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<th>Title</th>
<th>MOSER'S QUESTION ON A SIMULTANEOUS APPROXIMATION OF A SET OF NUMBERS AND A SIMULTANEOUS NORMAL FORMS OF MAPS (Diophantine phenomena in differential equations and dynamical systems)</th>
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</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yoshino, Masafumi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1377: 92-101</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25640">http://hdl.handle.net/2433/25640</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Moser's Question on a Simultaneous Approximation of a Set of Numbers and a Simultaneous Normal Forms of Maps

Masafumi Yoshino
Graduate School of Sciences
Hiroshima University

1. Introduction

In the paper [5] J. Moser studied the following problem. Let \( f_\nu \), \( \nu = 1, \ldots, d \) be the germs of commuting holomorphic functions \((\mathbb{C},0)\) satisfying

\[
\begin{align*}
(1.1) & \quad f_\nu \circ f_\mu = f_\mu \circ f_\nu, \quad \nu, \mu = 1, \ldots, d, \\
(1.2) & \quad f_\nu(0) = 0, \quad f_\nu'(0) \equiv \lambda_\nu = e^{2\pi i \alpha_\nu}, \quad \nu = 1, \ldots, d.
\end{align*}
\]

We want to seek a holomorphic function \( u(z) \) such that

\[
(1.3) \quad u(0) = 0, \ u'(0) = 1, \ (u^{-1} \circ f_\nu \circ u)(z) = \lambda_\nu z, \ \nu = 1, \ldots, d.
\]

Following Haeflinger [2] and Banghe-Haeflinger [1] the commuting example appears as a holonomy group of codimension one foliation.

In the case of a single map with \( \alpha_1 = \theta \) the following theorem is well known.

**Theorem 1.** (Siegel) If there exist \( C > 0 \) and \( \tau > 0 \) such that

\[
(1.4) \quad \| \theta q \| := \inf_{p \in \mathbb{Z}} | \theta q - p | \geq C q^{-\tau}, \forall q \geq 2, q \in \mathbb{Z}
\]

there exists a unique holomorphic solution \( u(z) \) such that

\[
(1.5) \quad u(0) = 0, \ u'(0) = 1, \ u(e^{2\pi i \theta} z) = f(u(z)).
\]

The difficult part of the proof of this theorem lies in proving the convergence of the formal power series solution \( u \) of the so-called homology equation. The condition (1.4) is a sufficient condition in order to show the convergence of the formal power series solution. On the other hand

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1Supported by Grant-in-Aid for Scientific Research (No. 11640183), Ministry of Education, Science and Culture, Japan and GNAMPA-INDAM, Italy. E-mail: yoshino@math.sci.hiroshima-u.ac.jp
it is a difficult and interesting problem to find a necessary condition for the convergence. We recall a classical result due to Cremer: if

\[
\lim_{k \to \infty} \sup_{\infty} \frac{1}{d^k} \log \left| \frac{1}{\lambda^k - 1} \right| = \infty, \quad d \geq 2, \text{integer}
\]

there exists a divergent formal solution $u$. We note that the left-hand side is expressed by using a Nevalina function. Therefore it is an interesting problem to understand the convergence without a Siegel condition.

We recall two approaches to this problem. The former one is to weaken the Diophantine condition. The typical one is a so-called Bruno condition: there exist $c > 0$ and $\tau > 0$ such that

\[
||\theta q|| \geq \exp \left( -\frac{cq}{(\log(q + 1))^{1+\tau}} \right), \quad q \in \mathbb{Z}_+.
\]

The latter one is to understand from the viewpoint of the symmetry, $\exists h, f \circ h = h \circ f$. Namely, if there exist sufficiently many symmetry then we can linearize our map without a Siegel condition or without any Diophantine condition. This approach is closely related with the work of Moser in [5].

We note that a similar Diophantine phenomena happen in the study of the Goursat problem. This was first noted by J. Leray in [3]. More precisely, let us consider the following Goursat problem.

\[
\frac{\partial^2}{\partial s \partial t} u = 0, \quad u|_{s+t=0} = u_1(s), \quad u|_{\lambda s+t=0} = u_2(s),
\]

where $\lambda \neq 0$ be a complex number and $u_j(s)$ are analytic functions near the origin $s = 0, t = 0$. Here $s$ and $t$ are real or complex variables. The Goursat problem is related with a moving boundary problem for a hyperbolic equation.

