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MOSER'S QUESTION ON A SIMULTANEOUS APPROXIMATION OF A SET OF NUMBERS AND A SIMULTANEOUS NORMAL FORMS OF MAPS

1. INTRODUCTION

In the paper [5] J. Moser studied the following problem. Let $f_{\nu}$, $\nu = 1, \ldots, d$ be the germs of commuting holomorphic functions $(\mathbb{C}, 0)$ satisfying

\begin{align}
(1.1) & \quad f_{\nu} \circ f_{\mu} = f_{\mu} \circ f_{\nu}, \quad \nu, \mu = 1, \ldots, d, \\
(1.2) & \quad f_{\nu}(0) = 0, \quad f'_{\nu}(0) \equiv \lambda_{\nu} = e^{2\pi i \alpha_{\nu}}, \quad \nu = 1, \ldots, d.
\end{align}

We want to seek a holomorphic function $u(z)$ such that

\begin{equation}
(1.3) \quad u(0) = 0, \quad u'(0) = 1, (u^{-1} \circ f_{\nu} \circ u)(z) = \lambda_{\nu} z, \quad \nu = 1, \ldots, d.
\end{equation}

Following Haeflinger [2] and Banghe-Haeflinger [1] the commuting example appears as a holonomy group of codimension one foliation.

In the case of a single map with $\alpha_{1} = \theta$ the following theorem is well known.

**Theorem 1.** (Siegel) If there exist $C > 0$ and $\tau > 0$ such that

\begin{equation}
(1.4) \quad \|q\| := \inf_{p \in \mathbb{Z}} |q - p| \geq Cq^{-\tau}, \quad \forall q \geq 2, \quad q \in \mathbb{Z}
\end{equation}

there exists a unique holomorphic solution $u(z)$ such that

\begin{equation}
(1.5) \quad u(0) = 0, \quad u'(0) = 1, \quad u(e^{2\pi i \theta} z) = f(u(z)).
\end{equation}

The difficult part of the proof of this theorem lies in proving the convergence of the formal power series solution $u$ of the so-called homology equation. The condition (1.4) is a sufficient condition in order to show the convergence of the formal power series solution. On the other hand

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it is a difficult and interesting problem to find a necessary condition for the convergence. We recall a classical result due to Cremer: if

\begin{equation}
\limsup_{k \to \infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|} = \infty, \quad d \geq 2, \text{integer}
\end{equation}

there exists a divergent formal solution $u$. We note that the left-hand side is expressed by using a Nevalina function. Therefore it is an interesting problem to understand the convergence without a Siegel condition.

We recall two approaches to this problem. The former one is to weaken the Diophantine condition. The typical one is a so-called Bruno condition: there exist $c > 0$ and $\tau > 0$ such that

\begin{equation}
||\theta q|| \geq \exp \left( -\frac{cq}{(\log(q + 1))^{1+\tau}} \right), \quad q \in \mathbb{Z}_{+}.
\end{equation}

The latter one is to understand from the viewpoint of the symmetry, $\exists h, f \circ h = h \circ f$. Namely, if there exist sufficiently many symmetry then we can linearize our map without a Siegel condition or without any Diophantine condition. This approach is closely related with the work of Moser in [5].

We note that a similar Diophantine phenomena happen in the study of the Goursat problem. This was first noted by J. Leray in [3]. More precisely, let us consider the following Goursat problem.

\begin{equation}
\frac{\partial^2}{\partial s \partial t} u = 0, \quad u|_{s+t=0} = u_1(s), \quad u|_{\lambda s+t=0} = u_2(s),
\end{equation}

where $\lambda \neq 0$ be a complex number and $u_j(s)$ are analytic functions near the origin $s = 0, t = 0$. Here $s$ and $t$ are real or complex variables. The Goursat problem is related with a moving boundary problem for a hyperbolic equation.

