On Extreme Ways and Curves of Extreme Value of Holomorphic Functions.

By

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(Received March 10, 1915.)

Let $Z=f(z)$ be a holomorphic function of z in a certain region of Gauss' plane, then we can expand the function about any point e.g. z_0 in the region into a power-series of z-z_0:

$$Z-Z_0=a_1(z-z_0)+a_2(z-z_0)^2+\dots,$$

where Z is the value of the function at $z=z_0$. If we write Z for $Z-Z_0$ and z for $z-z_0$, the series becomes

$$Z=a_1z+a_2z^2+\dots.$$

When a_1 is not zero, by the elementary theorem of the Theory of Functions, z can also be expanded as a power-series of Z,

$$z=A_1Z+A_2Z^2+\dots,\qquad A_1\neq 0.$$

Such a function as Z of z, we shall hereafter call a *reversibly holomorphic* function about the origin $z=0$.

Since for such a function, $\frac{dZ}{dz}$ does not vanish about the origin a circle about it is easily proved to be transformed by means of $Z=f(z)$ into a closed regular curve about the origin $Z=0$ of the plane of Z. Therofore when the absolute value $|Z|$ of Z is not equal to a constant, $|Z|$ must take at least a maximum and a minimum value on the circle. The locus of the point z_0 at which $|Z|$ is extreme, when z runs concentric circles, is the object of the present study. In the following we assume $Z=f(z)$, $(f(0)=0)$, to be reversibly holomorphic about the origin; and only this small region is taken into consideration.

1. The locus of the point z_0 at which $|Z|$ takes an extreme value when z describes a circle with the variable radius, whose centre is at

the origin, is defined as an *extreme way* of the function, while the corresponding locus of $Z(z_0)$ in the plane of Z is defined as a *curve of extreme value*. For maximum values, *maximum ways* resp. *curves of maximum value* are used and for minimum values, similar definitions. In the following, the relations between the ways and curves are discussed.

2. As usual in a reversibly holomorphic function, put

$$Z = a_1 z + a_2 z^2 + \dots, \qquad a_1 \neq 0,$$
$$z = x + iy, \; x = r\cos\theta, \qquad y = r\sin\theta,$$
$$Z = X + iY, \; X = R\cos\Theta, \qquad Y = R\sin\Theta.$$

It is known and easily proved that X as well as Y can be expanded into a Taylor's series of x and y about (0|0), and hence of r and θ.

Putting
$$R^2(x,y) = \varphi(r,\theta),$$
$$\varphi(r,\theta+h) = \varphi(r,\theta) + \varphi_\theta'(r,\theta)\frac{h}{1!} + \dots + \varphi_\theta^{(2n-1)}(r,\theta)\frac{h^{2n-1}}{(2n-1)!}$$
$$+ \varphi_\theta^{(2n)}(r,\theta)\frac{h^{2n}}{(2n)!} + \dots .$$

When θ changes from 0 to 2π, with respect to the circle $x^2+y^2=r_0^2$, R, therefore $\varphi(r_0,\theta)$ returns to its original value, hence it must take at least a maximum (resp. a minimum) value for certain θ, say θ_0 and assume that

$$\varphi_\theta'(r_0,\theta_0) = \dots = \varphi_\theta^{(2n-2)}(r_0,\theta_0) = 0,$$

and
$$\varphi_\theta^{(2n-1)}(r_0,\theta_0) = 0, \; \varphi_\theta^{(2n)}(r_0,\theta_0) \neq 0,$$

where n is an integer greater than or equal to 1. Suppose that these relations hold for parametric values of r about r_0, e.g., in an interval I_1 and for corresponding values of θ_0: then consider the function $\varphi^{(2n-1)}(r,\theta)$. This function satisfies the relations

$$\varphi_\theta^{(2n-1)}(r_0,\theta_0) = 0, \; \varphi_\theta^{(2n)}(r_0,\theta_0) \neq 0,$$

hence we obtain a continuous function $\theta = \psi(r)$ in a certain interval about r_0, say I_2. Let I be the smaller of the two intervals I_1 and I_2, then we have for r in I,

$$\varphi_\theta'(r,\psi(r)) = \dots = \varphi_\theta^{(2n-1)}(r,\psi(r)) = 0,$$

$$\varphi_\theta^{(2n)}(r, \psi(r)) \neq 0, \ \theta_0 = \psi(r_0)$$

Now put

$$\varphi^{(2n-1)}(r, \theta) \equiv F(r, \theta),$$

then $\psi(r)$ is the continuous solution for $r=r_0, \theta=\theta_0$ of the differential equation

