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Nevanlinna theoretical approach to a small divisor problem in complex dynamics

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1 Local theory in complex dynamics

A rational map f is a holomorphic endomorphism of the Riemann sphere $\hat{\mathbb{C}}$. The *Fatou set* $F(f)$ is the region of normality of $\{f^k := f^{\circ k}\}$, and the *Julia set* $J(f)$ its complement. Both of them are f-completely invariant, and the dynamics of f over them are tame and chaotic respectively.

Figure 1: Newton method for a cubic polynomial

In general, the following problem is quite difficult:

Problem 1. Determine which $F(f)$ or $J(f)$ a given point of $\hat{\mathbb{C}}$ belongs to.

We focus on a *periodic* point, so study a *local normal form* of analytic germs around a *fixed point*.

Problem 2. Determine the moduli of analytic conjugacy classes of analytic germs

$$M(\lambda) := \{f(z) = \lambda z + O(z^2)\}/(\text{analytic conjugacy fixing } 0).$$

λ is called the *multiplier* of f at the origin.

2 irrationally indifferent fixed points

In some cases, the moduli is determined and its structure is easy: In the *superattractive* case $\lambda = 0$ (Böttcher),

$$M(0) \cong \mathbf{N} \cong \{\text{the number of sheets of } f \text{ around } 0\}.$$

In the *attractive* case $0 < |\lambda| < 1$ and *repelling* one $|\lambda| > 1$ (Koenigs), $M(\lambda)$ is trivial, i.e., $f(z)$ is always analytically conjugate to λz.

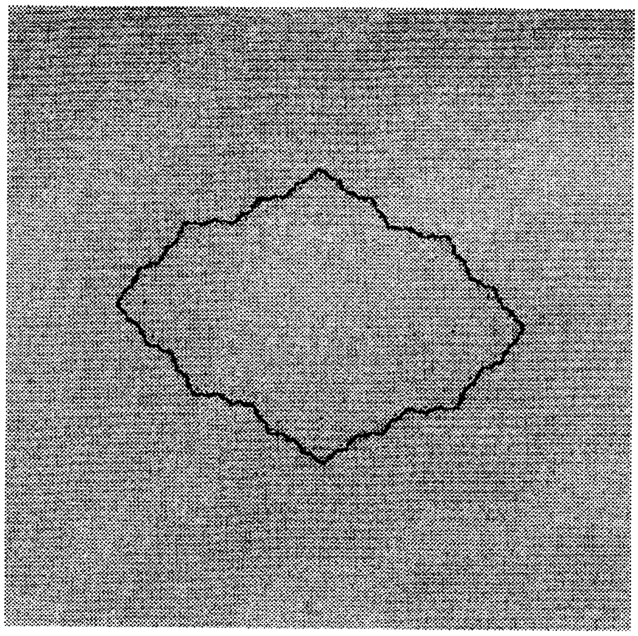

Figure 2: attractive basin

In another case, the moduli is determined but its structure is complicate: In the *parabolic* case that λ is a root of the unity (Ecalle [4] and Volonin [15]),

$$M(\lambda) \cong \mathbb{C}^* \times \mathbb{N} \times \{\text{an } \infty\text{-dimentional space}\}.$$

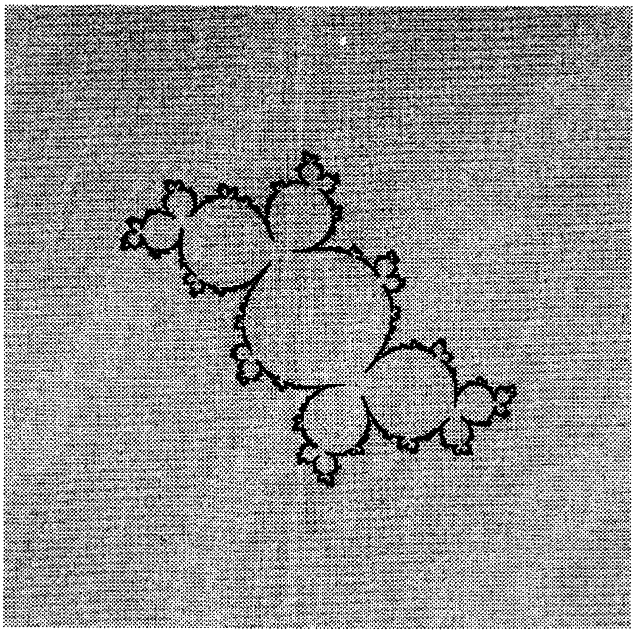

Figure 3: parabolic basin

In the other case, the moduli itself has not been determined yet: In the *irrationally indifferent* case $\lambda = e^{2i\pi\alpha}$ ($\alpha \in \mathbb{R} - \mathbb{Q}$), a small divisor problem occurs.

