The Geometry of Somas in Non-Euclidean Spaces.

BY

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CHAPTER I.

The Geometry of Somas in Elliptic Space.

SECTION I.

Soma and Its Transformations.

§ 1.

Soma and Its Coordinates.

When we consider a rigid body in the elliptic space, its position in the space is always determined by three oriental lines which are fixed in the body and cut orthogonally each other. This figure can be brought to ∞^6 different positions. Every position is called a 'Soma' and any determinate one the 'protosoma' by Study. From the protosoma every other soma is obtained by a determinate motion.

Consider three oriented lines $\overrightarrow{OX_1}$, $\overrightarrow{OX_2}$, $\overrightarrow{OX_3}$ which intersect orthogonally each other and any point P in the space. Let

$$\begin{aligned} \alpha &\equiv \measuredangle X_1OP_1, \\ \beta &\equiv \measuredangle X_2OP_2, \\ \gamma &\equiv \measuredangle X_3OP_3, \\ r &\equiv \frac{\overline{OP}}{K}., \end{aligned}$$

where $\frac{1}{K^2}$ is the measure of curvature[1] of the space.

If we define the coordinates of the point P by the ratio[2]

[1], [2] See Coolidge, 'Non-euclidean Geometry', p. 53, 68.

$$x_0 : x_1 : x_2 : x_3$$

such that

$$\cos r \equiv \frac{x_0}{\sqrt{(xx)}},$$

$$K \sin r \cos \alpha \equiv \frac{x_1}{\sqrt{(xx)}},$$

$$K \sin r \operatorname{cas} \beta \equiv \frac{x_2}{\sqrt{(xx)}},$$

$$K \sin r \cos \gamma \equiv \frac{x_3}{\sqrt{(xx)}},$$

where

$$(xx) \equiv x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

Then the motion in the elliptic space may be represented by the equation[1]

$$X' = PXQ \qquad (|P| \cdot |Q| \neq 0) \ldots\ldots\ldots \text{(i)},$$

where X', P, X, Q are the quaternions such that

$$X' \equiv x_0' + ix_1' + jx_2' + kx_3',$$

$$X \equiv x_0 + ix_1 + jx_2 + kx_3,$$

$$P \equiv p_0 + ip_1 + ip_2 + kp_3,$$

$$Q \equiv q_0 + iq_1 + jq_2 + kq_3.$$

and

$$|P| \equiv p_0^2 + p_1^2 + p_2^2 + p_3^2,$$

$$|Q| \equiv q_0^2 + q_1^2 + q_2^2 + q_3^2,$$

$$i^2 + 1 = j^2 + 1 = k^2 + 1 = ijk + 1$$

$$= jk + kj = ki + ik = ij + ji = 0.$$

From (i) we see that a soma can be represented by the parameters PQ. For the sake of brevity, we will assume hereafter that

$$|P| = |Q|$$

without loss of generality.

If a soma $P'Q'$ can be brought to another soma $P''Q''$ by a motion PQ, then the following equations will hold by (i):

$$PP' = P'', \quad \tilde{Q}\tilde{Q}' = \tilde{Q}'',$$

where

$$\tilde{Q} \equiv q_0 - iq_1 - jq_2 - kq_3, \text{ etc.}$$

[1] See Klein, 'Zur nicht-euklidischen Geometrie', Math. Ann. **27**.

Let us now put

$$
\begin{aligned}
{}_lX_0 &\equiv p_0, & {}_rX_0 &\equiv q_0\sqrt{-1}\,, \\
{}_lX_1 &\equiv p_1, & {}_rX_1 &\equiv -\,q_1\sqrt{-1}\,, \\
{}_lX_2 &\equiv p_2, & {}_rK_2 &\equiv -\,q_2\sqrt{-1}\,, \\
{}_lX_3 &\equiv p_3, & {}_rX_3 &\equiv -\,q_3\sqrt{-1}\,,
\end{aligned}
$$

then the following identity exists:

$$(XX) \equiv {}_rX_0^2 + {}_rX_1^2 + {}_rX_2^2 + {}_rX_3^2 + {}_lX_0^2 + {}_lX_1^2 + {}_lX_2^2 + {}_lX_3^2. = 0.$$

Hereafter we shall define the ratios

$$({}_lX) \equiv {}_lX_0 : {}_lX_1 : {}_lX_2 : {}_lX_3, \qquad ({}_rX) \equiv {}_rX_0 : {}_rX_1 : {}_rX_2 : {}_rX_3$$

as the coordinates of the soma PQ. $({}_lX), ({}_rX)$ are analogous to the coordinates of a cross.[1] For the sake of compactness of expression in some of the following formulæ, we will replace these two sets $({}_lX), ({}_rX)$ by a single set of parameters of dual sort.

We will introduce dual quantities[2] of the sort $a_le + b_re\ (A)$ which are subjected to the following rules of multiplication of units:

$${}_le^2 = {}_le,\ {}_re^2, = {}_re,\ {}_le_re = {}_re_le.$$

Or shown with the diagram

	${}_le$	${}_re$
${}_le$	${}_le$	o
${}_re$	o	${}_re$

For two such dual numbers, the following relations are true:

$$(a_le + b_re) \times (a'_le + b'_re) = aa'_le + bb'_re,$$

$$(a_le + b_re) \div (a'_le + b'_re) = \frac{a}{a'}\,{}_re + \frac{b}{b'}\,{}_re.$$

The dual quantities satisfy associative, commutative and distributive laws, but when

1 See Coolidge, 'Noneuclidean Geometry,' p. 124.

2 Scheffers, Ann., 39, 297.

$$A \times B = 0,$$
$$A \text{ or } B = 0$$

is not always true.

We shall call the dual numbers which are formed from $({}_lX), ({}_rX)$ such that

$$\mathscr{X}_0 \equiv {}_lX_0\, {}_le + {}_rX_0\, {}_re,$$
$$\mathscr{X}_1 \equiv {}_lX_1\, {}_le + {}_rX_2\, {}_re,$$
$$\mathscr{X}_2 \equiv {}_lX_2\, {}_le + {}_rX_2\, {}_re,$$
$$\mathscr{X}_3 \equiv {}_lX_3\, {}_le + {}_rX_3\, {}_re$$

the 'dual coordinates' of the soma $({}_lX), ({}_rX)$.

Let us now put

$$\mathscr{X} \equiv \mathscr{X}_0 + i\mathscr{X}_1 + j\mathscr{X}_2 + k\mathscr{X}_3,$$
$$\mathscr{Y} \equiv \mathscr{Y}_0 + i_1\mathscr{Y} + j\mathscr{Y}_2 + k\mathscr{Y}_3,$$

then the quaternions with dual coefficients (biquaternions) may be written in the form

$$\mathscr{X} \equiv {}_lX\, {}_le + {}_rX_r e,$$

where

$${}_lX \equiv {}_lX_0 + i\, {}_lX_1 + j\, {}_lX_2 + k\, {}_lX_3,$$
$${}_lX \equiv {}_rX_0 + i\, {}_rX_1 + j\, {}_rX_2 + k\, {}_rX_3.$$

The motion $\mathscr{A}$ by which a Soma $(\mathscr{X})$ is brought to another soma $(\mathscr{Y})$ may be given by the equation

$$\mathscr{A} = \mathscr{Y}\, \widetilde{\mathscr{X}},$$

where

$$\widetilde{\mathscr{X}} \equiv \mathscr{X}_0 - i\mathscr{X}_1 - j\mathscr{X}_2 - k\mathscr{X}_3,$$
$$\mathscr{A} \equiv \mathscr{A}_0 + i\mathscr{A}_1 + j\mathscr{A}_2 + k\mathscr{A}_3.$$

Or

$$\mathscr{A}_0 = \mathscr{X}_0\, \mathscr{Y}_0 + \mathscr{X}_1\, \mathscr{Y}_1 + \mathscr{X}_2\, \mathscr{Y}_2 + \mathscr{X}_3\, \mathscr{Y}_3,$$
$$\mathscr{A}_1 = \mathscr{X}_0\, \mathscr{Y}_1 - \mathscr{X}_1\, \mathscr{Y}_0 + \mathscr{X}_2\, \mathscr{Y}_3 - \mathscr{X}_3\, \mathscr{Y}_2,$$
$$\mathscr{A}_2 \equiv \mathscr{X}_0\, \mathscr{Y}_2 - \mathscr{X}_1\, \mathscr{Y}_3 - \mathscr{X}_2\, \mathscr{Y}_0 + \mathscr{X}_3\, \mathscr{Y}_1,$$
$$\mathscr{A}_3 \equiv \mathscr{X}_0\, \mathscr{Y}_3 + \mathscr{X}_1\, \mathscr{Y}_2 - \mathscr{X}_2\, \mathscr{Y}_1 - \mathscr{X}_3\, \mathscr{Y}_0.$$

The totality of all ∞^6 somas can be put one to one correspondence with the totality of all ∞^6 paiis of points, or with that of all ∞^6 pairs of planes, one in each of real, projectively defined spaces.

For the sake of clearness we shall assume our soma-space is doubly overlaid. We shall say that a soma belongs to the 'upper layer,' when it is represented by a pair of points; when it is repiesented by a pair of planes, we shall speak of a soma of the lower layer. The two spaces shall be called in either case the 'representing spaces.'

When

$$(x)\,\{A\}\,(x');^{1}$$

the cosines of the K^{th} parts of the amounts of the left and right translations in the motion are equal to

$$\frac{{}_lA_0}{\sqrt{({}_lA_lA)}},\ \frac{{}_rA_0}{\sqrt{({}_rA_rA)}}$$

respectively.