$$\frac{\partial F}{\partial r} + \frac{\partial F}{\partial \theta}\frac{d\theta}{dr} = 0,$$

where $\frac{\partial F}{\partial r}, \frac{\partial F}{\partial \theta}$ are power-series of r and θ. Moreover $\frac{\partial F}{\partial \theta}$ does not vanish in I. Hence $\theta=\psi(r)$ will be obtained as a power-seiies of r in I. The equation of the extreme way is, therefore,

$$\xi = r\cos\{\psi(r)\}, \ \eta = r\sin\{\psi(r)\}, \text{ about } r_0;$$

and

$$d\xi^2 + d\eta^2 = (1 + r^2\psi'^2)\, dr^2.$$

So the extreme way is a regular curve. From the properties of the reversibly holomorphic function, the corresponding curve of extreme value is also a regnlar curve whose equation is

$$\Xi = R(r, \psi(r)) \cos\{\Theta(r, \psi(r))\},$$
$$H = R(r, \psi(r)) \sin\{\Theta(r, \psi(r))\}.$$

Thus, if our conditions are satisfied. we see that in this case the following relations must be established:

$$\varphi_\theta'(r, \theta) = \{\theta - \psi(r)\}^{2n-1} K_1(r, \theta),$$
$$\dots = \dots$$
$$\varphi_\theta^{(2n-1)}(r, \theta) = \{\theta - \psi(r)\}\, K_{2n-1}(r, \theta),$$

where the functions K do not vanish in the neighborhood of (r_0, θ_0). But if for only the point (r_0, θ_0),

$$\varphi_\theta'(r_0, \theta_0) = \dots = \varphi_\theta^{(2n-1)}(r_0, \theta_0) = 0,$$

while

$$\varphi_\theta^{(2n)}(r_0, \theta_0) \neq 0.$$

Then from the differential equation

$$\varphi_\theta'' + \frac{dr}{d\theta}\frac{\partial\varphi_\theta'(r, \theta)}{\partial r} = 0,$$

we obtain the solution:

$$r-r_0 = A_{2n-1}(\theta-\theta_0)^{2n-1} + A_{2n}(\theta-\theta_0)^{2n} + \dots,$$

provided
$$\frac{\partial \varphi_0'(r_0, \theta_0)}{\partial r} \neq 0;$$

whence it follows that

$$\theta-\theta_0 = B_1(r-r_0)^{\frac{1}{2n-1}} + B_2(r-r_0)^{\frac{2}{2n-1}} + \dots .$$

This solution represents $2n-1$ determinations. If all the coefficients of a certain determination $P(r-r_0)$ for $r>r_0$ be real, then by the analytical continuation about r_0, we can find another determination $Q(r_0-r)$ for $r_0>r$, whose coefficients are all real. Hence before and after r_0, we shall obtain a real extreme way. Therefore when the number of the extreme ways changes, we must have at least

$$\frac{\partial \varphi_n'(r.\theta)}{\partial r} = 0.$$

From such an equation we will obtain in general solutions of the type,

$$\theta-\theta_0 = \theta(t),$$ [1]

$$r-r_0 = r(t).$$

Among these will be given several extreme ways. As illustration consider the following function:

$$Z = z - z^3.$$

The analytic continuation of the function to the point $z=\frac{1}{3}$ gives:

$$Z = \frac{8}{27} + \frac{2}{3}(z-\frac{1}{3}) - (z-\frac{1}{3})^2 - (z-\frac{1}{3})^3.$$

Writing the series as usual:

$$Z = \frac{2}{3}z - z^2 - z^3,$$

reversibly holomorphic about $z=0$, $Z=0$.

