isomophism from $\alpha^{-1}\mathscr{O}_X^*$ to $\mathscr{O}_X^*$. A triple (X, M, α) consisting of a scheme X and a log structure (M, α) on X is called a log scheme. We usually denote it by (X, M) or X for short, and often denote by $\mathring{X}$ the underlying scheme of a log scheme X. We regard the sheaf $\mathscr{O}_X^*$ as a subsheaf of M and set $\overline{M} = M/\mathscr{O}_X^*$. It is proven that, for each pre-log structure M, we can construct its associated log structure M^a, as a universal object for morphisms of pre-log structures from M to log structures.

Note that any scheme X can be considered to be a log scheme *via* the natural inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$. This is called the trivial log structure.

A morphism of log schemes are defined naturally, i.e. a pair of a morphism of underlying schemes and a homomorphism of monoid sheaves satisfying a natural compatibility.

A log scheme X is called Noetherian, quasi-compact, regular and so on, if its underlying scheme $\overset{\circ}{X}$ is so. Similarly we often say, for example, "f is of finite type" when no confusions occur.

The following are the first typical examples.

Example A.1. Let X be a regular scheme, $D \subset X$ a divisor with normal crossings and $j : U = X \setminus D \hookrightarrow X$ the open immersion. Then the inclusion $M_X = j_*\mathcal{O}_U^* \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X$ with the scheme X becomes a log scheme. We call it the log scheme associated with (X, D), and denote it by (X, D) if no confusion occurs.

Example A.2. Let k be a ring and P a monoid. A natural morphism of monoids $P \hookrightarrow k[P]$ induces a pre-log structure $P_X \to \mathcal{O}_X$ on $X = \operatorname{Spec} k[P]$ with a constant sheaf P_X. We often denote by $\operatorname{Spec} k[P]$ the log scheme associated with the pre-log structure.

In general, given a log scheme (Y, N) and a morphism of schemes $f : X \to Y$, we define f^*N to be the log structure associated with a pre-log structure $f^{-1}N \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. For a morphism of log schemes $f : (X, M) \to (Y, N)$, we say that f is strict if the natural morphism $f^*N \to M$ is an isomorphism.

Example B. Let R be a discrete valuation ring, k its residue field and π a uniformizer of R. Put $X = \operatorname{Spec} R$ and $D = V(\pi) \cong \operatorname{Spec} k$. As in Example A.1, we have a log scheme (X, D), the log structure of which is described as $\mathcal{O}_X^*\pi^{\mathbb{N}} \hookrightarrow \mathcal{O}_X$. When we pull-back the log structure with respect to the closed immersion $i : D \hookrightarrow X$, we have a log structure on $\operatorname{Spec} k$,

$$\begin{array}{rcl} i^*M_X = \mathcal{O}_D^* t^{\mathbb{N}} & \to & \mathcal{O}_D \\ t^n & \mapsto & \begin{cases} 0 & (\text{if } n > 0) \\ 1 & (\text{if } n = 0), \end{cases} \end{array}$$

where "$t = i^*\pi$". Such a log scheme is called a log point.

Next, we give the definition of fs log scheme. A monoid P is called integral if the canonical morphism from P to its group envelope P^{gp} is injective, and saturated if it is integral and satisfies the following condition:

For any $p \in P^{\mathrm{gp}}$, if there exists a non-negative integer n such that $p^n \in P$, p itself belongs to P.

A log scheme X is called fine and saturated (or fs for short) if, étale locally, it admits a strict morphism of log schemes to $\operatorname{Spec} \mathbb{Z}[P]$ with

P a finitely generated and saturated monoid. This strict morphism is called a (local) chart.

Example. Log schemes appearing in Example A.1 and in Example B are fs. The log scheme $\operatorname{Spec} k[P]$ in Example A.2 is fs if P is finitely generated and saturated.

Remark. Both in the category of log schemes and in the category of fs log schemes, there exist fibre products, but the two concepts do not coincide in general (cf. Example C below).

A morphism of monoids $h : Q \to P$ is called Kummer if h is injective and for all $p \in P$ there exists a non-negative integer n such that $p^n \in h(Q)$. For a morphism of fs log schemes $f : X \to Y$, one says that f is Kummer if, for any $x \in X$, a natural morphism of monoids $f_x^* : \overline{M}_{Y,\overline{f(x)}} \to \overline{M}_{X,\overline{x}}$ is Kummer. Finally, the morphism f is Kummer étale (or shortly Két) if it is log étale and Kummer. Here log étaleness is defined in terms of local infinitesimal liftings as in the classical case (See, for details, [Kat89]).