The equation of motion given in (1) may also be written in the forms

$$\begin{aligned} x'_0 &= S_{00}x_0 + S_{01}x_1 + S_{02}x_2 + S_{03}x_3, \\ x'_1 &= S_{10}x_0 + S_{11}x_1 + S_{12}x_2 + S_{13}x_3, \\ x'_2 &= S_{20}x_0 + S_{21}x_1 + S_{22}x_2 + S_{23}x_3, \\ x'_3 &= S_{30}x_0 + S_{31}x_1 + S_{32}x_2 + S_{33}x_3, \end{aligned}$$

where

$$\begin{aligned}
&S_{00} \equiv p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3, \quad S_{01} \equiv -p_0q_1 - p_1q_0 + p_2q_3 - p_3q_2, \\
&S_{02} \equiv p_0q_2 - p_1q_3 - p_2q_0 + p_3q_1, \quad S_{03} \equiv p_0q_3 + p_1q_2 - p_5q_1 - p_3q_0, \\
&S_{10} \equiv p_2q_0 + p_0q_1 - p_3q_2 + p_2q_3, \quad S_{11} \equiv p_1q_1 + p_0q_0 + p_3q_3 + p_5q_2, \\
&S_{12} = -p_1q_2 + p_0q_3 - p_3q_0 - p_2q_1, \quad S_{13} \equiv -p_3q_3 - p_0q_2 - p_3q_1 + p_2q_0. \\
&S_{20} \equiv p_2q_0 + p_3q_1 + p_0q_2 - p_1q_3, \quad S_{21} \equiv p_2q_1 + p_3q_0 - p_0q_3 - p_1q_2, \\
&S_{22} \equiv p_2q_2 + p_3q_3 + p_0q_0 + p_1q_1, \quad S_{23} \equiv p_2q_3 - p_3q_2 + p_0q_1 - p_1q_0, \\
&S_{30} \equiv p_3q_0 - p_2q_1 + p_1q_2 + p_0q_3, \quad S_{31} \equiv -p_3q_1 - p_2q_0 + p_1q_3 + p_3q_2, \\
&S_{32} \equiv -p_3q_2 - p_2q_3 + p_1q_0 - p_0q_1, \quad S_{33} \equiv p_3q_3 + p_2q_2 + p_1q_1 + p_0q_0.
\end{aligned}$$

Now let $(X), (p_{ij})$[2] be Klein's coordinates and Plücker's coordinates of a line and $(X'), (p'_{ij})$ be that of the transformed line in the above motion respectively, then

$$2p_{01}' = {}_lX_1' + {}_rX_1'\sqrt{-1} = 2\begin{vmatrix} y_0' & x_1' \\ y_0' & y_1' \end{vmatrix}$$

---

1 This notation means that the soma $(\mathscr{X})$ is brought to the soma $(\mathscr{X}')$ by the motion $(\mathscr{A})$.

2 $(X) \equiv ({}_lX_1 : {}_lX_2 : {}_lX_3 : {}_rX_1 : {}_rX_2 : {}_rX_3)$,

$p_{ij} \equiv \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}.$

$$= 2\begin{vmatrix} s_{00} & s_{01} & s_{02} & s_{03} \\ s_{10} & s_{11} & s_{12} & s_{13} \end{vmatrix} \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{vmatrix}$$

$$= 2\sum \begin{vmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{vmatrix} \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + 2\sum \begin{vmatrix} s_{02} & s_{03} \\ s_{12} & s_{13} \end{vmatrix} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}$$

$$= 2\sum \begin{vmatrix} s_{00} & s_{01} \\ s_{10} & s_{10} \end{vmatrix} p_{01} + 2\sum \begin{vmatrix} s_{02} & s_{03} \\ s_{12} & s_{13} \end{vmatrix} p_{23}$$

$$= 2\sum \begin{vmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{vmatrix} ({}_lX_1 + {}_lX_1\sqrt{-1}) + 2\sum \begin{vmatrix} s_{02} & s_{03} \\ s_{12} & s_{13} \end{vmatrix} ({}_lX_1 - {}_rX_1\sqrt{-1}).$$

But, if

$${}_lX_1 : {}_lX_2 : {}_lX_3 : {}_rX_1 : {}_rX_2 : {}_rX_3$$

$$= p_1 : p_2 : p_3 : -\sqrt{-1}\, q_1 : -\sqrt{-1}\, q_2 : -\sqrt{-1}\, q_3,$$

then

$${}_lX_1' + {}_rX_1'\sqrt{-1} = (p_1 - q_1\sqrt{-1})|P|\cdot|Q| = ({}_rX_1 + {}_rX_1\sqrt{-1})|P|\cdot|Q|.$$

In the same manner, we can show that

$${}_lX_2' + {}_rX_2'\sqrt{-1} = ({}_lX_2 + {}_rX_2\sqrt{-1})|P|\cdot|Q|,$$

$${}_lX_3' + {}_rX_3'\sqrt{-1} = ({}_lX_3 + {}_rX_3\sqrt{-1})|P|\cdot|Q|.$$

Therefore the cross defined by the ratios $({}_lX_1 : {}_lX_1 : {}_lX_3)$, $({}_rX_1 : {}_rX_2 : {}_rX_3)$ remains itself in this motion. The cross $({}_lX), ({}_rX)$ shall be called the 'director-cross' of the motion.

§ 2.

The Special Position of Two Somas.

If a soma $(\mathscr{X}')$ is obtained from another soma $(\mathscr{X})$ by a motion in which the amounts of left and right translations are equal, then

$$\frac{({}_lX'{}_lX)^2}{({}_lX'{}_lX')({}_lX{}_lX)} = \frac{({}_rX'{}_rX)^2}{({}_rX'{}_rX')({}_rX{}_rX)}.$$

If

$$(\mathscr{X})\,\{\mathscr{A}\}\,(\mathscr{X}'),$$

then

$${}_lX' = {}_lA_l X, \quad {}_rX' = {}_rA_r X.$$

Therefore

$$
\begin{aligned}
{}_lX_0^1 &= {}_lA_0\,{}_lX_0 - {}_lA_1\,{}_lX_2 - {}_lA_2\,{}_lX_2 - {}_lA_3\,{}_lX_3,\\
{}_lX_0^1 &= {}_lA_1\,{}_lX_0 + {}_lA_0\,{}_lX_1 + {}_lA_3\,{}_lX_2 - {}_lA_2\,{}_lX_3,\\
{}_lX_3^1 &= {}_lA_2\,{}_lX_0 - {}_lA_3\,{}_lX_1 + {}_lA_0\,{}_lX_2 + {}_lA_1\,{}_lX_3,\\
{}_lX_3^1 &= {}_lA_3\,{}_lX_0 + {}_lA_2\,{}_lX_1 - {}_lA_1\,{}_lX_3 + {}_lA_0\,{}_lX_3,\\
{}_rX_0^1 &= {}_rA_0\,{}_rX_0 - {}_rA_1\,{}_rX_1 - {}_rA_2\,{}_rX_2 - {}_rA_3\,X_3,\\
{}_rX_1^1 &= {}_rA_1\,{}_rX_1 + {}_rA_0\,{}_rX_1 + {}_rA_3\,{}_rX_2 - {}_rA_2\,X_3,\\
{}_rX_2^1 &= {}_rA_2\,{}_rX_2 - {}_rA_3\,{}_rX_3 + {}_rA_0\,{}_rX_2 + {}_rA_1\,X_3.\\
{}_rX_3^1 &= {}_rA_3\,{}_rX_3 + {}_rA_2\,{}_rX_3 - {}_rA_1\,{}_rX_2 + {}_rA_0\,X_3.
\end{aligned}
$$

Since

$$({}_lX_lX) = -{}_r(X_rX),$$

it follows that

$$\frac{({}_lX'{}_lX')}{\sqrt{({}_lX'{}_lX')({}_lX_lX)}} \pm \frac{({}_rX'{}_rX)}{\sqrt{({}_rX'{}_rX')({}_rX_rX)}} = ({}_lA_0 \pm {}_rA_0)\frac{({}_lX_lX)}{\sqrt{({}_lA_lA)({}_lX_lX)}}.$$

But in our case

$${}_lA_0 \pm {}_rA_0 = 0,$$

provided that

$$({}_lA_lA) + ({}_rA_rA) = 0.$$

Hence, we have

$$\frac{({}_lX'{}_lX)}{\sqrt{({}_lX'{}_lX')({}_lX_lX)}} = \pm \frac{({}_rX'{}_rX)}{\sqrt{({}_rX'{}_rX')({}_rX_rX)}}.$$

This is the relation analogous to the condition[1] of the intersection of two crossses in the elliptic space. The converse of this theorem is also true. The motion here considered shall be called 'equitranslation in left and right' (or simply equitranslation) and the somas $(\mathscr{X})$, $(\mathscr{X}')$ 'equitranslational somas.'

A rotation about the origin of coordinates may be represented by the equation

$$X' = PX\widetilde{P}.$$

Therefore, it is easily seen that this motion is an equitranslation.

If a soma $(\mathscr{X}')$ can be obtained from another soma $(\mathscr{X})$ by a half translation, then

$${}_lA_0 = {}_rA_0 = 0.$$

---

[1] $$\sum_{i=1}^{3} \frac{{}_lX_i'\,{}_lX_i}{\sqrt{({}_lX'{}_lX')({}_lX_lX)}} = \pm \sum_{i=1}^{3} \frac{{}_rX_i'\,{}_rX_i}{\sqrt{({}_rX'{}_rX')({}_rX_rX)}}$$

In this case ${}_lA$ and ${}_rA$ must be vectors. Further, it follows that

$$(X'X)=(X'|X)=0,$$

where

$$(X'|X)\equiv {}_lX_0^{\,2}+{}_lX_1^{\,2}+{}_lX_2^{\,2}+{}_lX_3^{\,3}-{}_rX_9^{\,2}-{}_rX_1^{\,2}-{}_rX_2^{\,2}-{}_rX_3^{\,2}.$$

Or

$$({}_lX'{}_lX)=({}_rX'{}_rX)=0.$$

This is an analogous relation to the condition of orthogonal intersection of two crosses in the elliptic space. When one soma $(\mathscr{U})$ lies in the lower layer the condition takes the form

$$(\mathscr{U}\mathscr{X})=0.$$

Two orthogonal somas will be represented by two orthogonal points (planes) in the both representing spaces. When one lies in the lower layer two orthogonal somas will be represented by a point and a plane in the united position. Thus two orthogonal somas will be represented by a plane element (a system of a point and a plane in the united position) after Lie.

The two somas of which one can be brought to the other by a half translation shall be called 'orthogonal somas.'

We shall introduce the conception of parataxy of somas. Two somas shall be called 'left (right) paratactic' if one can be brought from the other by a right (left) translation. For a right (left) translation $Q\,(P)$ is merely a real number. Therefore, if two somas $(\mathscr{X})$, $(\mathscr{X}')$ be left (right) paratactic, then

$${}_lX_0:{}_lX_1:{}_lX_2:{}_lX_3={}_lX_0':{}_lX_1':{}_lX_2':{}_lX_3'\,({}_rX_0:{}_rX_1:{}_rX_2:{}_rX_3={}_rX_0':{}_rX_1':{}_rX_2':{}_rX_3').$$

§ 3.