Now $$R^2 = \frac{4}{9}r^2 - \frac{4}{3}r^3\cos\theta - \frac{4}{3}r^4\cos 2\theta + 2r^5\cos\theta + r^4 + r^6,$$

1 See e.g., E. Picard: Traite D'analyse III, pp. 36... .

putting $u \equiv \frac{3}{2r^3} R^2$,

$$\frac{\partial u}{\partial \theta} = \sin\theta\,(2+8r\cos\theta-3r^2) = 0.$$

The roots are
$$\theta = 0,\ \pi$$

and
$$\cos\theta = \frac{3r^2-2}{8r}$$

i) when r is sufficiently small, the roots are o and π:

and
$$\left(\frac{\partial^2 u}{\partial \theta^2}\right)_{\theta=0} = 2+8r-3r^2 > 0 \ldots \text{min.},$$

$$\left(\frac{\partial^2 u}{\partial \theta^2}\right)_{\theta=\pi} = -2+8r+3r^2 < 0 \ldots \text{max.},$$

i.e., we have only a maximum way $\theta=\pi$ and a minimum way $\theta=0$.
ii) when r is not so small, adding to the above extreme ways, we have

$$\cos\theta = \frac{3r^2-2}{8r}.$$

From the condition
$$|\cos\theta| \leqq 1,$$

we have
$$3r^2+8r-2 \geqq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$-3r^2+8r+2 \geqq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2);$$

from (1),
$$r \geqq \frac{\sqrt{22}-4}{3} \equiv r_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$\cos\theta_{r_1} = -1,\ \theta_{r_1} = \pi,$$

and from (2),
$$r \leqq \frac{\sqrt{22}+4}{3} \equiv r_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$\cos\theta_{r_2} = +1,\ \theta_{r_2} = 0.$$

By easy calculations,

$$\left(\frac{\partial^2 u}{\partial \theta^2}\right)_{\substack{\theta=\pi\\ r=r_1}} = \left(\frac{\partial^2 u}{\partial \theta^2}\right)_{\substack{\theta=0\\ r=r_2}} = \left(\frac{\partial^3 u}{\partial \theta^3}\right)_{\theta=0,\,\pi} = 0,$$

$$\left(\frac{\partial^4 u}{\partial \theta^4}\right)_{r=r_1,\,r=r_2} \neq 0.$$

These results show, (Fig. 1), that the x-axis is a minimum way from O to Q, ($OQ=r_2$); a maximum way from O to $P(OP=r_1)$; the circle $APBQ$ whose equation is $3r^2-8r\cos\theta-2=0$ represents two maximum ways; PR, a minimum way. At P and Q the number of the extreme ways changes, so we have,

$$\left(\frac{\partial^2 u}{\partial r\,\partial\theta}\right)_{\substack{r=r_1,\ r=r_2\\ \theta=\pi,\ \theta=0}} = 0,$$

moreover we notice that P is a maximum point.

z-plane.

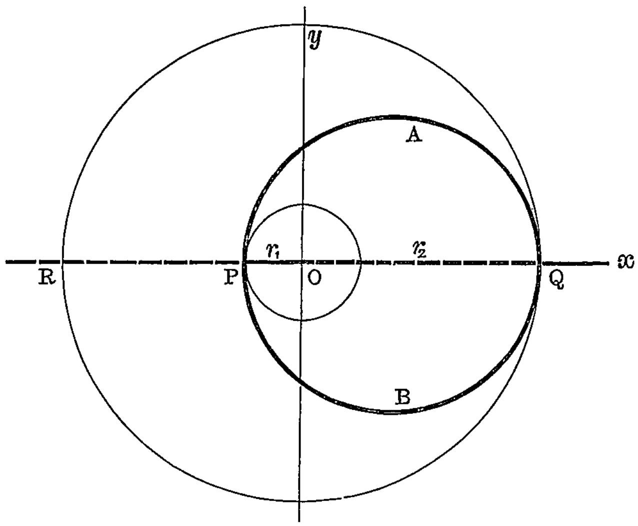

Fig. 1.

3 Let Γ be the curve in Z-plane corresponding to a circle γ in z-plane, ω being the angle between the radius vector and the tangent to Γ, we have

$$\tan\omega = \frac{R d_\theta \Theta}{d_\theta R}.$$

When R is extreme, $d_\theta R$ vanishes; but $d_\theta \Theta$ does not vanish; for if it vanish, from

$$\Theta = \arctan\left(\frac{Y}{X}\right),$$

we must have

$$X\frac{\partial Y}{\partial \theta}-Y\frac{\partial X}{\partial \theta}=0 \quad\dots\dots\dots\dots\dots\dots\dots (1),$$

from $d_\theta R=0$,

$$X\frac{\partial X}{\partial \theta}+Y\frac{\partial Y}{\partial \theta}=0 \quad\dots\dots\dots\dots\dots\dots\dots (2)$$