A Goursat problem is also related with the Schröder equation as follows. It follows from the equation $\partial_t \partial_s u = 0$ that $u = \exists \phi(t) + \exists \psi(s)$. By the boundary conditions we obtain

\begin{equation}
\phi(-s) + \psi(s) = u_1(s), \quad \phi(-\lambda s) + \psi(s) = u_2(s).
\end{equation}

It follows that

\begin{equation}
\phi(-\lambda s) - \phi(-s) = u_2(s) - u_1(s) \equiv v(-s).
\end{equation}

By setting $s \mapsto -s$ we obtain the Schröder equation

\begin{equation}
\phi(\lambda s) - \phi(s) = v(s).
\end{equation}
It is almost clear that we meet a Diophantine condition if we want to solve (1.11) in a class of analytic functions. Indeed, let

\[ \phi(s) = \sum_{n=1}^{\infty} \phi_n s^n, \quad v(s) = \sum_{n=1}^{\infty} v_n s^n \]

be the expansions of \( \phi \) and \( v \), respectively. By inserting the expansions into the equation we obtain

\[ \sum_{n=1}^{\infty} \phi_n (\lambda^n - \lambda) s^n = \sum_{n=1}^{\infty} v_n s^n. \]

Hence, if \( \lambda^n - \lambda \neq 0 \) \((n = 1, 2, \ldots)\) we can construct a formal solution. As to the convergence of a formal power series solution we need a Diophantine condition.

By a similar argument as in the above we can prove

**Theorem 2.** (Leray) If

\[ \rho(\lambda) := \lim_{k \to \infty} \sup \frac{1}{k} \log \frac{1}{|\lambda^k - 1|} < \infty \]

(1.8) has a unique analytic solution for any \( u_j(s) \).

We call \( \rho(\lambda) \) a Leray-Pisot function. (cf. [4]). The necessary part is given by

**Theorem 3.** (cf. [8]) If \( \rho(\lambda) = \infty \) then there exist \( u_1 \) and \( u_2 \) such that (1.8) has a formal power series solution \( u \) which does not converge in any neighborhood of the origin.

Hence it may happen that one can weaken the Cremer's condition for the divergence of a formal power series solution. Leray's result implies us this may be case since Goursat problem is closely related with Schröder's equation, a linearized homology equation.

If we consider the Goursat problem for third order equation we find that the Leray-Pisot function of two variables

\[ \rho(\lambda, \mu) := \lim_{k \to \infty} \sup \frac{1}{k} \log \frac{1}{|\lambda^k - 1| + |\mu^k - 1|} \]

plays the same role as \( \rho(\lambda) \) in the case of second order equation. In fact, the condition \( \rho(\lambda, \mu) > 0 \) is necessary and sufficient for the unique local solvability in some neighborhood of the origin for any right-hand side and any boundary conditions, while if \( \rho(\lambda, \mu) = 0 \) we have a divergence of a formal power series solution.
2. STATEMENT OF THE RESULTS

Simultaneous Diophantine condition. We say that the set of numbers \( \alpha_j (j = 1, \ldots, d) \) satisfies a simultaneous Diophantine condition if there exist \( \exists C > 0 \) and \( \exists \tau > 0 \) such that

\[
\max_{\nu=1,\ldots,d} \|q\alpha_\nu\| \geq Cq^{-\tau}, q = 1, 2, 3, \ldots,
\]

where

\[
\|q\alpha_\nu\| = \min_{p \in \mathbb{Z}} |q\alpha_\nu - p|.
\]

This condition is weaker than the so-called simultaneous Siegel condition:

\[
\exists C > \exists \tau > 0 ; \|q\alpha_\nu\| \geq Cq^{-\tau}, \nu = 1, \ldots, d, q = 1, 2, \ldots
\]

We say that \( \beta \) is a Liouville number if, for every \( \lambda > 0 \) there exist infinitely many integers \( q \in \mathbb{Z} \) such that

\[
0 < \|q\beta\| < q^{-\lambda}.
\]

Moser's question. Given the germs of commuting holomorphic functions \((\mathbb{C},0), f_\nu(z), \nu = 1, \ldots, d\) satisfying (1.1) and (1.3). We consider

\[
f(z) := f_1(z)^{g_1} \circ \cdots \circ f_d(z)^{g_d}, \quad g_1, \ldots, g_d \in \mathbb{Z}.
\]