The Fundamental Somas and Improper Somas.

Let us consider the protosoma represented by the coordinate axes $\overrightarrow{OX_1}$, $\overrightarrow{OX_2}$, $\overrightarrow{OX_3}$, then we know that its coordinates PQ are equal to the ratios

$$(1000)\ (1000).$$

We can obtain the three somas represented by the system of the oriental lines

$$(\overrightarrow{OX_1}, \overleftarrow{OX_2}, \overleftarrow{OX_3}),$$
$$(\overleftarrow{OX_1}, \overrightarrow{OX_2}, \overleftarrow{OX_3}),$$
$$(\overleftarrow{OX_1}, \overleftarrow{OX_2}, \overrightarrow{OX_3}),$$

by the half translations along to the crosses formed by the lines OX_1, OX_2, OX_3, and their absolute polars respectively. The coordinates $({}_lX)$ $({}_rX)$ of these sȯmas aıe equal to the ratios

$$(0100)\ (0100),$$
$$(0010)\ (0010),$$
$$(0001)\ (0001),$$

respectively. Of course, these somas are orthogonal to each other by the definition of orthogonality. We shall call the soma as well as the protosoma itself the 'fundamental somas.' The fundamental somas are analogous to the edge-crosses of the fundamental orthogonal tetrahedron[1] in the elliptic space.

Sometimes it is convenient to consider certain imaginaıy somas called 'improper somas.'

The imaginary somas defined by the equation

$$({}_lX_l X) = 0\ (({}_lX_l X) \neq 0),\ ({}_rX_r X) \neq 0\ (({}_rX_r X) = 0)$$

shall be called left (right improper somas. The left (and right) improper somas will constitute a system which shall be called the 'improper somas of the first sort.'

We shall consider another sort of improper somas whose coordinates satisfy the relations

$$({}_lX_l X) = ({}_rX_r X) = 0.$$

Or

$$(\mathscr{X}\mathscr{X}) = 0.$$

Such somas shall be called 'improper somas of the second sort' The figure formed by the totality of the improper somas of the second soıt shall be called 'the soma-absolute.' Thus we see that the equation to it

$$(\mathscr{X}\mathscr{X}) = 0.$$

[1] A tetrahedron whose opposite edges are formed by crosses cutting orthogonally each other.

A left (right) improper soma will be represented by a point in the absolute quadric in the left (right) representing space and a point not in the absolute quadric in the right (left) representing space.

An improper soma of the second sort will be represented by a pair of points in the absolute quadrics in the both representing spaces.

The left (right) improper soma $(\mathscr{U})$ in the lower layer will be represented by a tangent plane in the absolute quadric in the left (right) representing space and a plane not touching the absolute quadric in the right (left) representing space.

An improper soma of the second sort will be represented by a pair of tangent planes to the absolute quadric in the both representing spaces.

The orthogonality of an improper soma $(\mathscr{X})$ and a soma $(\mathscr{U})$ in the lower layer shall be defined by the equation

$$(\mathscr{U}\mathscr{X}) = 0.$$

§ 4.

The Functions of the Dual Coordinates of Somas.

If $(\mathscr{X})$ and $(\mathscr{X}')$ be two proper somas, then

$$\cos\frac{{}_lD}{K} = \frac{({}_lX'{}_lX)}{\sqrt{({}_lX'{}_lX')({}_lX{}_lX)}}, \quad \cos\frac{{}_rD}{K} = \frac{({}_rX'{}_rX)}{\sqrt{({}_rX'{}_rX')({}_rX{}_rX)}},$$

where ${}_lD$ and ${}_rD$ are the amounts of the left and right translations in the motion by which the soma $(\mathscr{X}')$ is brought from the soma $(\mathscr{X})$. Now we see that

$$\cos\frac{{}_lD}{K}\,{}_le + \cos\frac{{}_rD}{K}\,{}_re$$

$$= \frac{({}_lX'{}_lX)}{\sqrt{({}_rX'{}_rX')({}_rX{}_rX)}}\,{}_le + \frac{({}_rX'{}_rX)}{\sqrt{({}_rX'{}_rX')({}_rX{}_rX)}}\,{}_re$$

$$= \frac{(\mathscr{X}'\mathscr{X})}{\sqrt{(\mathscr{X}'\mathscr{X}')(\mathscr{X}\mathscr{X})}}.$$

We shall define the dual distance of the two somas $(\mathscr{X}), (\mathscr{X}')$ by the dual quantity

$$\mathfrak{D} \equiv {}_lD\,{}_le + {}_rD\,{}_re.$$

And we shall further define the cosine of the dual distance by the dual quantity

$$\cos\frac{\mathfrak{D}}{K} = \frac{(\mathfrak{X}'\mathfrak{X})}{\sqrt{(\mathfrak{X}'\mathfrak{X}')(\mathfrak{X}\mathfrak{X})}}.$$

When the somas $(\mathfrak{X}), (\mathfrak{X}')$ are orthogonal

$$\frac{{}_lD}{K} = \frac{{}_rD}{K} = \frac{\pi}{2}.$$

Therefore

$$\cos\frac{\mathfrak{D}}{K} = 0.$$

Notice that $\frac{{}_lD}{K}$ and $\frac{{}_rD}{K}$ are analogous to the Clifford's angles[1] of two crosses in the elliptic space.

Further we shall define $\sin\frac{\mathfrak{D}}{K}$ by the expression such that

$$\sin\frac{\mathfrak{D}}{K} = \sqrt{1-\cos^2\frac{\mathfrak{D}}{K}}.$$

Thus the expression for $\sin\frac{\mathfrak{D}}{K}$ becomes

$$\frac{\sqrt{\begin{Vmatrix} \mathfrak{X}_0' & \mathfrak{X}_1' & \mathfrak{X}_2' & \mathfrak{X}_3' \\ \mathfrak{X}_0 & \mathfrak{X}_1 & \mathfrak{X}_2 & \mathfrak{X}_3 \end{Vmatrix}^2}}{\sqrt{(\mathfrak{X}'\mathfrak{X}')(\mathfrak{X}\mathfrak{X})}}$$

Let the expression $J(XY)$ be such that

$$J(XY) = \frac{(X|Y)}{(XY)},$$

then

$$\frac{(X|Y)}{\sqrt{({}_lX_lX)({}_lY_lY)}} = \frac{({}_lX_lY)}{\sqrt{({}_lX_lX)({}_lY_lY)}} - \frac{({}_rX_rX)}{\sqrt{({}_rX_rX)({}_rY_rY)}}$$

$$= 2\sin\frac{{}_rD+{}_lD}{K}\,.\,\sin\frac{{}_rD-{}_lD}{K},$$

$$\frac{(XY)}{\sqrt{({}_lX_lX)({}_lY_lY)}} = \frac{({}_lX_lY)}{\sqrt{({}_lX_lX)({}_lY_lY)}} + \frac{({}_rX_rY)}{\sqrt{({}_rX_rX)({}_rY_rY)}}$$

$$2\cos\frac{{}_rD+{}_lD}{K}\,.\,\sin\frac{{}_rD-{}_lD}{K}.$$

[1] See Coolidge's 'Non-euclidean Geometry,' p. 126.

Hence

$$\mathcal{J}(XY) = \mathrm{tg}\,\frac{{}_rD + {}_lD}{K}\,.\,\mathrm{tg}\,\frac{{}_rD - {}_lD}{K}\,.$$

Thus we can say that the absolute invariant $\mathcal{J}(XY)$ is equal to the product of the tangents of the sum and difference of the K^{th} part of the amounts of the right and left translations in the motion by which the soma $(\mathscr{Y})$ is brought from the soma $(\mathscr{X})$.

The figure formed by the totality of somas whose dual distances from a fixed soma are constant shall be called a 'soma-sphere.' The equation to a soma sphere is

$$\frac{(\mathscr{X}\mathscr{A})}{\sqrt{(\mathscr{X}\mathscr{X})(\mathscr{A}\mathscr{A})}} = \cos\frac{\mathscr{D}}{K}\,.$$

A soma sphere will be represented by a pair of two spheres in the representing spaces.

The figure formed by three somas shall be called a soma-triangle. This is represented by a pair of triangles in the both representing spaces.

Let us now consider a dual function formed by the dual coordinates of a soma tiiangle $[(\mathscr{X})(\mathscr{Y})(\mathscr{Z})]$ such that

$$\sin(\mathscr{X}\mathscr{Y}\mathscr{Z}) \equiv \frac{\sqrt{|\mathscr{X}\mathscr{Y}\mathscr{Z}|^2}}{\sqrt{(\mathscr{X}\mathscr{X})(\mathscr{Y}\mathscr{Y})(\mathscr{Z}\mathscr{Z})}}\,,$$

where

$$|\mathscr{X}\mathscr{Y}\mathscr{Z}| \equiv \left\| \begin{matrix} \mathscr{X}_0 & \mathscr{X}_1 & \mathscr{X}_2 & \mathscr{X}_3 \\ \mathscr{Y}_0 & \mathscr{Y}_1 & \mathscr{Y}_2 & \mathscr{Y}_3 \\ \mathscr{Z}_0 & \mathscr{Z}_1 & \mathscr{Z}_2 & \mathscr{Z}_3 \end{matrix} \right\|$$

We see that

$$\sin(\mathscr{X}\mathscr{Y}\mathscr{Z}) \equiv \sqrt{\frac{|{}_lX_lY_lZ|^2}{({}_lX_lX)({}_lY_lY)({}_lZ_lZ)}}\,{}_le + \sqrt{\frac{|{}_rX_rY_rZ|^2}{({}_rX_rX)({}_rY_rY)({}_rZ_rZ)}}\,{}_re$$

$$= \sin({}_lX_lY_lZ)\,{}_le + \sin({}_rX_rY_rZ)\,{}_re.$$

The function $\sin(\mathscr{X}\mathscr{Y}\mathscr{Z})$ shall be called sine-amplitude of the soma-triangle. This function is a dual quantity formed by the sine-amplitudes of the triangles in the representing spaces corresponding to the soma-triangle.