By (1) and (2)

$$\left(\frac{\partial X}{\partial \theta}\right)^2+\left(\frac{\partial Y}{\partial \theta}\right)^2=0.$$

Hence

$$\frac{\partial X}{\partial \theta}=-r\sin\theta\frac{\partial X}{\partial x}+r\cos\theta\frac{\partial X}{\partial y}=0$$

$$\frac{\partial Y}{\partial \theta}=-r\sin\theta\frac{\partial Y}{\partial x}+r\cos\theta\frac{\partial Y}{\partial y}=0,$$

which leads to the equation:

$$\left(\frac{\partial X}{\partial x}\right)^2+\left(\frac{\partial Y}{\partial x}\right)^2=\left|\frac{dZ}{dz}\right|^2=0,$$

which is impossible. We have $\omega=\frac{\pi}{2}$; i.e., the radius vector R is perpendicular to the curve Γ at the point of an extreme value.

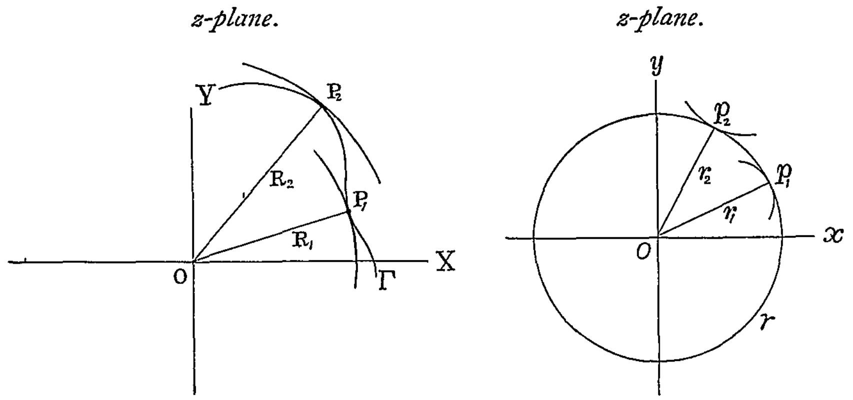

Fig. 2.

Fig. 2 shows that for $z=p_1, Z=P_1, R$ is minimum; for $z=p_2$, $Z=P_2, R$ is maximum. Since R is perpendicular to Γ at P, the part of the circle with centre O, which is drawn to touch the curve Γ at P, lies on one side of Γ, so also is its corresponking curve in z-plane. When P_1 is the point of minimum value, then the part of the circle lies within Γ, touching it; therefore the corresponding curve in z-plane lies within γ and touches it at p_1; i.e., op_1 is maximum in the neighborhood of p_1; the same is true for every point of the locus of P_1. For P_2, op_2 is minimum; hence

a). *The curve of minimum (maximum) value of the function $Z=Z(z)$ is the maximum (minimum) way of the function $z=z(Z)$.*

b) *The minimum (maximum) way of the function $Z=Z(z)$ is the curve of maximum (minimum) value of the function $z=z(Z)$.*

4. Example. $Z=z(2-z)$.

This function is reversibly holomorphic in the circle with radius $=1$ about the origin.

$$R^2 = r^2(4-4r\cos\theta+r^2),$$

$$\frac{\partial R^2}{\partial\theta} = 4r^3\sin\theta = 0,\ \theta = 0 \text{ or } \pi.$$

For

$$\theta = 0,\quad R_0 = r(2-r)\ \dots \text{min.}$$

$$\theta = \pi,\quad R_\pi = r(2+r)\ \dots \text{max.}$$

In Fig. 3 (right), the dotted line is the minimum way; the thick line the other. In Fig 3 (left),, the curve of minimum value is given by the dotted line, the other by the thick line.

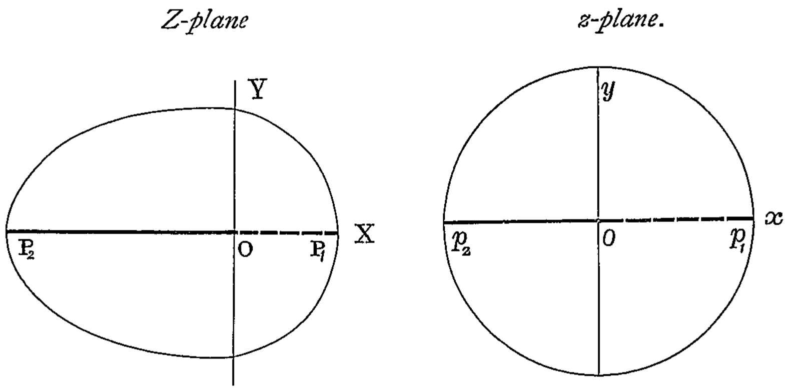

Fig. 4.