A Goursat problem is also related with the Schröder equation as follows. It follows from the equation $\partial_t \partial_s u = 0$ that $u = \exists \phi(t) + \exists \psi(s)$. By the boundary conditions we obtain

\[
\phi(-s) + \psi(s) = u_1(s), \quad \phi(-\lambda s) + \psi(s) = u_2(s).
\]

It follows that

\[
\phi(-\lambda s) - \phi(-s) = u_2(s) - u_1(s) = v(-s).
\]

By setting $s \mapsto -s$ we obtain the Schröder equation

\[
\phi(\lambda s) - \phi(s) = v(s).
\]
It is almost clear that we meet a Diophantine condition if we want to solve (1.11) in a class of analytic functions. Indeed, let

$$
\phi(s) = \sum_{n=1}^{\infty} \phi_n s^n, \quad v(s) = \sum_{n=1}^{\infty} v_n s^n
$$

be the expansions of $\phi$ and $v$, respectively. By inserting the expansions into the equation we obtain

$$
(1.12) \quad \sum_{n=1}^{\infty} \phi_n (\lambda^n - \lambda) s^n = \sum_{n=1}^{\infty} v_n s^n.
$$

Hence, if $\lambda^n - \lambda \neq 0$ ($n = 1, 2, \ldots$) we can construct a formal solution. As to the convergence of a formal power series solution we need a Diophantine condition.

By a similar argument as in the above we can prove

**Theorem 2.** (Leray) If

$$
(1.13) \quad \rho(\lambda) := \limsup_{k \to \infty} \frac{1}{k} \log \frac{1}{|\lambda^k - 1|} < \infty
$$

(1.8) has a unique analytic solution for any $u_j(s)$.

We call $\rho(\lambda)$ a Leray-Pisot function. (cf. [4]). The necessary part is given by

**Theorem 3.** (cf. [8]) If $\rho(\lambda) = \infty$ then there exist $u_1$ and $u_2$ such that (1.8) has a formal power series solution $u$ which does not converge in any neighborhood of the origin.

Hence it may happen that one can weaken the Cremer's condition for the divergence of a formal power series solution. Leray's result implies us this may be case since Goursat problem is closely related with Schröder's equation, a linearized homology equation.

If we consider the Goursat problem for third order equation we find that the Leray-Pisot function of two variables

$$
(1.14) \quad \rho(\lambda, \mu) := \limsup_{k \to \infty} \frac{1}{k} \log \frac{1}{|\lambda^k - 1| + |\mu^k - 1|}
$$

plays the same role as $\rho(\lambda)$ in the case of second order equation. In fact, the condition $\rho(\lambda, \mu) > 0$ is necessary and sufficient for the unique local solvability in some neighborhood of the origin for any right-hand side and any boundary conditions, while if $\rho(\lambda, \mu) = 0$ we have a divergence of a formal power series solution.
2. Statement of the results

Simultaneous Diophantine condition. We say that the set of numbers \( \alpha_j \ (j = 1, \ldots, d) \) satisfies a simultaneous Diophantine condition if there exist \( \exists C > 0 \) and \( \exists \tau > 0 \) such that

\[
\max_{\nu=1,\ldots,d} ||q\alpha_{\nu}|| \geq C q^{-\tau}, q = 1, 2, 3, \ldots,
\]

where

\[
||q\alpha_{\nu}|| = \min_{p \in \mathbb{Z}} |q\alpha_{\nu} - p|.
\]

This condition is weaker than the so-called simultaneous Siegel condition:

\[
\exists C > 0, \exists \tau > 0; ||q\alpha_{\nu}|| \geq C q^{-\tau}, \nu = 1, \ldots, d, q = 1, 2, \ldots.
\]

We say that \( \beta \) is a Liouville number if, for every \( \lambda > 0 \) there exist infinitely many integers \( q \in \mathbb{Z} \) such that

\[
0 < ||q\beta|| < q^{-\lambda}.
\]

Moser's question. Given the germs of commuting holomorphic functions \((\mathbb{C}, 0), f_\nu(z), \nu = 1, \ldots, d\) satisfying (1.1) and (1.3). We consider

\[
f(z) := f_1(z)^{g_1} \circ \cdots \circ f_d(z)^{g_d}, \quad g_1, \ldots, g_d \in \mathbb{Z}.
\]