The figure formed by four somas shall be called a 'soma-tetrahedron.' A soma-tetrahedron will be represented by a pair of tetrahedrons each formed by the four points in the representing spaces corresponding to the given somas.

If we now consider the function formed by the dual coordinates of the four somas forming a soma-tetrahedron such that

$$\sin(\mathscr{X}\mathscr{Y}\mathscr{Z}w) \equiv \frac{|\mathscr{X}\mathscr{Y}\mathscr{Z}w|}{\sqrt{(\mathscr{X}\mathscr{X})(\mathscr{Y}\mathscr{Y})(\mathscr{Z}\mathscr{Z})(ww)}},$$

then we see that

$$\sin(\mathscr{X}\mathscr{Y}\mathscr{Z}w) = \frac{|{}_lX_lY_lZ_lW|}{\sqrt{({}_lX_lX)({}_lY_lY)({}_lZ_lZ)({}_lW_lW)}}\,{}_le$$

$$+ \frac{|{}_rX_rY_rZ_rW|}{\sqrt{({}_rX_rX)({}_rY_rY)({}_rZ_rZ)({}_rW_rW)}}\,{}_le$$

$$= \sin({}_lX_lY_lZ_lW)\,{}_le + \sin({}_lX_lY_lZ_lW)\,{}_le.$$

The dual quantity shall be called 'sine-amplitudes' of the soma-tetrahedron formed by the four somas. This is the dual quantity formed by the sine-amplitudes of the tetrahedrons in the representing spaces corresponding to the soma-tetrahedron.

Let us assume that the somas $(\mathscr{Y}), (\mathscr{Z}), (\mathscr{T}), (w)$ are such somas that no two of these are paratactic and satisfying the condition

$$|\mathscr{Y}\mathscr{Z}w| = 0.$$

Such an aggregates of somas shall be called a 'normal net of somas.' There is a soma orthogonal to all the somas of the normal net of somas which is given by the dual coordinates $(\mathscr{U})$ such that

$$\mathscr{U}_0 : \mathscr{U}_1 : \mathscr{U}_2 : \mathscr{U}_3 = |\mathscr{Y}\mathscr{Z}w|.$$

This soma shall be called the 'nucleus' of the normal net of somas. Let us consider four somas $(\mathscr{Y}), (\mathscr{Z}), (w), (\mathscr{T})$ of which no two are paratactic with the conditions

$$({}_lY_lZ_lW_lT) = 0\;(({}_rY_rZ_rW_rT) = 0),$$

$$({}_rY_rZ_rW_rT) = 0\;(({}_lY_lZ_lW_lT) = 0).$$

These indicate that $({}_lY), ({}_lZ), ({}_lW)({}_lT)$ $[({}_rY), ({}_rZ), ({}_rW), ({}_rT)]$ are the solutions of the equation

$$({}_lU_lX) = 0\;(({}_rU_rX) = 0) \;\ldots\ldots\ldots\ldots\ldots\; \text{(i)}.$$

Conversely, the somas whose coordinates satisfy the equation (i) possess the property that they can be equitranslational only when they are orthogonal witn those somas which belong to the system of the left (right) paratactic somas obtained by giving the left (right) system of coordinates the fixed values $({}_lU)$ $(({}_rU))$.

We shall call the assemblage of all the somas satisfying such an equation as (i) a 'left (right) normal net' of somas.

Now we will define the dual double ratio. Let $(\mathfrak{X}), (\mathfrak{Y}), (\mathfrak{Z}), (\mathfrak{T})$ be the dual coordinates of any four somas of which no two are paratactic belonging to a one dimentional soma chain[1] whose equations are

$$\mathfrak{X}_i = a\mathfrak{P}_i + b\mathfrak{Q}_i$$

and $(R), (S)$ be those of any two somas which do not belong to any left (right) normal net of somas with any two of them, We shall define the ratio of the dual quantities.

$$\frac{\sin(\mathfrak{R}S\mathfrak{X}\mathfrak{Z})}{\sin(\mathfrak{R}S\mathfrak{Y}\mathfrak{Z})} : \frac{\sin(\mathfrak{R}S\mathfrak{X}\mathfrak{T})}{\sin(\mathfrak{R}S\mathfrak{Y}\mathfrak{T})}$$

the dual double ratio of the four somas $(\mathfrak{X}), (\mathfrak{Y}), (\mathfrak{Z}), (\mathfrak{T})$ and denote it by the symbol

$$(\mathfrak{X}\mathfrak{Y}\mathfrak{Z}\mathfrak{T}).$$

The dual double ratio is independent of the somas $(\mathfrak{R}), (S)$. This may be written in the form

$$\frac{|\mathfrak{R}S\mathfrak{X}\mathfrak{Z}|}{|\mathfrak{R}S\mathfrak{Y}\mathfrak{Z}|} : \frac{|\mathfrak{R}S\mathfrak{X}\mathfrak{T}|}{|\mathfrak{R}S\mathfrak{Y}\mathfrak{T}|}$$

$$= ({}_lX_lY_lZ_lT)\,{}_le + ({}_rX_rY_rZ_rT)\,{}_re.$$

§ 5.

The Transformation-group of the Soma-space in the Elliptic Space.

The group of dual projective geometry contains 30 essentiai parameters and is isormorphic with the 30 parameter group of two projective spaces. The group is constituted from two parts;

(1) the half simple subgroup $\mathfrak{G}_{30}$ under which left and right parataxy of somas are invariant,

(2) $\mathfrak{H}_{30}$ where the two sorts of parataxy are interchanged.

[1] This will be defined in §6.

Somas of different layers aro subjected to contragradient transformations under $\mathfrak{G}_{30}\,\mathfrak{H}_{30}$, and the orthogonality of two somas of different layers in an invariant under all transformations of the group.

The transformations of $\mathfrak{H}_{30}$ form no group, two compound transformation of the transformations of $\mathfrak{H}_{30}$ being a transformation of the group $\mathfrak{G}_{30}$. Every transformation of $\mathfrak{G}_{30}$ has the form

$$\mathcal{X}'_i = (\mathcal{A}_i\,\mathcal{X}): \qquad i = 0,\ 1,\ 2,\ 3,$$

where

$$\mathcal{A}_{ij} = a_{ij\ l}e + b_{ij\ r}e,$$

$$|\mathcal{A}_{ij}| \neq 0.$$

This transformation shall be called 'dual collineation.' This equations may also be written in the form

$$_lX_i' = (a_i\,{}_lX_i), \qquad {}_rX_i' = (b\,{}_rX_i).$$

Dual double ratio is an invariant for the transformation of $\mathfrak{G}_{30}$. We shall define b_le+a_re as the conjugate dualitic quantity to a_le+b_re and denote by $\bar{\mathcal{A}}$. Thus the general transformation of $\mathfrak{H}_{30}$ is

$$\mathcal{X}_i' = (\mathcal{A}_i\,\mathcal{X}).$$

This may be obtained from the general transformation of $\mathfrak{G}_{30}$ combined the transformation

$$\mathcal{X}_i' = \bar{\mathcal{X}}_i.$$

The transformation shall be called 'dual anticollineation.' Dual anticollineation interchanges left and right parataxy of somas.

The transformation of $\mathfrak{G}_{30}$ are obtained by combining those of $\mathfrak{G}_{30}$ with an interchange of left and right parataxy of somas. This combination will produce a transformation of $\mathfrak{G}_{30}$ when applied in their order. But it is not, in general commutative.

There is another sort of transformation by which dual soma coordinates of one layer are expressed as linear function of the other, i.e.,

$$\mathcal{X}_i' = (\mathcal{A}_i\mathcal{U}), \quad \text{.........................} \quad \text{(i)}$$

$$|\mathcal{A}_{ij}| \neq 0.$$

This is formed by a transformation of $\mathfrak{G}_{30}$ compounded with the transformation

$$\mathcal{X}_i' = \mathcal{U}_i. \text{...........................} \quad \text{(ii)}$$

This simple transformation is merely replacing each soma of one layer by the some of the other conincident with it. The transformation (1) shall be called 'dual correlation.'

We have another transformation

$$\mathfrak{X}_i' = (\mathcal{A}_i \overline{\mathfrak{U}}).$$

This is composed from a transformation of $\mathfrak{G}_{30}$ and a transformation

$$\mathfrak{X}_i' = \overline{\mathfrak{U}}_i.$$

Or this may be composed from a dual correlation and transformation

$$\mathfrak{U}_i' = \overline{\mathfrak{U}}_i.$$

The transformation $\mathfrak{U}_i = \overline{\mathfrak{U}}_i$ is the interchange of left and right translations in a soma in the lower layer. This transformation shall be called 'dual anticorrelation.'

The determinant formed by the dual coordinates of any four somas is evidently invariant under the group $\mathfrak{G}_{30}$.

$\mathfrak{G}_{30}$ has two invariants 15 parametric subgroups ${}_l\mathfrak{G}_{15}$, ${}_r\mathfrak{G}_{15}$ which are formed by the most general transformations, transforming each of somas into another, respectively left and right paratactic to it. These groups are intransitive and each is isormorphic with the general quaternary projective group.

Another subgroups $\mathfrak{G}_{13}$, $\mathfrak{H}_{12}$ of $\mathfrak{G}_{30}$ are the 12 parametric groups constituted of all orthogonal transformations among $({}_lX)$ and $({}_rX)$. Here $\cos\frac{{}_lD}{K}$ and $\cos\frac{{}_rD}{K}$ of two somas are invariable for the transformation of the groups $\mathfrak{G}_{12}$, $\mathfrak{H}_{12}$; hence two orthogonal somas are transformed into such ones. The groups $\mathfrak{G}_{12}$, $\mathfrak{H}_{12}$ make invariable the equation:

$$(xx) \equiv ({}_lX_l X)\,{}_le + ({}_rX_r X)\,{}_re = 0.$$

And any soma-sphere is an invariant figure for these transformations of the group. This may also be considered as a dual collineation of soma which leaves the soma absolute absolutely invariant or the absolute quadrics in the representing spaces invariant.