The reciprocal function is

$$z = 1-\sqrt{1-Z}\,.$$

When R is sufficiently small (Fig. 4).

For $$\theta = 0,\ r_0 = 1 - \sqrt{1-R} \ldots \text{max.},$$

$$\theta = \pi,\ r_\pi = \sqrt{1+R} - 1 \ .\because \text{min.}$$

op_1 is the curve of maximum value; OP_1, the maximum way.

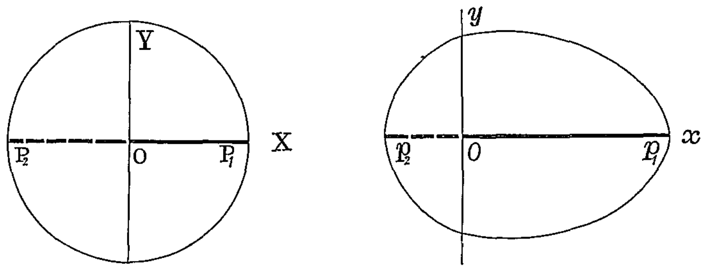

Fig 4.

5. Next we shall find the number of the extreme ways and of the curves of extieme value of the function about the oiigin:

$$Z = a_1 z + a_2 z^2 + \ldots, \quad a \neq 0$$

Put $$a_1 = \rho_1 e^{i\alpha_1}, \ldots a_n = \rho_n e^{i}\alpha_n, \ldots .$$

Then $$X = \sum_{n=1}^{\infty} \rho_h r^h \cos(h\theta + \alpha_h),$$

$$Y = \sum_{k=1}^{\infty} \rho_k r^k \sin(k\theta + \alpha_k)$$

Hence $$XX_\theta' + YY_\theta' = \left\{\sum_{h=1}^{\infty} \rho_h r^h \cos(h\theta + \alpha_h)\right\}\left\{-\sum_{k=1}^{\infty} k\rho_k r^k \sin(k\theta + \alpha_k)\right\}$$

$$+ \left\{\sum_{k=1}^{\infty} \rho_k r^k \sin(k\theta + \alpha_k)\right\}\left\{\sum_{h=1}^{\infty} h\rho_h r^h \cos(h\theta + \alpha_h)\right\}$$

$$= \sum_{h\ k=1}^{\infty} (h-k)\rho_h \rho\ r^{h+k} \cos(h\theta + \alpha_h) \sin(k\theta + \alpha_k)$$

Changing h into k, k into h, and adding we obtain

$$2\,(XX_\theta'+YY_\theta')=\sum_{h,\,k=1}^{\infty}(h-k)\,\rho_h\,\rho_k\,r^{h+k}\sin\left\{(k-h)\,\theta+\alpha_k-\alpha_h\right\}$$

We may divide the double sum into three parts of

i) sum of terms for $h=k$, (equal to zero),

ii) sum of terms for $h<k$,

iii) sum of terms for $h>k$.

If we interchange h and k, the thiıd sum is equal to the second. Therefore we obtain

$$XX_\theta'+YY_\theta'=\sum_{h,\,k=1}^{\infty}(h-k)\,\rho_h\,\rho_k\,r^{h+k}\sin\left\{(k-h)\,\theta+\alpha_k-\alpha_h\right\},$$

where h and k take all positive integeıs $k>h$.

Let α_n be the first coefficient (except α_1) which does not vanish, then

$$XX_\theta'+YY_\theta'=(1-n)\,\rho_1\,\rho_n\,r^{n+1}\sin\{(n-1)\,\theta+\alpha_n-\alpha_1\}+\ldots;$$

for the extreme value, this must vanish even for very small r, therefore in the limit,

$$\sin\{(n-1)\,\theta+\alpha_n-\alpha_1\}=0.$$

The roots of the equation are given by

$$\frac{m\pi+\alpha_1-\alpha_n}{n-1},\quad m=0,1,2,\ldots 2n-3.$$

All the roots give extreme values for small r, since for these values

$$\frac{\partial}{\partial\theta}(XX'+YY_\theta')\neq 0.$$

Hence, *the number of the extreme ways and that of the curves of extreme value are each equal to* $2(n-1)$, *where the nth coefficient is the first one not zero. Moreover by the conformity of the function, the angle between two consecutive ways is equal to that of the corresponding curves of extreme value.*