Suppose that \( \alpha_j (j = 1, \ldots, d) \) satisfy the simultaneous Diophantine condition. Then Moser asked whether there exist \( g_1, \ldots, g_d \in \mathbb{Z} \) such that \( f(z) \) satisfies a Diophantine condition. If this is the case, the linearization problem in a commuting case is reduced to the case of a single map, hence to Siegel's theorem. The answer to this question is negative. In fact, Moser proved:

**Theorem 4.** (Moser) For \( d \geq 2 \) and a given \( \tau > 2/(d-1) \) there exists a set of cardinality of \((\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \) such that the simultaneous Diophantine condition holds, but such that, for all \( g = (g_1, \ldots, g_d) \in \mathbb{Z}^d \setminus \{0\} \)

\[
r := g_1\alpha_1 + \cdots + g_d\alpha_d
\]

are Liouville numbers (i.e., non Diophantine).

In [5], Moser raised the question whether this theorem can be extended to case where \( \alpha_j (j = 1, \ldots, d) \) are \( n \)-dimensional vectors, \( \alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n}) \). More precisely we consider a commuting system of maps

\[
f_\nu : (\mathbb{C}^n,0) \longrightarrow (\mathbb{C}^n,0), f_\nu(z) = A_\nu z + O(z^2), \nu = 1, \ldots, d.
\]
Let $\lambda_j^\nu$, $(j = 1, \ldots, n)$ be the eigenvalues of $A_\nu$ with multiplicity, $(\nu = 1, \ldots, d)$. We write

\begin{equation}
\lambda_j^\nu = \exp(2\pi i \theta_j^\nu), \quad 0 \leq \theta_j^\nu \leq 1,
\end{equation}

and set $\theta^\nu = (\theta_1^\nu, \ldots, \theta_n^\nu)$. We define

\begin{equation}
\langle \alpha, \theta^\nu \rangle := \sum_{j=1}^{n} \alpha_j \theta_j^\nu, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n.
\end{equation}

We say that $\{\theta^\nu\}_{\nu=1}^{d}$ satisfies a simultaneous Diophantine condition if there exist $C > 0$ and $\tau > 0$ such that

\begin{equation}
\min_{k=1,\ldots,n} \sum_{\nu=1}^{d} \|\langle \alpha, \theta^\nu \rangle - \theta_k^\nu \| \geq C|\alpha|^{-\tau}, \quad \forall |\alpha| \geq 2, \alpha \in \mathbb{Z}_+^n,
\end{equation}

where $\|t\| = \inf_{p \in \mathbb{Z}} |t - p|$.

Let $p_\nu \in \mathbb{Z}$, $(\nu = 1, \ldots, d)$ and set

\begin{equation}
\delta_j = \sum_{\nu=1}^{d} \theta_j^\nu p_\nu, \quad \delta = (\delta_1, \ldots, \delta_n).
\end{equation}

We say that $\delta$ is a Liouville vector, if for every $\lambda > 0$ the inequality

\begin{equation}
0 < \min_{k=1,\ldots,n} \|\langle \alpha, \delta \rangle - \delta_k \| < |\alpha|^{-\lambda}
\end{equation}

holds for infinitely many $\alpha \in \mathbb{Z}_+^n$. Note that $\delta$ gives the eigenvalues of a map $f = f_1^{p_1} \circ \cdots \circ f_d^{p_d}$. Then we have

**Theorem 5.** Suppose that $d > n \geq 2$. Then there exists a set of linearly independent vectors $\theta_j = (\theta_j^1, \ldots, \theta_j^d)$ $(j = 1, \ldots, n)$ with the density of continuum satisfying a simultaneous Diophantine condition for which, for any $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d \setminus 0$ the $\delta = (\delta_1, \ldots, \delta_n)$, $\delta_j = \sum_{\nu=1}^{d} \theta_j^\nu p_\nu$ is a Liouville vector.

We note that $f_\nu(z)$, $\nu = 1, \ldots, d$ satisfies a simultaneous Diophantine condition while, for any $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d$ $f := f_1^{p_1} \circ \cdots \circ f_d^{p_d}$ does not satisfy a Diophantine condition.