The groups $\mathfrak{G}_{12}$, $\mathfrak{H}_{12}$ are composed of all transformations of somas $\mathfrak{G}_{12}$ by which all the generators of the absolute quadric in the representing spaces are transformed in itself, and $\mathfrak{H}_{12}$ by which all generators of the different sets are permutated. The group $\mathfrak{G}_{12}$ may be written in the form

$$\mathscr{X}' = \mathscr{A}\mathscr{X}\mathscr{B}.$$

The transformations may be decomposed into two sorts of transformations;

$$\mathscr{X}' = \mathscr{A}\mathscr{X},$$

$$\mathscr{X}' = \mathscr{X}\mathscr{B}.$$

Therefore $\mathfrak{G}_{12}$ can be decomposed into two commutative groups $\mathfrak{G}_6$ and $\mathfrak{G}_6'$, which are defined by the two kinds of transformations of the above respectively.

$\mathfrak{G}_6$ is the group of motions of somas.

Further the transformation of the group $\mathfrak{G}_6$ can be classified into the two kinds :

$${}_lX' = {}_lA_l X,$$

$${}_rX' = {}_rX\,{}_rB.$$

The transformations define two subgroups ${}_l\mathfrak{G}_3, {}_r\mathfrak{G}_3$ of the group $\mathfrak{G}_6$. ${}_l\mathfrak{G}_3$ and ${}_r\mathfrak{G}_3$ are evidently groups of the left and right translation respectively. By these transformations of ${}_l\mathfrak{G}_3$ $({}_r\mathfrak{G}_3)$, any soma is transformed to a soma right (left) paratactic to the soma.

SECTION II.

The Soma-manifoldness in the Elliptic Space.

Let us consider a soma-manifoldness whose dual coordinates are linearly dependent for a fixed different somas, i.e.

$$\rho\mathscr{X}_i = a_0\mathscr{X}_i^{(0)} + a_1\,\mathscr{X}_i^{(1)}\ldots\ldots\ldots \quad + a_n\mathscr{X}_i^{(n)} \qquad i=0,\ 1,\ 2,\ 3$$

$$a_i\ :\ \text{real}.$$

We shall define the above soma-manifoldness an 'n-dimensional soma-chain' and $(\mathscr{X}^{(0)}), (\mathscr{X}^{(1)}), \ldots, (\mathscr{X}^{(n)})$ shall be called the 'bases' of the soma-chain. The fundamental soma-chains are one, two, three dimensional soma chains. We shall use the notation C_n to represent a n dimensional soma chain. Any $n+1$ somas of C_n

$$\rho\,\mathscr{X}_i = a_0^{(1)}\mathscr{X}_i^{(0)} + a_1^{(1)}\mathscr{X}_i^{(1)} + \ldots \qquad + a_n^{(1)}\mathscr{X}_i^{(n)},$$

$$\rho\,\mathscr{Y}_i = a_0^{(2)}\mathscr{X}_i^{(0)} + a_1^{(2)}\mathscr{X}_i^{(1)} + \ldots \qquad + a_n^{(2)}\mathscr{X}_i^{(n)},$$

$$=a_0^{(n+1)}\mathscr{X}_i^{(0)}+\qquad\qquad+a_n^{(n+1)}\mathscr{X}_i^{(1)}.$$

may be taken as its bases, provided that

$$\begin{vmatrix} a_0^{(1)} & a^{(1)} & \ldots\ldots & a_n^{(1)} \\ a_0^{(n+1)} & a_1^{(n+1)} & & a_n^{(n+1)} \end{vmatrix} \neq 0.$$

If of two families of somas of different layers each consists of all the somas which are orthogonal to all the somas of the other, then the both families are soma-chains.

We shall call the soma-chain arranged to pair 'reciprocal soma chain' to each other.

§ 6.

The One-dimensional Soma-chain.

The one-dimensional soma-chain has the equations

$$\rho\mathscr{X}_i=a\mathscr{Y}_i+b\mathscr{Z}_i \qquad (i=0,\ 1,\ 2,\ 3)$$

$$a,\ b:\ \text{real}$$

Or

$$\rho_l X_i=a_l Y_i+b_l Z_i,\qquad \rho_r X_i=a_r Y_i+b_r Z_i.$$

Let us consider the case in which

$$|\,{}_lY_lZ\,|^{(1)}\neq 0,\qquad |\,{}_rY_rZ\,|\neq 0.$$

In this case $({}_lY)$, $({}_lZ)$ are not linearly depent and $({}_rY)$ $({}_rZ)$ are also such ones. Such a soma-chain shall be called a 'soma-chain regulus.' A soma-chain regulus will be represented by a pair of rows of points (sheaves of planes) projectively related in the both representing spaces.

Every soma-chain regulus has at least one pair of real somas, called 'principal somas,' which are orthogonal. To find these, we must solve the equation

$$(a\mathscr{Y}+b\mathscr{Z},\ a'\mathscr{Y}+b'\mathscr{Z})=0.$$

Or

$$aa'({}_lY_lY)+(ab'+a'b)({}_lY_lZ)+bb'({}_lZ_lZ)=0,$$

$$aa'({}_rY_rY)+(ab'+a'b)({}_rY_rZ)+bb'({}_rZ_rZ)=0.$$

---

[1] I means $|\,{}_lY_lZ|\neq 0$, etc. that all the determinants of the matrics $\begin{Vmatrix} {}_lY_0 & {}_lY_1 & {}_lY_2 & {}_lY_3 \\ {}_lZ_0 & {}_lZ_1 & {}_lZ_2 & {}_lZ_3 \end{Vmatrix}$ are not equal to zero.

By the elimination of $a' : b'$ from these equations, we have a quadratic in $a : b$; i e.,

$$a^2\begin{vmatrix}({}_lY_lY) & ({}_lY_lZ)\\ ({}_rY_rY) & ({}_rY_rZ)\end{vmatrix} + ab\begin{vmatrix}({}_lY_lY) & ({}_lZ_lZ)\\ ({}_rY_rY) & ({}_rZ_rZ)\end{vmatrix} + b^2\begin{vmatrix}({}_lY_lZ) & ({}_lZ_lZ)\\ ({}_rY_rZ) & ({}_rZ_rZ)\end{vmatrix} = 0.$$

The discriminant of the equation is

$$\begin{vmatrix}({}_lY_lY) & ({}_lZ_lZ)\\ ({}_rY_rY) & ({}_rZ_rZ)\end{vmatrix}^2 + 4\begin{vmatrix}({}_lY_lY) & ({}_lY_lZ)\\ ({}_rY_rY) & ({}_rY_rZ)\end{vmatrix}\cdot\begin{vmatrix}({}_lZ_lZ) & ({}_lY_lZ)\\ ({}_rZ_rZ) & ({}_rY_rZ)\end{vmatrix}$$

$$= \begin{vmatrix}({}_lY_lY) & ({}_lZ_lZ)\\ ({}_rY_rY) & ({}_rZ_rZ)\end{vmatrix}^2 + 4\,({}_lY_lY)({}_lZ_lZ)\{({}_rY_rZ) + ({}_rY_rZ)\}^2$$

$$\not< 0.$$

Therefore the roots can not be complex. The discriminant may be considered as the simultaneous invariant of the forms

$$(a_lY + b_lZ,\ a_lY + b_lZ) = 0,$$
$$(a_rY + b_rZ,\ a_rY + b_rZ) = 0$$

for $a : b$. When our somas have real coordinates, the roots of each of these two quadratic equations are conjugate complex pairs and two such pairs can not separate one another harmonically: hence the simultaneous invariant is not zero. Thus the somas are distinct ones.

In general, one of the principal somas are not the soma which is obtained by half translation from the other along one of the crosses formed by a line of these three oriented lines representing the other and its absolute polar. Let us consider the case in which one of the principal somas is obtained by a half translation from the other. In this case, let us take the principal somas as the fundamental somas (0010) (0010) and (0001) (0001), then the equations to the soma-chain may be reduced to its canonical form

$${}_lX_0 = {}_rX_0 = 0,$$

$${}_lX_1 = {}_rX_1 = 0,$$

$$X_2 = a_2\,{}_rX_2,$$

$${}_lX_3 = a_3\,{}_rX_3.$$

Such a sort of soma-chain can be generated by the half translations from the protosoma along every cross of a cross-chain,[1] provided that the soma is represented by the two oriented lines of the principal crosses[2] of the cross-chain and of the cross cutting orthogonally the principal crosses. From the cross-chain

$$
\begin{aligned}
{}_lX_1&={}_rX_1=0,\\
{}_lX_2&=a_2\ {}_rX_2,\\
{}_lX_3&=a_3\ {}_rX_3.
\end{aligned}
$$

a soma-chain of such a sort that

$$
\begin{aligned}
{}_lX_0&={}_rX_0=0,\\
{}_lX_1&={}_2X_1=0,\\
{}_lX_2&=a_2\ {}_rX_2,\\
{}_lX_3&=a_3\ {}_rX_3.
\end{aligned}
$$

may be generated by the half translations from the protosoma properly taken.

Let us consider a special case in which

$$a_2^2=a_3^2=a^2.$$

We can write the equation of the soma-chain in the form

$$
\begin{aligned}
{}_lX_0&={}_rX_0=0,\\
{}_lX_1&={}_rX_1=0,\\
{}_lX_2&=a\ {}_4X_2,\\
{}_lX_3&=a\ {}_rX_3.
\end{aligned}
$$

Any soma in this system is transformed into another of the same system by the equitranslation about X_1-axis, thus

$$
\begin{aligned}
&\rho\ {}_lX_0'=0, && \rho\ {}_rX_0'=0,\\
&\rho\ {}_lX_1'=0, && \rho\ {}_rX_1'=0,\\
&\rho\ {}_lX_2'=a\,(p_0\ {}_rX_2-p_1\ {}_rX_3), && \rho\ {}_rX_2'=p_0\ {}_rX_2-p_1\ {}_rX_3.\\
&\rho\ {}_lX_3'=a\,(p_1\ {}_rX_2+p_0\ {}_rX_3), && \rho\ {}_rX_3'=p_1\ {}_rX_2+p_0\ {}_rX_3.
\end{aligned}
$$

[1, 2] See Coolidge's dissertation, 'Dual Projective Geometry,' p. 24, 25.