In Fig. 5, the dotted lines represent the minimum ways resp. the curves of minimum value of the function,

$$Z = z\,(1 - z^4)\,;$$

the thick lines, those of maximum.

$$R\ \text{min.} = r\,(1 - r^4),$$
$$R\ \text{max.} = r\,(1 + r^4).$$

Z-plane. *z-plane.*

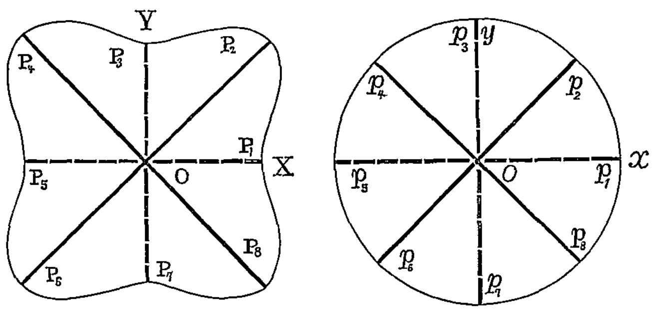

Fig. 5.

Fig. 6 is for the inverse function of the above, approximately. The number of the ways is 8.

Z-plane. *z-plane.*

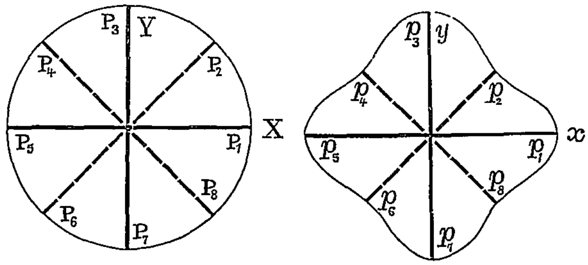

Fig. 6.

6. Let us call a power-series with real coefficients, which is reversible, a *real power-series;* then a *real power-series has the x-axis for an extreme way and the X-axis, for the corresponding curve of extreme value and vice versa.* For such a function is symmetrical with respect to the x- resp. X-axis.

When a reversibly holomorphic function $Z = a_1 z + a_2 z^2 + \ldots$ *has the x-axis for extreme way, then the quotients :*

$$\frac{a_2}{a_1}, \frac{a_3}{a_1} \cdots$$

are all real and vice versa.

For suppose a_n the first coefficient (except a_1) not zero, in the formula of §5 put $\theta = 0$ (or π),

$$XX_\theta' + YY_\theta' = (1-n)\, r^{n+1} \rho_1 \rho_n \sin(\alpha_n - \alpha_1) - nr^{n+2} \rho_1 \rho_{n+1} \sin(\alpha_{n+1} - \alpha_1) + \dots$$

$$= 0.$$

Diminishing r we must have,

$$\alpha_n - \alpha_1 = m_{n,1}\, \pi,$$

wheie $m_{n,1}$ is an integer ; hence

$$\alpha_{n+i} - \alpha_1 = m_{n+i,1}\, \pi, \quad i = 0, 1, 2, \dots n-1, \quad \dots\dots (1)$$

moreover the coefficient of r^{2n+s},

$$(1 - \overline{2n+s-1}) \rho_1 \rho_{2n+s-1} \sin(\alpha_{2n+s-1} - \alpha_1) + (n - \overline{n+s}) \rho_n \rho_{n+s} \sin(\alpha_n - \alpha_{n+s}) + \dots$$

is equal to zero; then by (1), we have

$$\alpha_{n+i} - \alpha_1 = m_{n+i,1}\, \pi,$$

$$\alpha_{n+j} - \alpha_1 = m_{n+j,1}\, \pi$$

Therefore $\alpha_{n+j} - \alpha_{n+i}$ is a multiple of π, hence

$$\alpha_{2n+s-1} - \alpha_1 = m_{2n+s-1,1}\, \pi, \quad s = 1, 2, \dots n-1.$$

Proceeding in this way, we can prove in general

$$\alpha_s - \alpha_1 = m_{s,1}\, \pi,$$

where $m_{s,1}$ is an integer; so that our proposition is proved.