Suppose that \( \alpha_j \ (j = 1, \ldots, d) \) satisfy the simultaneous Diophantine condition. Then Moser asked whether there exist \( g_1, \ldots, g_d \in \mathbb{Z} \) such that \( f(z) \) satisfies a Diophantine condition. If this is the case, the linearization problem in a commuting case is reduced to the case of a single map, hence to Siegel's theorem. The answer to this question is negative. In fact, Moser proved:

**Theorem 4.** (Moser) For \( d \geq 2 \) and a given \( \tau > 2/(d-1) \) there exists a set of cardinality of \((\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \) such that the simultaneous Diophantine condition holds, but such that, for all \( g = (g_1, \ldots, g_d) \in \mathbb{Z}^d \setminus 0 \)

\[
r := g_1\alpha_1 + \cdots + g_d\alpha_d
\]

are Liouville numbers (i.e., non Diophantine).

In [5], Moser raised the question whether this theorem can be extended to case where \( \alpha_j \ (j = 1, \ldots, d) \) are \( n \)-dimensional vectors, \( \alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n}) \). More precisely we consider a commuting system of maps

\[
f_\nu : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0), f_\nu(z) = A_\nu z + O(z^2), \nu = 1, \ldots, d.
\]
Let $\lambda_j^\nu$, $(j = 1, \ldots, n)$ be the eigenvalues of $A_\nu$ with multiplicity, $(\nu = 1, \ldots, d)$. We write

$$
\lambda_j^\nu = \exp(2\pi i \theta_j^\nu), \quad 0 \leq \theta_j^\nu \leq 1,
$$

and set $\theta^\nu = (\theta_1^\nu, \ldots, \theta_n^\nu)$. We define

$$
\langle \alpha, \theta^\nu \rangle := \sum_{j=1}^{n} \alpha_j \theta_j^\nu, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n.
$$

We say that $\{\theta^\nu\}_{\nu=1}^{d}$ satisfies a simultaneous Diophantine condition if there exist $C > 0$ and $\tau > 0$ such that

$$
\min_{k=1,\ldots,n} \sum_{\nu=1}^{d} \|\langle \alpha, \theta^\nu \rangle - \theta_k^\nu\| \geq C|\alpha|^{-\tau}, \quad \forall |\alpha| \geq 2, \alpha \in \mathbb{Z}_+^n,
$$

where $\|t\| = \inf_{p \in \mathbb{Z}} |t - p|$.

Let $p_\nu \in \mathbb{Z}$, $(\nu = 1, \ldots, d)$ and set

$$
\delta_j = \sum_{\nu=1}^{d} \theta_j^\nu p_\nu, \quad \delta = (\delta_1, \ldots, \delta_n).
$$

We say that $\delta$ is a Liouville vector, if for every $\lambda > 0$ the inequality

$$
0 < \min_{k=1,\ldots,n} \|\langle \alpha, \delta \rangle - \delta_k\| < |\alpha|^{-\lambda}
$$

holds for infinitely many $\alpha \in \mathbb{Z}_+^n$. Note that $\delta$ gives the eigenvalues of a map $f = f_1^{p_1} \circ \cdots \circ f_d^{p_d}$. Then we have

**Theorem 5.** Suppose that $d > n \geq 2$. Then there exists a set of linearly independent vectors $\theta_j = (\theta_j^1, \ldots, \theta_j^d)$ $(j = 1, \ldots, n)$ with the density of continuum satisfying a simultaneous Diophantine condition for which, for any $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d \setminus 0$ the $\delta = (\delta_1, \ldots, \delta_n)$, $\delta_j = \sum_{\nu=1}^{d} \theta_j^\nu p_\nu$ is a Liouville vector.

We note that $f_\nu(z)$, $\nu = 1, \ldots, d$ satisfies a simultaneous Diophantine condition while, for any $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d$ $f := f_1^{p_1} \circ \cdots \circ f_d^{p_d}$ does not satisfy a Diophantine condition.