Therefore we have

$${}_lX_0'={}_rX_0'=0,$$

$${}_lX_1'={}_rX_1'=0,$$

$${}_lX_2'=a\,{}_rX_2',$$

$${}_lX_3'=a\,{}_rX_3'.$$

General C_1

$$\rho\,{}_2X_i=a\,{}_2Y_i+b\,{}_2Z_i,\ \rho\,{}_rX_i=a\,{}_rY_i+b\,{}_rZ_i,\ (i=1,2,3),\ \ldots\ \text{(i)}$$

can be generated by half translations of a soma with respect to every cross of the chain[1] from a fixed soma $(\mathcal{T})$ with respect to the cross (X) has the coordinates

$$\begin{aligned}
\tau\,{}_lX_0' &= -{}_lX_1\,{}_lT_1-{}_lX_2\,{}_lT_2-{}_lX_3\,{}_lT_3, & \tau\,{}_rX_0' &= -{}_rX_1\,{}_rT_1-{}_rX_2\,{}_rT_2-{}_rX_3\,{}_rT_3,\\
\tau\,{}_lX_1' &= {}_lX_2\,{}_lT_0-{}_lX_3\,{}_lT_2+{}_lX_2\,{}_lT_3, & \tau\,{}_rX_1' &= {}_rX_2\,{}_rT_0-{}_rX_3\,{}_rT_2+{}_rX_2\,{}_rT_3,\\
\tau\,{}_lX_2' &= {}_lX_2\,{}_lT_0+{}_lX_3\,{}_lT_1-{}_lX_1\,{}_lT_3, & \tau\,{}_rX_2' &= {}_rX_2\,{}_rT_0+{}_rX_3\,{}_rT_1-{}_rX_1\,{}_rT_3,\\
\tau\,{}_lX_3' &= {}_lX_3\,{}_lT_0-{}_lX_2\,{}_lT_1+{}_lX_1\,{}_lT_2, & \tau\,{}_rX_3' &= {}_rX_3\,{}_rT_0-{}_rX_2\,{}_rT_1+{}_rX_1\,{}_rT_2\quad \text{(ii)}
\end{aligned}$$

By substituting the values of (X) in (i) to (ii) we have

$$\begin{aligned}
\rho\,{}_lX_0' &= -(a\,{}_lY_1+b\,{}_lZ_1)\,{}_lT_1-(a\,{}_lY_2+b\,{}_lZ_2)\,{}_lT_2-(a\,{}_lY_3+b\,{}_lZ_3)\,{}_lT_3,\\
\rho\,{}_lX_1' &= (a\,{}_lY_1+b\,{}_lZ_1)\,{}_lT_0-(a\,{}_lY_3+b\,{}_lZ_3)\,{}_lT_2+(a\,{}_lY_2+b\,{}_lZ_2)\,{}_lT_3,\\
\rho\,{}_lX_2' &= (a\,{}_lY_3+b\,{}_lZ_3)\,{}_lT_0+(a\,{}_lY_2+b\,{}_lZ_2)\,{}_lT_2-(a\,{}_lY_1+b\,{}_lZ_1)\,{}_lT_3,\\
\rho\,{}_lX_3' &= (a\,{}_lY_3+b\,{}_lZ_3)\,{}_lT_0-(a\,{}_lY_2+b\,{}_lZ_2)\,{}_lT_1+(a\,{}_lY_1+b\,{}_lZ_1)\,{}_lT_2.\\
\rho\,{}_rX_0' &= -(a\,{}_rY_1+b\,{}_rZ_1)\,{}_rT_1-(a\,{}_rY_2+b\,{}_rZ_2)\,{}_rT_2-(a\,{}_rY_3+b\,{}_rZ_3)\,{}_rT_3,\\
\rho\,{}_rX_1' &= (a\,{}_rY_1+b\,{}_rZ_1)\,{}_rT_0-(a\,{}_rY_3+b\,{}_rZ_3)\,{}_rT_2+(a\,{}_rY_2+b\,{}_rZ_2)\,{}_rT_3,\\
\rho\,{}_rX_2' &= (a\,{}_rY_3+b\,{}_rZ_3)\,{}_rT_0+(a\,{}_rY_2+b\,{}_rZ_2)\,{}_rT_1-(a\,{}_rY_1+b\,{}_rZ_2)\,{}_rT_3.\\
\rho\,{}_rX_3' &= (a\,{}_rY_3+b\,{}_rZ_3)\,{}_rT_0-(a\,{}_rY_2+b\,{}_rZ_2)\,{}_rT_1+(a\,{}_rY_1+b\,{}_rZ_1)\,{}_rT_2
\end{aligned}$$

Or,

[1] See Coolidge's dissertation, "Dual Projective Geometry," 1904, p. 24.

$$
\begin{aligned}
\rho\, {}_lX_0' &= a(-{}_lY_1\,{}_lT_1-{}_lY_2\,{}_lT_2-{}_lY_3\,{}_lT_3)+b(-{}_lZ_1\,{}_lT_1-{}_lZ_2\,{}_lT_2-{}_lZ_3\,{}_lT_3),\\
\rho\, {}_lX_1' &= a(\ \ {}_lY_1\,{}_lT_0-{}_lY_3\,{}_lT_2+{}_lY_2\,{}_lT_3)+b(\ \ {}_lZ_1\,{}_lT_0-{}_lZ_3\,{}_lT_2+{}_lZ_2\,{}_lT_3),\\
\rho\, {}_lX_2' &= a(\ \ {}_lY_2\,{}_lT_0+{}_lY_3\,{}_lT_1-{}_lY_1\,{}_lT_3)+b(\ \ {}_lZ_2\,{}_lT_0+{}_lZ_3\,{}_lT_1-{}_lZ_1\,{}_lT_3),\\
\rho\, {}_lX_3' &= a(\ \ {}_lY_3\,{}_lT_0-{}_lY_2\,{}_lT_1+{}_lY_1\,{}_lT_2)+b(\ \ {}_lZ_3\,{}_lT_0-{}_lZ_2\,{}_lT_1+{}_lZ_1\,{}_lT_2),\\
\rho\, {}_rX_0' &= a(-{}_rY_1\,{}_rT_1-{}_rY_2\,{}_rT_2-{}_rY_3\,{}_rT_3)+b(-{}_rZ_1\,{}_rT_1-{}_rZ_2\,{}_rT_2-{}_rZ_3\,{}_rT_3),\\
\rho\, {}_rX_1' &= a(\ \ {}_rY_1\,{}_rT_0-{}_rY_3\,{}_rT_2+{}_rY_2\,{}_rT_3)+b(\ \ {}_rZ_1\,{}_rT_0-{}_rZ_3\,{}_rT_2+{}_rZ_2\,{}_rZ_3),\\
\rho\, {}_rX_2' &= a(\ \ {}_rY_2\,{}_rT_0+{}_rY_3\,{}_rT_1-{}_rY_1\,{}_rT_3)+b(\ \ {}_rZ_2\,{}_rT_0+{}_rZ_3\,{}_rT_1-{}_rZ_1\,{}_rT_3),\\
\rho\, {}_rX_3' &= a(\ \ {}_rY_3\,{}_rT_0-{}_rY_2\,{}_rT_1+{}_rY_1\,{}_rT_2)+b(\ \ {}_rZ_3\,{}_rT_0-{}_rZ_2\,{}_rT_1+{}_rZ_1\,{}_rT_1).
\end{aligned}
$$

Let us consider the case in which the matrices

$$
\left.\begin{aligned}
&|{}_lY{}_lZ| = 0, \qquad (|{}_rY{}_rZ| \neq 0),\\
&|{}_lY{}_lZ| \neq 0, \qquad (|{}_rY{}_rZ| = 0).
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}
$$

In such cases $({}_lY), ({}_lZ), (({}_rY), ({}_rZ))$ are linearly independent; the somas in this chain are all left (right) paratactic. Such a family can be generated from a fixed soma ($\mathscr{P}$) by the half translations, with respect to each cross of a left (right) strip,[1] i.e., from the left strip

$$
\begin{aligned}
&\rho\, {}_lX_i = a\, {}_lY_i + b\, {}_lZ_i, \quad \rho\, {}_rX_i = a\, {}_rY_i + b\, {}_rZ_i,\\
&{}_lY_i = \gamma\, {}_lZ_i,
\end{aligned}
$$

the following soma-system can be generated:

$$
\begin{aligned}
\mu\, {}_lX_0 &= (-{}_lY_1 p_1-{}_lY_2 p_2-{}_lY_3 p_3)(a+br),\\
\mu\, {}_lX_1 &= (\ \ {}_lY_1 p_0-{}_lY_3 p_2+{}_lY_2 p_3)(a+br),\\
\mu\, {}_lX_2 &= (\ \ {}_lY_2 p_0+{}_lY_3 p_1-{}_lY_1 p_3)(a+br)\\
\mu\, {}_lX_3 &= (\ \ {}_lY_3 p_0-{}_lY_2 p_1+{}_lY_1 p_2)(a+br).
\end{aligned}
$$

Or

$$
\begin{aligned}
\mu\, {}_rX_0 &= a(-{}_rY_1 q_0-{}_rY_2 q_1-{}_rY_3 q_2)+b(-{}_rZ_1 q_0-{}_rZ_2 q_1-{}_rY_3 q_2),\\
\mu\, {}_rX_1 &= a(\ \ {}_rY_1 q_0+{}_rY_3 q_2-{}_rY_2 q_3)+b(\ \ {}_rZ_1 q_3+{}_rZ_3 q_2-{}_rY_2 q_0),\\
\mu\, {}_rX_2 &= a(\ \ {}_rY_2 q_5-{}_rY_0 q_1+{}_rY_1 q_3)+b(\ \ {}_rZ_2 q_0-{}_rZ_3 q_1+{}_rZ_1 q_3),\\
\mu\, {}_rX_3 &= a(-{}_rY_1 q_0+{}_rY_2 q_1-{}_rY_3 q_2)+b(\ \ {}_rZ_3 q_0+{}_rZ_2 q_1-{}_rZ_1 q_2),
\end{aligned}
$$

But the lines of left strip are the generators of a Clifford's surface.[1] Thus the family of somas can be generated by the half-translations along each pair of generators of a Clifford's surface.

---

1 See Coolidge, 'Non-euclidean Geometry.' p, 128.

The system of the somas defined by (i) shall be called left (right) " one dimensional soma-strip " or merely soma-strip.

The reciprocal soma-system to a soma-strip is given by the equations

$$_lU_i=a\,_lV_i+b\,_lP_i+c\,_lQ_i,\ _rU_i=a\,_rV_i+b\,_rP_2;\ i=0,1,2,3.$$

§ 7.

The Two-dimensional Soma-chain.