It is proven that if $f : X \to Y$ is a morphism of schemes, regarded also as a morphism of log schemes with trivial log structures, then f is log étale if and only if f is classically étale.

It is also well-known that f is Kummer étale if and only if, étale locally on X and Y, we can construct the diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[dr, "f"'] & Y' \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[P] \arrow[d, "\operatorname{Spec} \mathbb{Z}[h]"] \\ & Y \arrow[r] & \operatorname{Spec} \mathbb{Z}[Q], \end{tikzcd}$$

where P and Q are finitely generated and saturated, the right square Cartesian, all horizontal arrows strict, f' (classically) étale, and $h : Q \to P$ is a Kummer map such that the order of $\operatorname{Coker} h^{\mathrm{gp}}$ is invertible on X (Note that $\operatorname{Coker} h^{\mathrm{gp}}$ is finite).

Kummer étale morphism is a generalisation of tamely ramified morphism in classical algebraic geometry, as the next example also suggests.

Example C. In Example A.2, suppose further that k is a separably closed field and that $P = \mathbb{N}$. Then $\operatorname{Spec} k[\mathbb{N}]$ is isomorphic to $(X, D) = (\operatorname{Spec} k[t], V(t))$ in the sense of Example A.1. Take a non-negative integer n prime to the characteristic of k. Then a natural morphism of fs log schemes

$$(X_n, D_n) = (\operatorname{Spec} k[t^{\frac{1}{n}}], V(t^{\frac{1}{n}})) \longrightarrow (X, D)$$

is Kummer étale. Moreover we see that, in the category of fs log schemes, it is a Galois cover with Galois group μ_n *via* its action on

(X_n, D_n):

$$\begin{array}{ccc}\mu_n & \longrightarrow & \mathrm{Aut}\,((X_n, D_n)/(X, D)) \\ \zeta_n & \longmapsto & \mathrm{Spec}\,(t^{\frac{1}{n}} \mapsto \zeta_n t^{\frac{1}{n}}).\end{array}$$

Indeed, as is easily checked, we have

$$X_n \times_X X_n \cong \mathrm{Spec}\, k[P],$$

where the monoid P is a push-out of the diagram $\mathbb{N} \xleftarrow{n} \mathbb{N} \xrightarrow{n} \mathbb{N}$ in the category of monoids. It is not an fs log scheme for $n \geq 2$, and the fibre product $X_n \times_X^{\mathrm{fs}} X_n$ in the category of *fs* log schemes is proven to be isomorphic to $\mathrm{Spec}\, k[\mathbb{N} \oplus \mathbb{Z}/n\mathbb{Z}]$ (Notice that its underlying scheme is the normalisation of $X_n \times_X X_n$), or more canonically, isomorphic to the disjoint union of X_n indexed by μ_n.

By a base change with respect to $D \hookrightarrow X$, we have another important example of Kummer étale morphisms

$$(\mathrm{Spec}\, k[t^{\frac{1}{n}}]/(t), \text{ some log str.}) \longrightarrow (\text{log point}).$$

Similarly, this is a Galois cover with Galois group μ_n.

Now we are ready to construct a Kummer etale site. Let X be an fs log scheme. The Kummer étale site of X, denoted by X_{Ket}, is defined as follows:

The underlying category is that of fs log schemes Kummer étale over X. A family of morphisms $\{\phi_i : U_i \to U\}_{i \in I}$ is defined to be a covering if and only if $U = \bigcup_{i \in I} \phi_i(U_i)$ set-theoretically.

We denote by $\widetilde{X_{\mathrm{Ket}}}$ the associated topos.

3. Kummer étale K-group

In this section we define the Kummer étale K-group, the main theme of this note. The idea is very simple and natural: First construct a structure sheaf on the Kummer étale site, and then define the K-group of vector bundles over the ringed topos.

Let X be an fs log scheme. We define a ring object $\mathcal{O}_{X_{\mathrm{Ket}}}$ in $\widetilde{X_{\mathrm{Ket}}}$ as follows: For an fs log scheme X' Kummer étale over X, $\mathcal{O}_{X_{\mathrm{Ket}}}(X') = \Gamma(\overset{\circ}{X'}, \mathcal{O}_{\overset{\circ}{X'}})$.