3. SKETCH OF THE PROOF

We will give the sketch of the proof of Theorem 5. We need lemmas in [5]. (For the detail, see [5]). Let $E^n \subset \mathbb{R}^d$ be a real subspace in $\mathbb{R}^d$. With the standard Euclidean norm $| \cdot |$ in $\mathbb{R}^n$ we define

\[ \text{dist}(x, E^n) = \min_{y \in E^n} |x - y|, \quad x \in \mathbb{R}^n. \]
Definition. We define $\mu := \mu(E^n)$ as the supremum of the numbers $\lambda$ for which

$$\text{dist}(j, E^n) < |j|^{-\lambda}, \quad j \in \mathbb{Z}^d$$

possesses infinitely many solutions. Here $\mu = \infty$ is admitted.

Clearly, the definition is independent of the norm. Note that, if $\mathbb{Z}^d \cap E^n = \{0\}$ and $\tau > \mu$ then there exists a positive constant $c$ such that

$$\text{dist}(j, E^n) \geq c|j|^{-\tau}, \quad \text{for all } j \in \mathbb{Z}^d \setminus \{0\}.$$  

(3.2)

A subspace $E^n$ satisfying $\mathbb{Z}^d \cap E^n = \{0\}$ and (3.2) is called a Diophantine subspace with respect to $\mathbb{Z}^d$. The following theorem is given in Moser [Theorem 2.1, 5]. (See also [6]).

Theorem. For almost all $E^n$ in the Grassmann manifold $G_n(\mathbb{R}^d)$ one has $\mu(E^n) = \frac{n}{d-n}$.

Proof of Theorem 5. Let us assume that there exists a subspace $E^n$ in $\mathbb{R}^d$ generated by the linearly independent vectors $\theta_j = (\theta_j^1, \ldots, \theta_j^d)$, $(j = 1, \ldots, n)$ such that $\mu(E^n) = \frac{n}{d-n}$. Let $\tau$ be such that $\tau > \frac{n}{d-n}$. Then we have (3.2). We consider the left-hand side of (2.8)

$$\text{dist}(j, E^n) \geq \min_{1 \leq k \leq n} \sum_{\nu=1}^{d} \inf_{p_{\nu} \in \mathbb{Z}} |\langle \alpha, \theta^\nu \rangle - \theta_k^\nu - p_{\nu}|.$$  

(3.3)

We set

$$y = y_k = (\alpha, \theta^\nu) = \theta_k^\nu, \quad (\nu, k) \in \mathbb{Z}^d \setminus \{0\}.$$

Let $j = (p_{\nu})_{\nu=1}^{d} \in \mathbb{Z}^d$ be a multiinteger for which the infimum in the right-hand side of (3.3) is taken. Then the right-hand side of (3.3) is bounded from the below by $c_1 \min_{1 \leq k \leq n} |j - y_k|$ for some positive constant $c_1$ independent of $j$ and $k$. By the inequality $|j - y_k| \geq \text{dist}(j, E^n)$ for $k = 1, \ldots, n$ and (3.2) we can estimate the right-hand side of (3.3) from the below in the following way

$$\geq c_1 \min_{1 \leq k \leq n} |j - y_k| \geq c_1 \text{dist}(j, E^n) \geq c_2 |j|^{-\tau},$$

(3.4)

for some positive constant $c_2$ independent of $j$. Because the infimum in (3.2) is taken for $j$ such that $|j - y_k| \leq M|y_k|$ for some constant $M$ independent of $k$, we obtain, by the condition $|\alpha| \geq 2$

$$|j| \leq (1 + M)|y_k| \leq c'(1 + |\alpha|) \leq c''|\alpha|$$

for some positive constants $c'$ and $c''$. It follows that the right-hand side of (3.3) is bounded from the below by $c|\alpha|^{-\tau}$ for some positive constant $c$ independent of $\alpha$. This proves (2.8).
We want to show that there exists $E^n$ satisfying $\mu(E^n) = \frac{n}{d-n}$ and the Liouville property (2.10) for any $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d \setminus 0$. For the detail we refer to [10].