**3. Sketch of the proof**

We will give the sketch of the proof of Theorem 5. We need lemmas in [5]. (For the detail, see [5]). Let $E^n \subset \mathbb{R}^d$ be a real subspace in $\mathbb{R}^d$. With the standard Euclidean norm $| \cdot |$ in $\mathbb{R}^n$ we define

$$
\text{dist}(x, E^n) = \min_{y \in E^n} |x - y|, \quad x \in \mathbb{R}^n.
$$
**Definition.** We define \( \mu := \mu(E^n) \) as the supremum of the numbers \( \lambda \) for which

\[
\text{dist}(j, E^n) < |j|^{-\lambda}, \quad j \in \mathbb{Z}^d
\]

possesses infinitely many solutions. Here \( \mu = \infty \) is admitted.

Clearly, the definition is independent of the norm. Note that, if \( \mathbb{Z}^d \cap E^n = \{0\} \) and \( \tau > \mu \) then there exists a positive constant \( c \) such that

\[
\text{dist}(j, E^n) \geq c|j|^{-\tau}, \quad \text{for all } j \in \mathbb{Z}^d \setminus \{0\}.
\]

A subspace \( E^n \) satisfying \( \mathbb{Z}^d \cap E^n = \{0\} \) and (3.2) is called a Diophantine subspace with respect to \( \mathbb{Z}^d \). The following theorem is given in Moser [Theorem 2.1, 5]. (See also [6]).

**Theorem.** For almost all \( E^n \) in the Grassmann manifold \( G_n(\mathbb{R}^d) \) one has \( \mu(E^n) = \frac{n}{d-n} \).

**Proof of Theorem 5.** Let us assume that there exists a subspace \( E^n \) in \( \mathbb{R}^d \) generated by the linearly independent vectors \( \theta_j = (\theta_j^1, \ldots, \theta_j^d) \), \( (j = 1, \ldots, n) \) such that \( \mu(E^n) = \frac{n}{d-n} \). Let \( \tau \) be such that \( \tau > \frac{n}{d-n} \). Then we have (3.2). We consider the left-hand side of (2.8)

\[
\min_{1 \leq k \leq n} \sum_{\nu=1}^{d} ||(\alpha, \theta^\nu) - \theta_k^\nu|| = \min_{1 \leq k \leq n} \sum_{\nu=1}^{d} \inf_{\nu \in \mathbb{Z}} |(\alpha, \theta^\nu) - \theta_k^\nu - p|.
\]

We set

\[
y = y_k = ((\alpha, \theta^\nu) - \theta_k^\nu)_{\nu=1}^{d} \in E^n, \quad k = 1, \ldots, n.
\]

Let \( j = (p_\nu)_{\nu=1}^{d} \in \mathbb{Z}^d \) be a multiinteger for which the infimum in the right-hand side of (3.3) is taken. Then the right-hand side of (3.3) is bounded from the below by \( c_1 \min_{1 \leq k \leq n} |j - y_k| \) for some positive constant \( c_1 \) independent of \( j \) and \( k \). By the inequality \( |j - y_k| \geq \text{dist}(j, E^n) \) for \( k = 1, \ldots, n \) and (3.2) we can estimate the right-hand side of (3.3) from the below in the following way

\[
\geq c_1 \min_{1 \leq k \leq n} |j - y_k| \geq c_1 \text{dist}(j, E^n) \geq c_2 |j|^{-\tau},
\]

for some positive constant \( c_2 \) independent of \( j \). Because the infimum in (3.2) is taken for \( j \) such that \( |j - y_k| \leq M|y_k| \) for some constant \( M \) independent of \( k \), we obtain, by the condition \( |\alpha| \geq 2 \)

\[
|j| \leq (1 + M)|y_k| \leq c'(1 + |\alpha|) \leq c''|\alpha|
\]

for some positive constants \( c' \) and \( c'' \). It follows that the right-hand side of (3.3) is bounded from the below by \( c|\alpha|^{-\tau} \) for some positive constant \( c \) independent of \( \alpha \). This proves (2.8).
We want to show that there exists $E^n$ satisfying $\mu(E^n) = \frac{n}{d-n}$ and the Liouville property (2.10) for any $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d \setminus 0$. For the detail we refer to [10].