This object, which is *a priori* a presheaf, in fact becomes a sheaf of rings ([Hag03]). So we obtain a ringed topos $(\widetilde{X_{\mathrm{Ket}}}, \mathcal{O}_{X_{\mathrm{Ket}}})$ naturally associated with an fs log scheme X. We also denote it by $(X, \mathcal{O}_X)$ or $\widetilde{X_{\mathrm{Ket}}}$ if no confusion occurs. Note that we have a canonical morphism ϵ_X of ringed topoi from $\widetilde{X_{\mathrm{Ket}}}$ to $\widetilde{X_{\mathrm{Zar}}}$ (Subscript X is often omitted).

We have the natural notion of $\mathcal{O}_{X_{\mathrm{Ket}}}$-modules and define $Mod(X_{\mathrm{Ket}})$ to be the category of $\mathcal{O}_{X_{\mathrm{Ket}}}$-modules on the ringed topos $(X, \mathcal{O}_X)$. The following definitions are also very natural:

Definition. Let X be an fs log scheme and $\mathscr{F}$ an $\mathscr{O}_{X_{\mathrm{Ket}}}$-module. We say that $\mathscr{F}$ is a Két vector bundle if it is isomorphic to the direct sum of $\mathscr{O}_{X_{\mathrm{Ket}}}$ Kummer étale locally. We call a Két vector bundle of rank 1 a Két line bundle. We denote by $Vect(X_{\mathrm{Ket}})$ the full subcategory of $Mod(X_{\mathrm{Ket}})$ consisting of Két vector bundles.

Example. In Example C, we have a fully faithful functor:

$$\{\mu_n\text{-equivariant } \mathscr{O}_{X_{n,\mathrm{Zar}}}\text{-vector bundle}\} \hookrightarrow Vect(X_{\mathrm{Ket}}).$$

For instance, for an integer i we can define a Kummer étale line bundle $\mathscr{O}_X(\frac{i}{n})$ corresponding to the μ_n-$k[t^{\frac{1}{n}}]$-submodule $t^{-\frac{i}{n}}k[t^{\frac{1}{n}}]$ of $k(t^{\frac{1}{n}})$.

More generally,

(1) In Example A.1, let $\{D_i | i \in I\}$ be the set of irreducible components of D and assume that X is a variety over a separably closed field k of characteristic p. Then we can define a Két line bundle $\mathscr{O}_X(\sum_{i \in I} \alpha_i D_i)$ for $\alpha_i \in \mathbb{Z}_{(p)} (i \in I)$.

(2) Let $X' \to X$ be a Galois Két cover of fs log schemes with a Galois group G. Then we have a fully faithful functor:

$$\{G\text{-equivariant } \mathscr{O}_{X'_{\mathrm{Zar}}}\text{-vector bundle}\} \hookrightarrow Vect(X_{\mathrm{Ket}}).$$

It is easily checked that $Vect(X_{\mathrm{Ket}})$ becomes an exact category in the sense of D. Quillen (cf. [Qui73]). So we can define its K-group $K_q(X_{\mathrm{Ket}})$ according to his recipe. We call it a Kummer étale K-group, or briefly a Két K-group. On the other hand, $K_q(X)$ stands for the K-group of a scheme X in the usual sense.

Question. *Calculate $K_q(X_{\mathrm{Ket}})$.*

A partial solution to this question is the main result of this note.

Remark. We have an exact functor $\epsilon^* : Vect(X_{\mathrm{Zar}}) \longrightarrow Vect(X_{\mathrm{Ket}})$, which induces group homomorphisms $\epsilon^* : K_q(X) \to K_q(X_{\mathrm{Ket}})$ for each $q \geq 0$. Moreover, for a log scheme with the trivial log structure, the functor induces an equivalence of categories and so leads to isomorphisms of K-groups, because of the étale descent.

4. Main Theorem

Now we state Main Theorem

Theorem. *Let X be a scheme smooth, separated and of finite type over a separably closed field k of characteristic p, D a simple normal crossing divisor and $\{D_i | i \in I\}$ its irreducible components. We endow X with the associated log structure. Then we have an isomorphism of Abelian groups:*

$$\eta_X : \bigoplus_{J \subset I} K_q(D_J) \otimes_{\mathbb{Z}} \widetilde{\Lambda}'_J \xrightarrow{\cong} K_q(X_{\mathrm{Ket}}),$$

for any non-negative integer q. Here for $J = \{i_1, \cdots, i_r\}$ we put

$$D_J = D_{i_1} \cap \cdots \cap D_{i_r} (D_\emptyset = X)$$

and $\widetilde{\Lambda}'_J$ is defined to be the free abelian group generated by the set

$$\{\xi : J \to (\mathbb{Q}/\mathbb{Z})' \mid \xi(j) \neq 0, \textit{ for any } j\ \},$$

where $(\mathbb{Q}/\mathbb{Z})' = \mathbb{Z}_{(p)}/\mathbb{Z}$.