General two-dimensional soma-chain is a system of somas which has the coordinates linearly dependent with real coefficients on those of three given somas, i.e.,

$$\rho\,\mathfrak{X}_i=a\mathfrak{Y}_i+b\,\mathfrak{Z}_i+cw_i,\qquad i=0,1,2,3\ \ldots\ldots\ (\text{i}).$$

Or this may be written in the form

$$\rho\,_lX_i=a\,_lY_i+b\,_lZ_i+c\,_lW_i,\ \rho\,_rX_i=a\,_rY_i+b\,_rZ_i+c\,_rW_i.$$

There occur many cases :

case I.

$$|\mathfrak{Y}\,\mathfrak{Z}\,w|\neq 0,$$

Or

$$|_lY_lZ_lW|\neq 0,\ |_rY_rZ_rW|\neq 0.$$

Such C_2 shall be called 'soma-chain congruence.' In this case the coordinates $(_lY)$, $(_lZ)$, $(_lW)$ are linearly independent. $(_rY)$, $(_rZ)$, $(_rW)$ are also such ones. The system of somas are represented by two projective fields of points, (or two projective hundles of planes in the lower layer) in the representing spaces. And the somas (u) in the lower layer such that

$$(\mathfrak{U})=|\mathfrak{Y}\,\mathfrak{Z}\,w|$$

is orthogonal to every somas of the soma-chain. For

$$(\mathfrak{U}\mathfrak{X})=(_lU_lX)\,_le+(_rU_rX)\,_re=0.$$

A soma, orthogonal to all the somas of the soma-chain, shall be called 'soma-nucleus' of the soma-chain congruence. If we take the soma-nucleus as the protosoma (1000)(1000), then the equation of the soma-chain congruence may be writien in the form

[1] See Klein 'Zur Nichteuclidischen Geometrie,' Math. Ann., **27** (1890).

$$\rho\, {}_lX_0=0, \qquad \rho\, {}_rX_0=0,$$
$$\rho\, {}_lX_1=a\, {}_lY_1+b\, {}_lZ_1+c\, {}_lT_1, \quad \rho\, {}_rX_1=a\, {}_rY_1+b\, {}_rZ_1+c\, {}_rT_1.$$
$$\rho\, {}_lX_1=a\, {}_lY_2+b\, {}_lZ_2+c\, {}_lT_2, \quad \rho\, {}_rX_2=a\, {}_rY_2+b\, {}_rZ_2+c\, {}_rT_2.$$
$$\rho\, {}_lX_3=a\, {}_lY_3+b\, {}_lZ_3+c\, {}_lT_3, \quad \rho\, {}_rX_3=a\, {}_rY_3+b\, {}_rZ_3+c\, {}_rT_3,$$

Solving for a, b, c the equations in the right and substituting in the left, we have

$${}_lX_0={}_rX_0=0$$
$${}_lX_1=(a_1\, {}_rX_1),$$
$${}_lX_2=(a_2\, {}_rX),$$
$${}_lX_3=(\,{}_3\, {}_rX).$$

Taking the nucleus as the protosoma, if the somas (0100)(0100), (0010)(0010), (0001)(0001) are included in the soma-chain congruence and these somas are taken as the bases of the soma-chain congruence, then we have the equations

$$\rho\, {}_lX_0={}_rX_0=0$$
$$\rho\, {}_lX_1=a\, {}_rX_1,$$
$$\rho\, {}_lX_2=a\, {}_rX_2,$$
$$\rho\, {}_lX_3=a\, {}_rX_3.$$

Such a special soma-chain congruence can be generated by half translations along every cross of cross chain congruence by taking the lines of cross which coincide with its reciprocal cross-chain congruence as the representation of the protosoma, i.e , from a cross-chain con, gruence

$$\rho\, {}_lX_1=a_1\, {}_rX_1,$$
$$\rho\, {}_lX_2=a_2\, {}_rX_2,$$
$$\rho\, {}_lX_3=a_3\, {}_rX_3,$$

the soma-chain

$${}_lX_0={}_rX_0=0$$
$${}_lX_1=a_1\, {}_rX_1$$
$${}_lX_2=a\, {}_rX_2$$
$${}_lX_3=a_3\, {}_rX_2$$

can be generated

General soma chain C_2 may be generated by the half translations from a fixed soma ($\mathcal{T}$) along every cross of chain congruence[1] whose equations are

$$\rho_l\,{}_lX_i = a\,{}_lY_i + b\,{}_lZ_i + c\,{}_lW_i,\quad \rho\,{}_rX_i = a\,{}_rY_i + b\,{}_rZ_i + c\,{}_rW_i,$$

The proof is analogous with the case of the generation of C_1. As the result of the generation, we have the soma-chain C_2 which has the equations

$$\begin{aligned}
\sigma\,{}_lX_0 &= a\,(-{}_lY_1\,{}_lT_1 - {}_lY_2\,{}_lT_2 - {}_lY_3\,{}_lT_3) + b\,(-{}_lZ_1\,{}_lT_1 - {}_lZ_2\,{}_lT_2 - {}_lZ_3\,{}_lT_3)\\
&\qquad + c\,(-{}_lW_1\,{}_lT_1 - {}_lW_2\,{}_lT_2 - {}_lW_2\,{}_lT_3),\\
\sigma\,{}_lX_1 &= a\,({}_lY_1\,{}_lT_0 - {}_lY_3\,{}_lT_2 + {}_lY_3\,{}_lT_3) + b\,({}_lZ_1\,{}_lT_0 - {}_lZ_3\,{}_lT_2 + {}_lZ_3\,{}_lT_3)\\
&\qquad + c\,({}_lW_1\,{}_lT_0 - {}_lW_3\,{}_lT_2 + {}_lW_3\,{}_lT_3),\\
\sigma\,{}_lX_2 &= a\,({}_lY_2\,{}_lT_0 + {}_lY_3\,{}_lT_1 - {}_lY_1\,{}_lT_3) + b\,({}_lZ_2\,{}_lT_1 + {}_lZ_3\,{}_lT_1 - {}_lZ_1\,{}_lT_3)\\
&\qquad + c\,({}_lW_2\,{}_lT_0 + {}_lW_3\,{}_lT_1 - {}_lW_1\,{}_lT_3),\\
\sigma\,{}_lX_3 &= a\,({}_lY_3\,{}_lT_0 - {}_lY_2\,{}_lT_1 + {}_lY_1\,{}_lT_2) + b\,({}_lZ_3\,{}_lT_0 - {}_lZ_1\,{}_lT_2 + {}_lZ_1\,{}_lT_2)\\
&\qquad + c\,({}_lW_3\,{}_lT_0 - {}_lW_3\,{}_lT_1 + {}_lW_1\,{}_lT_2).\\
\sigma\,{}_rX_0 &= a\,(-{}_rY_1\,{}_rT_1 - {}_rY_2\,{}_rT_2 - {}_rY_3\,{}_rT_3) + b\,(-{}_rZ_1\,{}_rT_1 - {}_rZ_2\,{}_rT_2 - {}_rZ_3\,{}_rT_3)\\
&\qquad + c\,(-{}_rW_1\,{}_rT_1 - {}_rW_2\,{}_rT_2 - {}_rW_3\,{}_rT_3),\\
\sigma\,{}_rX_1 &= a\,({}_rY_1\,{}_rT_0 - {}_rY_3\,{}_rT_2 + {}_rY_3\,{}_rT_3) + b\,({}_rZ_1\,{}_rT_0 - {}_rZ_3\,{}_rT_2 + {}_rZ_3\,{}_rT_3)\\
&\qquad + c\,({}_rW_1\,{}_rT_0 - {}_rW_3\,{}_rT_2 + {}_rW_3\,{}_rT_3),\\
\sigma\,{}_rX_2 &= a\,({}_rY_2\,{}_rT_0 + {}_rY_3\,{}_rT_1 - {}_rY_1\,{}_rT_3) + b\,({}_rZ_2\,{}_rT_1 + {}_rZ_3\,{}_rT_1 - {}_rZ_1\,{}_rT_3)\\
&\qquad + c\,({}_rW_2\,{}_rT_0 + {}_rW_3\,{}_rT_1 - {}_rW_1\,{}_rT_3),\\
\sigma\,{}_rX_3 &= a\,({}_rY_3\,{}_rT_0 - {}_rY_2\,{}_rT_1 + {}_rY_1\,{}_rT_2) + b\,({}_rZ_3\,{}_rT_0 - {}_rZ_2\,{}_rT_1 + {}_rZ_1\,{}_rT_2)\\
&\qquad + c\,({}_rW_3\,{}_rT_0 - {}_rW_2\,{}_rT_1 + {}_rW_1\,{}_rT_2).
\end{aligned}$$

The soma chain congruence, represented by the equation (i), are represented by a pair of fields of points projectively related in the representing spaces.

Case 2.

$$|{}_lY_lZ_lW| = 0\ (|{}_rY_rZ_rW| = 0);\ |{}_rY_rZ_rW| \neq 0,\ (|{}_rY_rZ_rW| \neq 0).$$

In this case the coordinates $({}_lY)$, $({}_lZ)$, $({}_lW)$ $({}_rY)$, $({}_rZ)$. $({}_rW)$ are linearly dependent. This system is represented by a row of points in the left (right) representing space and a field of points in the right (left) representing space. Or this is represented by a sheaf of planes in the left (right) representing space and a bundle of planes in the right (left) representing space in the lower layer. Such a system contains clearly a one-dimensional 'soma-strip.' Since $({}_lY)$, $({}_lZ)$, $({}_lW)$, $({}_rY)$, $({}_rZ)$, $({}_rW)$ are linearly dependent,

$${}_lW_i = \lambda_l Y_i + \mu\, {}_lZ_i \;({}_rW_i = \lambda\, {}_rY_i + \mu\, {}_rZ_i).$$

Therefore

$$\rho\, {}_lX_i = (a + c\lambda)\, {}_lY_i + (b + c\mu)\, {}_lZ_i \;(\rho\, {}_rX_i = (a + c\lambda)\, {}_rY_i + (b + c\mu)_r Z_i).$$

Let us now put

$$a + c\lambda l = , \quad b + c\mu = m,$$

then

$$\rho\, {}_lX_i = l\, {}_lY_i + m\, {}_lZ_i \;({}_rX_i = l\, {}_rY_i + m\, {}_rZ_i).$$

$$\rho\, {}_rX_i = (l - c\lambda)_r Y_i + (m - c\mu)_r Z_i ({}_lX_i = (l - c\lambda)_l Y_i + (m - c\mu)_l Z_i).$$

Therefore the soma-chain can be grouped into paratactic soma-strips.