4. COMMUTING SYSTEM OF VECTOR FIELDS

In the case of a commuting vector fields the situation is completely different from the case of maps. For the sake of simplicity, let us consider a system of holomorphic commuting system of vector fields $\mathcal{X}_\nu$ ($\nu = 1, \ldots, d$), $[\mathcal{X}_\nu, \mathcal{X}_\mu] = 0$ ($\nu, \mu = 1, \ldots, n$) which are singular at the origin. With a standard coordinate in $\mathbb{C}^n$ we write $\mathcal{X}_\mu = \sum_{j=1}^{n} X_j^\mu(x) \partial_{x_j}$ ($\mu = 1, \ldots, d$). Define $X^\mu := (X_1^\mu, \ldots, X_n^\mu)$ and $\Lambda^\mu = \nabla_x X^\mu(0)$. Note that $x \Lambda^\mu$ is the linear part of $X^\mu$. We assume that $\mathcal{X}$ is singular at the origin. Hence we can write

\begin{equation}
X^\mu(x) := X^\mu = (X_1^\mu(x), \ldots, X_n^\mu(x)) = x \Lambda^\mu + R^\mu(x), \quad 1 \leq \mu \leq d,
\end{equation}

where $R^\mu(x)$ is analytic in $x$ in some neighborhood of the origin such that

\begin{equation}
R^\mu(0) = \partial_x R^\mu(0) = 0, \quad 1 \leq \mu \leq d.
\end{equation}

Let $\lambda_j^\nu$ ($j = 1, \ldots, n, \mu = 1, \ldots, d$) be the eigenvalues with multiplicities of $\Lambda^\mu$. We set $\lambda^\mu = (\lambda_1^\mu, \ldots, \lambda_n^\mu)$ ($\mu = 1, \ldots, d$). For a multiinteger $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ we set $\langle \lambda^\nu, \alpha \rangle = \sum_{j=1}^{n} \lambda_j^\nu \alpha_j$ and define

\begin{equation}
\omega(\alpha) = \min_{1 \leq j \leq n} \sum_{\nu=1}^{d} |\langle \alpha, \lambda^\nu \rangle - \lambda_j^\nu|.
\end{equation}

**Definition.** We say that $\mathcal{X} := \{\mathcal{X}_\nu; \nu = 1, \ldots, d\}$ is non simultaneously resonant if $\omega(\alpha) \neq 0$ for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geq 2$. The set of $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geq 2$ such that $\omega(\alpha) = 0$ is called a simultaneous resonance of $\mathcal{X}$.

**Definition.** Let $\omega_k$ ($k = 2, 3, \ldots$) be given by

\begin{equation}
\omega_k = \inf \{\omega(\alpha); \omega(\alpha) \neq 0, \alpha \in \mathbb{Z}_+^n, 2 \leq |\alpha| < 2^k\}.
\end{equation}

We say that the system $\mathcal{X}$ satisfies a simultaneous Siegel condition, a simultaneous Bruno type condition and a simultaneous Bruno condition respectively if,

\begin{equation}
\omega_k \geq C(1 + 2^k)^{-\tau},
\end{equation}

\begin{equation}
\omega_k \geq \exp(-C2^k/(k+1)^{1+\tau}),
\end{equation}

where $C$ is a constant.
for some constants $C > 0$ and $\tau > 0$ independent of $k$, and

$$-\sum_{k=2}^{\infty} \ln \omega_k/2^k < \infty.$$ 

In the case $d = 1$ we say that the vector field $X = X_1$ satisfies a Siegel condition, a Bruno type condition and a Bruno condition, respectively if the corresponding simultaneous condition is verified. Then we have

**Theorem 6.** The system $X_{\nu}$ ($\nu = 1, \ldots, d$) satisfies one of a simultaneous Siegel condition, a simultaneous Bruno condition and a simultaneous Bruno type condition if and only if there exist numbers $c_{\nu}(\nu = 1, \ldots, d)$ such that the following conditions are satisfied:

(i) the vector field $X_0 := \sum_{\nu=1}^{d} c_{\nu} X_{\nu}$ satisfies a Siegel condition, a Bruno condition and a Bruno type condition, respectively.

(ii) the resonance of $X_0$ coincides with the simultaneous resonance of the system $X_{\nu}$ ($\nu = 1, \ldots, d$).