4. COMMUTING SYSTEM OF VECTOR FIELDS

In the case of a commuting vector fields the situation is completely different from the case of maps. For the sake of simplicity, let us consider a system of holomorphic commuting system of vector fields $\mathcal{X}_\nu (\nu = 1, \ldots, d)$, $[\mathcal{X}_\nu, \mathcal{X}_\mu] = 0$ $(\nu, \mu = 1, \ldots, n)$ which are singular at the origin. With a standard coordinate in $\mathbb{C}^n$ we write $\mathcal{X}_\mu = \sum_{j=1}^{n} X_j^\mu(x) \partial_{x_j}$ $(\mu = 1, \ldots, d)$.

Define $X^\mu := (X_1^\mu, \ldots, X_n^\mu)$ and $\Lambda^\mu := \nabla x X^\mu(0)$. Note that $x \Lambda^\mu$ is the linear part of $X^\mu$. We assume that $\mathcal{X}$ is singular at the origin. Hence we can write

(4.1)
$$X^\mu(x) := X^\mu = (X_1^\mu(x), \ldots, X_n^\mu(x)) = x \Lambda^\mu + R^\mu(x), \quad 1 \leq \mu \leq d,$$

where $R^\mu(x)$ is analytic in $x$ in some neighborhood of the origin such that

(4.2)
$$R^\mu(0) = \partial x R^\mu(0) = 0, \quad 1 \leq \mu \leq d.$$

Let $\lambda_j^\mu (j = 1, \ldots, n, \mu = 1, \ldots, d)$ be the eigenvalues with multiplicities of $\Lambda$. We set $\lambda^\mu = (\lambda_1^\mu, \ldots, \lambda_n^\mu)$, $(\mu = 1, \ldots, d)$. For a multiinteger $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ we set $\langle \lambda^\nu, \alpha \rangle = \sum_{j=1}^{n} \lambda_j^\nu \alpha_j$ and define

(4.3)
$$\omega(\alpha) = \min_{1 \leq j \leq n} \sum_{\nu=1}^{d} |\langle \lambda^\nu, \alpha \rangle - \lambda_j^\nu|. $$

**Definition.** We say that $\mathcal{X} := \{\mathcal{X}_\nu; \nu = 1, \ldots, d\}$ is non simultaneously resonant if $\omega(\alpha) \neq 0$ for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geq 2$. The set of $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geq 2$ such that $\omega(\alpha) = 0$ is called a simultaneous resonance of $\mathcal{X}$.

**Definition.** Let $\omega_k$ $(k = 2, 3, \ldots)$ be given by

(4.4)
$$\omega_k = \inf \{\omega(\alpha); \omega(\alpha) \neq 0, \alpha \in \mathbb{Z}_+^n, 2 \leq |\alpha| < 2^k \}.$$

We say that the system $\mathcal{X}$ satisfies a simultaneous Siegel condition, a simultaneous Bruno type condition and a simultaneous Bruno condition respectively if,

$$\omega_k \geq C(1 + 2^k)^{-\tau},$$
$$\omega_k \geq \exp(-C2^k/(k+1)^{1+\tau}),$$
for some constants $C > 0$ and $\tau > 0$ independent of $k$, and
\[- \sum_{k=2}^{\infty} \ln \omega_k / 2^k < \infty.\]

In the case $d = 1$ we say that the vector field $\mathcal{X} = \mathcal{X}_1$ satisfies a Siegel condition, a Bruno type condition and a Bruno condition, respectively if the corresponding simultaneous condition is verified. Then we have

**Theorem 6.** The system $\mathcal{X}_v$ $(v = 1, \ldots, d)$ satisfies one of a simultaneous Siegel condition, a simultaneous Bruno condition and a simultaneous Bruno type condition if and only if there exist numbers $c_v (v = 1, \ldots, d)$ such that the following conditions are satisfied:

(i) the vector field $\mathcal{X}_0 := \sum_{v=1}^{d} c_v \mathcal{X}_v$ satisfies a Siegel condition, a Bruno condition and a Bruno type condition, respectively.

(ii) the resonance of $\mathcal{X}_0$ coincides with the simultaneous resonance of the system $\mathcal{X}_v$ $(v = 1, \ldots, d)$.

We note that the case of vector fields shows a sharp contrast to that of maps. Because we can choose a Diophantine vector field from the Lie algebra generated by a system of vector fields if the given system satisfies a simultaneous Diophantine condition.