In the rest of this paper, we study only such λ.

Problem 3. Solve the functional equation (Schröder equation)

$$\phi \circ f(z) = \lambda \cdot \phi(z)$$

in the formal power series sense. Here

$$\phi(z) = z + \sum_{j>1} \phi_j z^j \quad \text{(unknown formal power series)},$$

$$f(z) = \lambda z + \sum_{j>1} f_j z^j \quad \text{(power series expansion)}.$$

It can be solved as $\phi_2 = f_2/(\lambda-\lambda^2)$, $\phi_3 = (f_3+2\lambda f_2\phi_2)/(\lambda-\lambda^3), \ldots$, inductively. We notice that

$$\phi_j = \frac{(\text{a polynomial of } f_2, f_3, \ldots, f_j, \phi_2, \phi_3, \ldots, \phi_{j-1})}{\lambda - \lambda^j}.$$

Problem 4. Determine when the formal power series solution ϕ converges.

It involves a *difficulty of small divisors*:

$$\liminf_{j\to\infty} |\lambda - \lambda^j| = 0. \tag{1}$$

Problem 5. For every $\lambda = e^{2i\pi\alpha}$ ($\alpha \in \mathbb{R} - \mathbb{Q}$), find a non-linear f such that ϕ converges.

It in not difficult. However, the following is not easy, and as we shall see in the next section, it is studied in detail by Cremer around 1930.

Problem 6. For every $\lambda = e^{2i\pi\alpha}$ ($\alpha \in \mathbb{R} - \mathbb{Q}$), does there exist a non-linear f such that ϕ diverges? If exists, how is it?

We shall study mainly this Problem 6.

Definition 2.1. f is (analytically) *linearizable* if ϕ converges.

3 Diophantine conditions related to complex analysis

In studying Problem 6, we encounter several Diophantine(-type) conditions.

Theorem 3.1 (Cremer [3] 1931). *Assume that $f \in M(\lambda)$ is a rational map of degree $d \geq 2$. If λ satisfies*

$$\liminf_{j\to\infty} |\lambda - \lambda^j|^{(1/d)^j} = 0,$$

(*compare* (1)) *or equivalently,*

$$\limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|} = \infty, \tag{Cr}$$

then f is not linearizable.

Theorem 3.2 (Siegel [13] 1942). *If λ satisfies*

$$|\lambda^q - 1| > \frac{C}{q^\kappa} \quad (q \in \mathbb{N}) \tag{S}$$

for some $C > 0$ and some order $\kappa \geq 1$, then $f \in M(\lambda)$ is always linearizable.

Cremer's argument was algebraic: he estimated the absolute values of roots of $f^j(z)-z$ from *above* by $|\lambda-\lambda^j|^{(1/d)^j}$. Such an estimate gives an *upper* estimate of the radius of convergence of ϕ. Consequently, Cremer obtained his non-linearizability result under the condition (Cr).

Contrastively, Siegel's argument was analytic: he constructed an absolutely convergent power series dominating ϕ under the condition (S). Its radius of convergence gives a *lower* estimate of that of ϕ. Consequently, Siegel obtained his linearizability result.

Remark 3.1. Every algebraic irrational number of order κ satisfies (S) of order κ.

Now we try rewriting (Cr) and (S) in a *Diophantine approximation formalism*, which is appropriate to a *geometric* study of our problem.

Let $\{p_n/q_n\}$ denote the convergent derived from the continued fraction expansion of α. Then the former is equivalent to

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{d^{q_n}} = \infty,$$

and the latter

$$\lim_{n\to\infty} \frac{\log q_{n+1}}{\log q_n} = 0.$$

Later, Brjuno ([1] 1971) improved (S) to

$$\sum_{n\in\mathbb{N}} \frac{\log q_{n+1}}{q_n} < \infty. \tag{B}$$

However (B) cannot be improved anymore.

Theorem 3.3 (Yoccoz [16] 1996). *If λ does not satisfy (B), some $f \in M(\lambda)$ is not linearizable.*

On the other hand, by Yoccoz and Ilyashenko, it is known that when the *quadratic* polynomial $P_2(z) = \lambda z + z^2$ is linearizable, then any element of $M(\lambda)$ is linearizable. Hence,

Theorem 3.4 (Yoccoz [16] 1996, Cheritat 2001). *If λ does not satisfy (B), then P_2 is not linearizable.*

4 a new argument – the Nevanlinna theory

Problem 7. Fill the gap between (Cr) and (B).