This theorem gives the complete description of the Két K-group in terms of the classical K-group, at least with respect to its (Abelian) group structure.

Example. Let C be a smooth curve over a separably closed field k of characteristic p and $P_1, \ldots, P_r$ distinct closed points. Then we have:

$$\eta_C : K_0(C) \oplus \bigoplus_{i=1}^{r} \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})' \setminus \{0\}] \xrightarrow{\cong} K_0(C_{\mathrm{Ket}}).$$

Here η_C is the map characterized by

$$\eta_C([\mathscr{F}]) = [\epsilon^* \mathscr{F}] \text{ for } [\mathscr{F}] \in K_0(C)$$

and

$$\eta_C([\alpha] \text{ at the } i\text{-th component}) = [\mathscr{O}_C] - [\mathscr{O}_C((\widetilde{\alpha} - 1)P_i)]$$

for $\alpha \in (\mathbb{Q}/\mathbb{Z})' \setminus \{0\}$, where $\widetilde{\alpha}$ denotes the rational number lifting α satisfying $0 < \widetilde{\alpha} < 1$.

5. The Sketch of the Proof

In this section we sketch out the proof of Main Theorem. For a detailed explanation see [Hag03]. First we introduce the notion of Két coherent sheaves of $\mathscr{O}_X$-modules.

Definition. Let X be an fs log scheme and $\mathscr{F}$ an $\mathscr{O}_{X_{\mathrm{Ket}}}$-module. We say $\mathscr{F}$ is a Két coherent sheaf of $\mathscr{O}_X$-modules if there exists a Kummer étale covering $\{X_i \longrightarrow X\}_{i \in I}$ of X such that each $\mathscr{F}|_{X_i}$ is of the form $\epsilon^*_{X_i}\mathscr{F}'_i$ for some coherent $\mathscr{O}_{X_i}$-module $\mathscr{F}'_i$ on $X_{i,Zar}$. We denote by $Coh(X_{\mathrm{Ket}})$ the full subcategory of $Mod(X_{\mathrm{Ket}})$ consisting of Két coherent sheaves of $\mathscr{O}_X$-modules.

The category $Coh(X_{\mathrm{Ket}})$ often, although not always, behaves well.

For example, consider an fs log scheme X such that $\overline{M}_{X,\bar{x}}$ is isomorphic to the direct sum of $\mathbb{N}$ for all $x \in X$. Then $Coh(X_{\mathrm{Ket}})$ becomes an abelian category and the canonical functor $\epsilon^* : Coh(X_{\mathrm{Zar}}) \longrightarrow Coh(X_{\mathrm{Ket}})$ exact. In particular we can define K'-theory of log schemes $K'(X_{\mathrm{Ket}})$ by the Quillen's method and obtain group homomorphisms $K'_q(X) \to K'_q(X_{\mathrm{Ket}})$ from the usual K'-groups.

Furthermore, for an fs log scheme X satisfying the assumptions in Main Theorem, we obtain a canonical group isomorphism $K_q(X_{\mathrm{Ket}}) \cong K'_q(X_{\mathrm{Ket}})$.

K'-theory has some advantages over K-theory. Among them is the existence of the localisation sequence, i. e.

Proposition. *Let X be a Noetherian equi-characteristic fs log scheme, Y a strictly closed subscheme and U its complement, which we endow with the induced log structure. We suppose that $\overline{M}_{X,\overline{x}}$ is isomorphic to a direct sum of $\mathbb{N}$ for all $x \in X$.*

Then we have a long exact sequence

$$\begin{gathered}\cdots \xrightarrow{\partial} K'_q(Y_{\mathrm{Ket}}) \xrightarrow{i_*} K'_q(X_{\mathrm{Ket}}) \xrightarrow{j^*} K'_q(U_{\mathrm{Ket}}) \xrightarrow{\partial} K'_{q-1}(Y_{\mathrm{Ket}}) \xrightarrow{i_*} \cdots \\ \cdots \xrightarrow{\partial} K'_0(Y_{\mathrm{Ket}}) \xrightarrow{i_*} K'_0(X_{\mathrm{Ket}}) \xrightarrow{j^*} K'_0(U_{\mathrm{Ket}}) \longrightarrow 0.\end{gathered}$$

Another advantage of K'-theory over K-theory is its calculability in the case of dimension 0.