5. **SKETCH OF THE PROOF**

We will give a sketch of the proof of Theorem 6. We will show the necessity of (i) and (ii). We note that the commutativity of $\mathcal{X}_v$ implies that the linear parts of $\mathcal{X}_v$ are pairwise commuting. Without loss of generality we may assume that the linear part $A_1$ of $\mathcal{X}_1$ is put in a Jordan normal form.

Let $c_1, \ldots, c_d$ be complex numbers. By the commutativity, the eigenvalues of the linear part of $\mathcal{X}_0 := \sum_{v=1}^{d} c_v \mathcal{X}_v$ are given by $\sum_{v=1}^{d} c_v \lambda_j^v$ $(j = 1, \ldots, n)$. For $c = (c_1, \ldots, c_d) \in \mathbb{C}_{+}^d$ and $\alpha \in \mathbb{Z}_+^n$ we define

\[(5.1) \quad \Omega(\alpha, c) = \min_{1 \leq j \leq n} \left| \sum_{\nu=1}^{d} c_{\nu} (\langle \alpha, \lambda_j^\nu \rangle - \lambda_j^\nu) \right|.
\]

Let $\omega(\alpha)$ and $\omega_k$ be given by (4.3) and the definition in the above, respectively. Then we define

\[(5.2) \quad A_k = \{ c = (c_1, \ldots, c_d) \in \mathbb{C}_{+}^d; \exists \alpha \in \mathbb{Z}_+^n, 2 \leq |\alpha| < 2^k \text{ such that } \omega(\alpha) \neq 0, \Omega(\alpha, c) < 2^{-nk-k} \omega_k \}. \]
We can easily show that the Lebesgue measure of the set $A := \lim_{k \to \infty} A_k$ is equal to zero. Therefore, if $c \notin A$ there exists $k_0 \geq 1$ such that

$$\Omega(\alpha, c) > \omega_k 2^{-n-k}, \quad \forall k \geq k_0.$$  

This proves that $\mathcal{X}_0$ satisfies a Siegel, a Bruno type and a Bruno condition, respectively.

In order to show (ii) we note that if $\alpha$ is not in a simultaneous resonance set of $\mathcal{X}_\nu (\nu = 1, \ldots , d)$, the set of $c \in \mathbb{C}^n$ such that $\sum_{\nu=1}^{d} c_\nu (\langle \alpha, \lambda^\nu \rangle - \lambda_{j}^\nu) = 0$ is a hyperplane for each $j$. The Lebesgue measure of the sum of these hyperplanes is zero. By adding $A$ to the sum of these hyperplanes we can choose $c \notin A$ such that the resonance of $\mathcal{X}_0$ is equal to the simultaneous resonance of $\mathcal{X}_\nu (\nu = 1, \ldots , d)$.

We will prove the sufficiency. We define $\tilde{\omega}(\alpha)$ by

$$\tilde{\omega}(\alpha) = \min_j |\langle \alpha, \sum_{\nu} c_\nu \lambda^\nu \rangle - \sum_{\nu} c_\nu \lambda_{j}^\nu|.$$  

We also define $\tilde{\omega}_k$ by (4.4) with $\omega(\alpha)$ replaced by $\tilde{\omega}(\alpha)$. We can easily show that $\tilde{\omega}(\alpha) \leq M \omega(\alpha)$ for some $M > 0$ independent of $\alpha$. It follows from the assumption (ii) that $\tilde{\omega}_k \leq M \omega_k$. This implies that if $\mathcal{X}_0$ satisfies a Siegel condition (or Bruno type condition) the system $\mathcal{X}$ also satisfies a simultaneous Siegel and Bruno type condition, respectively. Now, let us assume that $\mathcal{X}_0$ satisfies a Bruno condition. Because $\ln \tilde{\omega}_k < \ln M + \ln \omega_k$, it follows that $-\sum_k \ln \tilde{\omega}_k/2^k > -\sum_k (\ln M + \ln \omega_k)/2^k$. Hence $\mathcal{X}$ satisfies a simultaneous Bruno condition. This ends the proof.

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Present address: Graduate School of Sciences, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan. e-mail: yoshino@math.sci.hiroshima-u.ac.jp

This work is supported by Grant-in-Aid for Scientific Research (No. 14340042), Ministry of Education, Science and Culture, Japan.