We introduce the *Nevanlinna* theory to our problem. Assume that $f \in M(\lambda)$ is a rational map of degree $d \geq 2$.

The spherical area measure and the chordal distance on $\hat{\mathbb{C}}$ are

$$\sigma(w) = \frac{|dw|}{\pi(1+|w|^2)^2} \quad \text{and} \quad [z,w] = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

respectively. We note that they are normalized as $\sigma(\hat{\mathbb{C}}) = 1$ and $[0, \infty] = 1$.

The *Valiron exceptionality* of $\mathrm{Id}_{\hat{\mathbb{C}}}$ for $\{f^k\}$ is defined by

$$\mathrm{VE}(\mathrm{Id}_{\hat{\mathbb{C}}}; \{f^k\}) := \limsup_{k\to\infty} \frac{\int_{\hat{\mathbb{C}}} \log \frac{1}{[f^k(w),w]} d\sigma(w)}{d^k}.$$

Then we can characterize the left hand side of (Cr) in terms of the Nevanlinna theory.

Main Theorem 1 (natural equality). *If f is linearizable, then*

$$\limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|} = \mathrm{VE}(\mathrm{Id}_{\hat{\mathbb{C}}}; \{f^k\}). \tag{2}$$

On the other hand, the linearizability of f means the existence of the *Siegel disk*, where f acts as an irrational rotation, hence the tame dynamics of f actually exists.

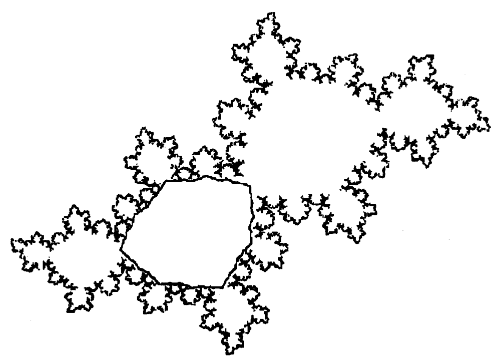

Figure 4: Siegel disk

Main Theorem 2 (Vanishing theorem). *When* $F(f) \neq \emptyset$,

$$\mathrm{VE}(\mathrm{Id}_{\hat{\mathbb{C}}}; \{f^k\}) = 0. \tag{3}$$

Combining them, we succeed in improving (Cr) greatly.

Main Theorem 3 ([12] 2002). *If*

$$\limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|} > 0 \tag{$\heartsuit$}$$

then f is not linearizable.

5 dynamical and Nevanlinna exceptional sets

In this section, we shall survey our article [11], of which the results have been already implicitly applied to the proof of Main Theorem 3.

Notation. Rat denotes the set of all rational endomorphism of $\hat{\mathbb{C}}$. $\hat{\mathbb{C}}$ is identified as the set of all constant functions of $\hat{\mathbb{C}}$.

From definition, the dynamical system *around* $J(f)$ has an *almost covering* feature: There exists $E(f) \subset \hat{\mathbb{C}}$ such that for every neighborhood U of a point of $J(f)$, the union of the forward-images of U under iterations covers $\hat{\mathbb{C}} - E(f)$.

Definition 5.1 (dynamical exceptional set). $E(f)$ is called the *dynamical exceptional set* of f.

From this almost covering feature, naturally arises the Nevanlinna theoretical study, which treats *preimages* under iterations.

Definition 5.2 (value distribution). For $f, g \in$ Rat, the *value distribution* $\mu(f, g)$ of f for g is defined by the mass distribution on the $(\deg f + \deg g)$-roots of the equation $f = g$.

Definition 5.3 (dynamical Nevanlinna theory [14]). For $f, g \in$ Rat, the *pointwise proximity function* is defined by

$$(w(g, f))(z) := \log \frac{1}{[g(z), f(z)]} : \hat{\mathbb{C}} \to [0, \infty],$$

and the *mean proximity* by

$$m(g, f) := \int_{\hat{\mathbb{C}}} w(g, f) d\sigma \in [0, \infty).$$

Let $\mathcal{F}$ be a *rational sequence* $\{f_k\}_{k=0}^{\infty} \subset \mathrm{Rat}$ with increasing degrees $\{d_k := \deg f_k\}$. For $g \in \mathrm{Rat}$, the dynamical *Nevanlinna* and *Valiron exceptionalities* are defined by

$$\mathrm{NE}(g;\mathcal{F}) := \liminf_{k\to\infty} \frac{m(g, f_k)}{d_k} \in [0,\infty],$$
$$\mathrm{VE}(g;\mathcal{F}) := \limsup_{k\to\infty} \frac{m(g, f_k)}{d_k} \in [0,\infty]$$

respectively.