We note that the case of vector fields shows a sharp contrast to that of maps. Because we can choose a Diophantine vector field from the Lie algebra generated by a system of vector fields if the given system satisfies a simultaneous Diophantine condition.

5. **Sketch of the Proof**

We will give a sketch of the proof of Theorem 6. We will show the necessity of (i) and (ii). We note that the commutativity of $X_{\nu}$ implies that the linear parts of $X_{\nu}$ are pairwise commuting. Without loss of generality we may assume that the linear part $A_1$ of $X_1$ is put in a Jordan normal form.

Let $c_1, \ldots, c_d$ be complex numbers. By the commutativity, the eigenvalues of the linear part of $X_0 := \sum_{\nu=1}^{d} c_{\nu} X_{\nu}$ are given by $\sum_{\nu=1}^{d} c_{\nu} \lambda_j^\nu$ ($j = 1, \ldots, n$). For $c = (c_1, \ldots, c_d) \in \mathbb{C}_+^d$ and $\alpha \in \mathbb{Z}_+^n$ we define

$$\Omega(\alpha, c) = \min_{1 \leq j \leq n} \left| \sum_{\nu=1}^{d} c_{\nu}(\alpha, \lambda_j^\nu - \lambda_j^\nu) \right|.$$  

Let $\omega(\alpha)$ and $\omega_k$ be given by (4.3) and the definition in the above, respectively. Then we define

$$A_k = \{ c = (c_1, \ldots, c_d) \in \mathbb{C}_+^d; \exists \alpha \in \mathbb{Z}_+^n, 2 \leq |\alpha| < 2^k\text{ such that } \omega(\alpha) \neq 0, \Omega(\alpha, c) < 2^{-nk-k} \omega_k \}.$$
We can easily show that the Lebesgue measure of the set $A := \lim_{k \to \infty} A_k$ is equal to zero. Therefore, if $c \not\in A$ there exists $k_0 \geq 1$ such that

$$\Omega(\alpha, c) > \omega_k 2^{-nk-k}, \quad \forall k \geq k_0.$$  

This proves that $\mathcal{X}_0$ satisfies a Siegel, a Bruno type and a Bruno condition, respectively.

In order to show (ii) we note that if $\alpha$ is not in a simultaneous resonance set of $\mathcal{X}_\nu \ (\nu = 1, \ldots, d)$, the set of $c \in \mathbb{C}^n$ such that

$$\sum_{\nu=1}^{d} c_\nu (\langle \alpha, \lambda^\nu \rangle - \lambda_j^\nu) = 0$$

is a hyperplane for each $j$. The Lebesgue measure of the sum of these hyperplanes is zero. By adding $A$ to the sum of these hyperplanes we can choose $c \not\in A$ such that the resonance of $\mathcal{X}_0$ is equal to the simultaneous resonance of $\mathcal{X}_\nu \ (\nu = 1, \ldots, d)$.

We will prove the sufficiency. We define $\tilde{\omega}(\alpha)$ by

$$\tilde{\omega}(\alpha) = \min_j \left| \langle \alpha, \sum_\nu c_\nu \lambda^\nu \rangle - \sum_\nu c_\nu \lambda_j^\nu \right|.$$  

We also define $\tilde{\omega}_k$ by (4.4) with $\omega(\alpha)$ replaced by $\tilde{\omega}(\alpha)$. We can easily show that $\tilde{\omega}(\alpha) \leq M \omega(\alpha)$ for some $M > 0$ independent of $\alpha$. It follows from the assumption (ii) that $\tilde{\omega}_k \leq M \omega_k$. This implies that if $\mathcal{X}_0$ satisfies a Siegel condition (or Bruno type condition) the system $\mathcal{X}$ also satisfies a simultaneous Siegel and Bruno type condition, respectively. Now, let us assume that $\mathcal{X}_0$ satisfies a Bruno condition. Because $\ln \tilde{\omega}_k < \ln M + \ln \omega_k$, it follows that $-\sum_k \ln \tilde{\omega}_k/2^k > -\sum_k (\ln M + \ln \omega_k)/2^k$. Hence $\mathcal{X}$ satisfies a simultaneous Bruno condition. This ends the proof.

REFERENCES


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