Example. Let P be a log point as in Example B and assume that the underlying field k is of characteristic p and contains all roots of unity. Then we can easily obtain an equivalence of categories

$$\bigcup_{(p,n)=1} \{\mu_n\text{-equivariant } \mathscr{O}_{P_{n,\mathrm{Zar}}}\text{-vector bundle}\} \approx \mathit{Vect}(P_{\mathrm{Ket}}),$$

where P_n is the Kummer étale cover of P constructed at the end of Example C. This induces an isomorphism

$$K'_q(P_{\mathrm{Ket}}) \cong \varinjlim_{(p,n)=1} K'_q(\operatorname{Spec} k[t]/(t^n), \mu_n),$$

where the right hand side is the inductive limit of equivariant K'-groups in Zariski topology. By the dévissage theorem in K-theory (cf. [Qui73]), we can neglect "the nilpotent part" in K' to rewrite the above group as

$$\varinjlim_{(p,n)=1} K'_q(\operatorname{Spec} k, \mu_n),$$

and we can obtain an isomorphism

$$K'_q(P_{\mathrm{Ket}}) \cong K'_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})'] \cong K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})']$$

as Abelian groups.

Remark. Similarly we get an isomorphism

$$K_q(P_{\mathrm{Ket}}) \cong \varinjlim_{(p,n)=1} K_q(\operatorname{Spec} k[t]/(t^n), \mu_n),$$

but we can no longer ignore the effect of "the nilpotent part" in the right hand side. In fact, its explicit calculation seems excessively more difficult in general.

Let us begin the proof of Main Theorem. For any subset J of I and any fs log scheme X' over X whose structure morphism is strict, we define X'_J by the Cartesian diagram below:

$$\begin{array}{ccc} X'_J & \xrightarrow{i'} & X' \\ {\scriptstyle f}\downarrow & & \downarrow \\ D_J & \xrightarrow{i} & X, \end{array}$$

and we set

$$\widetilde{K}'_q(X') = \bigoplus_{J \subset I} K'_q(X'_J) \otimes_{\mathbb{Z}} \widetilde{\Lambda}'_J.$$

Of course, K' in the right hand side means the classical K'-group.

The point is that we will prove $\widetilde{K}'_q(X') \cong K'_q(X'_{\mathrm{Ket}})$ for all fs log schemes X' strict over X simultaneously.

For each $\xi : J \to (\mathbb{Q}/\mathbb{Z})'$, we define $\tilde{\xi} : J \to \{x \in \mathbb{Z}_{(p)} | 0 < x < 1\}$ to be the unique lifting of ξ and set

$$\mathscr{O}_{D_J}\{\xi\} = \mathrm{Image}\,(\mathscr{O}_{D_J} \to i^*\mathscr{O}_X(\sum_{j \in J} \tilde{\xi}(j) D_j)).$$

This is an object in $Coh(D_{J,\mathrm{Ket}})$.

Key Lemma A. *The functor*

$$\begin{array}{ccc} Coh(X'_{J,\mathrm{Zar}}) & \longrightarrow & Coh(X'_{\mathrm{Ket}}) \\ \mathscr{F} & \mapsto & i'_*(\epsilon^*_{X'_J}\mathscr{F} \otimes_{\mathscr{O}_{X'_J}} f^*\mathscr{O}_{D_J}\{\xi\}) \end{array}$$

is exact (Note that in the right hand side appears $\otimes$, not "$\otimes^{\mathbb{L}}$").

By Key Lemma A, we can construct group morphisms $K'_q(X'_J) \to K'_q(X'_{\mathrm{Ket}})$ for each q. Summing them up for all ξ, we obtain $\eta_{X'} : \widetilde{K}'_q(X') \to K'_q(X'_{\mathrm{Ket}})$.

Key Lemma B. *The localisation sequences for $\widetilde{K}'(-)$ and $K'(-_{\mathrm{Ket}})$ are compatible with η.*

Note that we have a localisation sequence also for $\widetilde{K}'(-)$, constructed by the direct sum of classical ones.

By Key Lemma B and standard arguments using inductive limits, in order to prove that $\eta_{X'}$ is an isomorphism for all X' over X, it suffices to deal with the case where X' is of the form $\mathrm{Spec}\,K$ for some field K. As is mentioned above, in this case we can calculate $K'(X'_{\mathrm{Ket}})$ explicitly, and therefore prove directly that $\eta_{X'}$ is an isomorphism.

As the final remark, notice that we are often interested only in the case where $q = 0$ and a log scheme X is the one as in Example A, to be sure, but in order to let the above argument work smoothly, it is inevitable to introduce more general log schemes, for example log points, and to deal with their higher K-groups simultaneously.

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