From now on, we consider the *iteration* sequence $\{f^k\}_{k=1}^{\infty}$ of a rational map f of degree $d \geq 2$.

Definition 5.4 (dynamical Nevanlinna and Valiron exceptional sets). The dynamical *Nevanlinna* and *Valiron* exceptional sets of f in $\hat{\mathbb{C}}$ are defined by

$$E_N(f) := \{p \in \hat{\mathbb{C}}; \mathrm{NE}(p;\{f^k\}) > 0\},$$
$$E_V(f) := \{p \in \hat{\mathbb{C}}; \mathrm{VE}(p;\{f^k\}) > 0\}$$

respectively.

It is known that $\{(f^k)^*\sigma/d^k\}$ converges weakly. The limit is also known as the unique maximal entropy measure (see [9] and [10]).

Definition 5.5 (the maximal entropy measure).

$$\mu_f := \lim_{k\to\infty} \frac{(f^k)^*\sigma}{d^k}.$$

Definition 5.6 (accumulation and convergence loci). The *accumulation* and *convergence* loci of the averaged value distributions of f in $\hat{\mathbb{C}}$ are defined by

$$A(f) := \{p \in \hat{\mathbb{C}};\ \text{a subsequence of } \{\mu(f^k, p)/d^k\} \text{ converges to } \mu_f\},$$
$$\mathrm{Conv}(f) := \{p \in \hat{\mathbb{C}}; \lim_{k\to\infty} \frac{\mu(f^k, p)}{d^k} = \mu_f\}$$

respectively.

Now we state Main Theorem.

Main Theorem 4 (characterizations of exceptional sets). *For $f \in \mathrm{Rat}$ of degree ≥ 2,*

$$\hat{\mathbb{C}} - E_V(f) = \mathrm{Conv}(f) \subset A(f) = \hat{\mathbb{C}} - E_N(f) \subset \hat{\mathbb{C}} - E(f).$$

Independently, known is the following remarkable theorem which was first proved for polynomials by Brolin [2] and later for rational maps by Lyubich [9] and independently by Freire-Lopes-Mañé [7]. See also [5], [8], [6] for the other proofs.

Theorem 5.1 (convergence of averaged value distributions). *For $f \in \mathrm{Rat}$ of degree ≥ 2,*

$$\hat{\mathbb{C}} - E(f) = \mathrm{Conv}(f).$$

Combining them, we have the following.

Corollary (All exceptional sets are same.). *For $f \in \mathrm{Rat}$ of degree ≥ 2,*

$$E_N(f) = E_V(f) = E(f) = \hat{\mathbb{C}} - \mathrm{Conv}(f) = \hat{\mathbb{C}} - A(f).$$

Remark 5.1. In [12], Main Corollary has been already implicitly applied to the Siegel-Cremer linearizability problem of rational maps.

The important consequence of Main Corollary is a convergence theorem of the *potentials* of the averaged value distributions.

Definition 5.7 (spherical potential). For a regular measure μ on $\hat{\mathbb{C}}$, the *potential* is defined by

$$V_\mu := \int_{\hat{\mathbb{C}}} -\log[\cdot, w]\mu(w) : \hat{\mathbb{C}} \to [0, \infty].$$

Remark 5.2. In the potential theory, the potential is usually defined as $-V_\mu$, but the definition will be more convenient in our study.

The (axiomatic) potential theory implies that when regular measures μ_k converges to μ, then

$$\liminf_{k\to\infty} V_{\mu_k} = V_\mu$$

quasieverywhere on $\hat{\mathbb{C}}$. For the averaged value distributions, we obtain the stronger conclusion.

Main Theorem 5 (convergence theorem of potentials). *Let $f \in \mathrm{Rat}$ be of degree $d \geq 2$. If $p \in \hat{\mathbb{C}} - E(f)$ is not a fixed point, then*

$$\liminf_{k\to\infty} V_{\mu(f^k,p)/d^k} = V_{\mu_f} \tag{4}$$

on $\hat{\mathbb{C}}$. Otherwise (4) *holds on $\hat{\mathbb{C}} - \bigcup_{k>0} f^{-k}(p)$.*

We also characterize such points that the potentials actually *converge* there.

Main Theorem 6 (convergence of potentials and pointwise behavior). *Let $f \in$ Rat be of degree $d \geq 2$. For $p \in \hat{\mathbb{C}} - E(f)$ and $q \in \hat{\mathbb{C}}$,*

$$\lim_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) = V_{\mu_f}(q) \tag{5}$$

if and only if

$$\lim_{k\to\infty} \frac{1}{d^k} \log \frac{1}{[p, f^k(q)]} = 0. \tag{6}$$

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