Conversely since

$$\frac{Z}{a_1} = z + \frac{a_2}{a_1} z^2 + \dots$$

is real, an extreme way is the x-axis, hence, of the function Z. Thus when an extreme way is a straight line, by the rotation of the axes, the line may become the x-axis; next rotating the X-axis, the series becomes real; hence, *when the extreme way is a straight line, the corresponding curve of extreme value is also a straight line.*

7. Application to the modular function $J(\tau)$.

τ-plane.

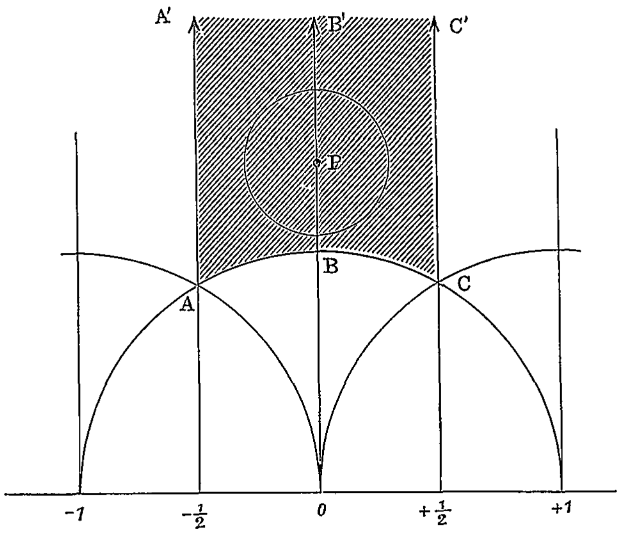

Fig. 7.

Fig. 7 shows the fundamental region of the modular function $J(\tau)$, in which we know that

$$\text{at } A,\quad J = J\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 0,$$

$$\text{at } B,\quad J = J(i) = 1,$$

$$\text{at } \infty,\quad J = J(\infty) = \infty$$

and that the function takes real values on the lines AA', AB, BB' and on their homologous lines. Now take any point $P(\tau_0, J_0)$ on the line BB', then we have an expansion:

$$J - J_0 = a_1(\tau - \tau_0) + a_2(\tau - \tau_0)^2 + \ldots,\qquad a_1 \neq 0.$$

Putting $\quad J - J_0 = Z,$

$$\tau - \tau_0 = iz,$$

it becomes $\quad Z = b_1 z + b_2 z^2 + \ldots,\qquad b_1 \neq 0.$

When z takes real values, τ takes values on the line BB', therefore the corresponding values of J resp. of Z are real and the series is a real series; hence the x-axis resp. the line BB' is an extreme way of the function $J(\tau)$. So also is the line AA'; but the line AB is not an extreme way; for the representation of the fundamental triangle in the plane of J is the real axis (Fig. 8). Now if the arc AB is an

J plane.

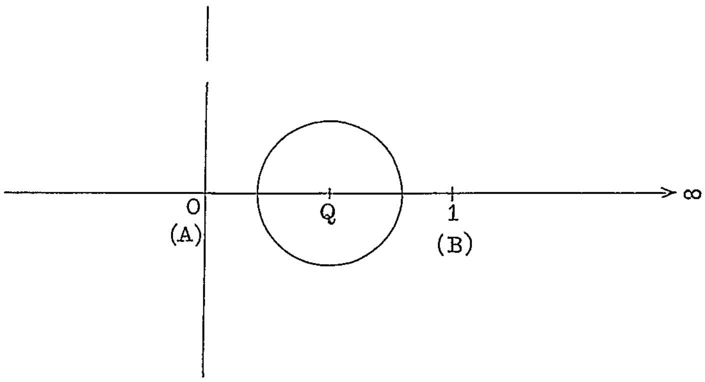

Fig. 8.

extreme way, then the part $\overline{01}$ of the real axis will be an extreme way of the inverse function $\tau(J)$. (§ 3). Since the extreme way is a straight line, the corresponding curve of extreme value must be also a straight line, (§ 6). But the line AB is an arc, i.e., the arc AB cannot be an extreme way of the modular function. The homologous lines of AA' resp. of BB' are extreme ways or not, according as they are straight or not, since the linear substitutions transform circle to circle but not in general centre to centre.

To Prof. J. Kawai, the author wishes to express thanks for his several remarks.