§ 8.

The Three-dimensional Soma-chain.

The three-dimensional soma-chain is formed by all the soma-manifoldness, which have the coordinates linearly dependent with real coefficients on those of four given somas. As before we write the equations in the form

$$\rho \mathscr{X}_i = a\mathscr{Y}_i + b\mathscr{Z}_i + c\mathscr{T}_i + dw, \quad i = 0, 1, 2, 3 \ldots\ldots\ldots\ldots (1).$$

These equations may be also written as follows:

$$\rho_l X_i = a_l Y_i + b_l Z_i + c_l T_i + d_l W_i, \quad \rho_r X_i = a_r Y_i + b_r Z_i + c_r T_i + d_r W_i.$$

Case I.

$$|\mathscr{Y}\, \mathscr{Z}\, \mathscr{T} w| = \text{proper dual number}^1.$$

Or

$$|{}_lY_l Z_l T_l W| \neq 0, \quad |{}_rY_r Z_r T_r W| \neq 0. \ldots\ldots\ldots\ldots\ldots (ii).$$

In this case the four given somas do not lie on one and the same plane in both of the representing spaces. Let us now solve $a : b . c : d$ from the equations

$$\rho\, {}_lX_i = \Sigma\, a\, {}_lY_i,$$

and substitute in the equations

$$\rho\, {}_rX_i = \Sigma\, a\, {}_rY_i,$$

then we have the equations

$${}_lX_i = (a_i\, {}_lX), \qquad |a_{ij}| \neq 0 \ldots\ldots\ \ldots\ldots (iii).$$

The last relation is easily seen, for

[1] A proper dual number is such one that $\alpha_l e + \beta_1 e$ $(\alpha \times \beta \neq 0)$

$$a_{ij}=\frac{|\,{}_lY_i\,{}_lZ_k\,{}_lT_l\,{}_lW_s\,|}{|\,{}_lY\,{}_lZ\,{}_lT\,{}_lW|}$$

and hence $|\,a_{ij}\,|=\dfrac{|\,{}_rY,{}_rZ,{}_rT,{}_rW\,|}{|\,{}_lY\,{}_lZ\,{}_lT\,{}_lW|}\neq 0.$ .... (iv)

Conversely, we can reach to the equations (i), (ii) from (iii), (iv).

We shall call the C_3 as 'homographic soma-chain (soma-chain complex)'.

$({}_lX)$ and $({}_rX)$ of a homographic three dimensional soma-chain have a collinear relation with non-vanishing discriminant. Conversely, every three dimensional soma-chain, so related, is homographic. Therefore the relation (iii) may be interpreted as a collinear relation between the two representing spaces. If, the points $({}_lX)$ trace a row of points, the corresponding points $({}_rX)$ will trace a second row, projectively related with the rows $({}_rX)$. and conversely. Further if the points $({}_lX)$ trace a field of points, then the corresponding points $({}_rX)$ will trace also a field of points, projectively related with $({}_lX)$, and conversely. Thus we obtain ∞^3 soma-chain congruences, contained in the soma chain complex. The somas of each soma-chain congruence are orthogonal to the nucleus of the soma-chain congruence, (in the other layer). We have to find the locus of the nuclei. Since the coordinates $({}_lU), ({}_rU)$ of this nucleus are plane-coordinates of the collinear fields in the representing spaces, they must be related by the transformation contragradient. Conversely, when we have given one soma of the lower layer $({}_lU), ({}_rU)$, it will be orthogonal with but a single soma of the given soma-complex, in the case when its $({}_lU)\,({}_rU)$ are connected by the relation contragredient to (iii), it will be the nucleus of a soma-chain congruence of the given soma-complex.

These nuclei will thus form a second three dimensional soma-chain, whose equations are

$$\mu\,,U_i=(A_i\,{}_rU)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)},$$

where A_{ij} is the cofactor of the element a_{ij} in the determinant $|a_{ij}|$. We shall call this soma-chain complex, the 'reciprocal soma-chain complex' of the given soma-chain complex.

No two somas of a homographic soma complex are paratactic. Each soma in space has a single left and a right paratactic soma in the homographic soma complex. For the equation

$$\rho\ {}_lX_i'=({}_la_l X)\ (\rho\ {}_rX_i'=({}_ra_r X))$$

have one and only one system of the solution.

If we suppose two reciprocal soma-chain complexes are superposed, that is to say, when considered as belonging to the same layer, do they have any common soma? To investigate this, we replace $({}_lU)$ and $({}_rU)$ in the equation (v) by $(\rho\ {}_lX)$ and $(\rho\ {}_rX)$ respectively, and substitute the value $({}_rX)$ expressed by $({}_lX)$ in the ıight hand side of the resulting equation, then we have

$$\tau X_i = (b_i\ {}_lX) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \text{(vi)},$$

where

$$b_{ij} = b_{ji} = \sum_{k=0}^{3} A_{ik}\, A_{jk}.$$

Therefore, we have a symmetric determinant equation by eliminating $({}_lX)$ from the equations (vi):

$$\begin{vmatrix} b_{00}-\tau & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11}-\tau & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22}-\tau & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33}-\tau \end{vmatrix} = 0 \ \ldots\ldots\ldots \ \text{(vii)}.$$

But as a symmetric determinant equation has always real roots, we have four real roots for τ. The somas corresponding to the four roots of the equation (vii) are orthogonal to one another.

Three dimensional soma chain C_3 can be generated from a chain congruence (i) by the equitranslations of a soma $(\mathcal{T})$ along every cross of the chain congruence as in the case with C_2. The resulting soma chain has the equations

$$\begin{aligned}
\rho\ {}_lX_0' &= {}_lT_{0l}X_0 + a(-{}_lT_{1l}Y_2 - {}_lT_{2l}Y_2 - {}_lT_{3l}Y_3) + b(-{}_lT_{1l}Z_1 - {}_lT_{2l}Z_2 - {}_lT_{3l}Z_3) \\
&\quad + c(-{}_lT_{1l}W_1 - {}_lT_{2l}W_2 - {}_lT_{3l}W_3), \\
\rho\ {}_lX_1' &= {}_lT_{1l}X_1 + a(\ {}_lT_{0l}Y_1 - {}_lT_{3l}Y_2 + {}_lT_{2l}Y_3) + b(\ {}_lT_{0l}Z_1 - {}_lT_{3l}Z_2 + {}_lT_{2l}Z_3) \\
&\quad + c(\ {}_lT_{0l}W_1 - {}_lT_{3l}W_2 + {}_lT_{2l}W_3), \\
\rho\ {}_lX_2' &= {}_lT_{2l}X_2 + a(\ {}_lT_{3l}Y_1 + {}_lT_{0l}Y_2 - {}_lT_{1l}Y_3) + b(\ {}_lT_{1l}Z_3 + {}_lT_{0l}Z_2 - {}_lT_{1l}Z_0) \\
&\quad + c({}_lT_{3l}W_1 + {}_lT_{0l}W_2 - {}_lT_{1l}W_3), \\
\rho\ {}_lX_3' &= {}_lT_{3l}X_3 + a(-{}_lT_{2l}Y_1 + {}_lT_{1l}Y_2 + {}_lT_{0l}Y_3) + b(-{}_lT_{2l}Z_1 + {}_lT_{1l}Z_2 + {}_lT_{0l}Z_3)
\end{aligned}$$

$$\rho\,{}_rX_3' = {}_rT_{3r}X_3 + a(-{}_rT_{2r}Y_1 + {}_rT_{1r}Y_2 + {}_rT_{0r}Y_3) + b(-{}_rT_{2r}Z_1 + {}_rT_{r1}Z_1 + {}_jT_{0r}Z_3)$$
$$+ c(-{}_rT_{2r}W_1 + {}_rT_{1r}W_2 + {}_rT_{0r}W_3),$$
$$\rho\,{}_rX_1' = {}_rT_{1r}X_1 + a(\ {}_rT_{0r}Y_1 - {}_rT_{3r}Y_2 + {}_rT_{2r}Y_3) + b(\ {}_rT_{0r}Z_1 - {}_rT_{3r}Z_2 + {}_rT_{2r}Z_3)$$
$$+ c({}_rT_{0r}W_1 - {}_rT_{3r}W_2 + {}_rT_{2r}W_3),$$
$$\rho\,{}_rX_2' = {}_rT_{2r}X_2 + a(\ {}_rT_{3r}Y_1 + {}_rT_{0r}Y_2 - {}_rT_{1r}Y_2) + b(\ {}_rT_{3r}Z_1 + {}_rT_{0r}Z_2 - {}_rT_{1r}Z_3)$$
$$+ c(\ {}_rT_{3r}W_1 + {}_rT_{0r}W_0 - {}_rT_{1r}W_3),$$
$$\rho\,{}_rX_3' = {}_rT_{3r}X_3 + a(-{}_rT_{2r}Y_1 + {}_{r1}T_rY_2 + {}_rT_{0r}Y_3) + b(-{}_{r2}T_rZ_1 + {}_{r1}T_rZ_2 + {}_rT_{0r}Z_3)$$
$$+ c(-{}_{r2}T_rW_1 + {}_rT_{1r}W_2 + {}_rT_{0r}W_3).$$

Therefore the soma-chain is C_3.

The above four somas found as the roots of the equation (vii) are not in general identical with the soma and the three somas obtained by the half translations along to the three lines representing it. But as a special case, if the four somas can be looked upon as the fundamental somas (1000)(10000), (0100)(0100), (0010)(0010), (0001)(0001), the equation to the soma-chain complex may be reduced to the canonical forms

$${}_lX_i = a_i\,{}_rX_i \qquad i=0,1,2,3\,;\ a_0a_1a_2a_3 \neq 0,$$

The equation of the reciprocal soma-chain complex must have the form

$${}_lU_i = \frac{1}{a_i}\,{}_rU_i, \qquad i=0,1,2,3.$$

Such a soma chain complex can be generated by the equitranslations along each cross of a homographic chain congruence from the somas defined by three orthogonal somas of the chain-congruence which are superposed with the somas of its reciprocal congruence.

There are two specially interesting cases among homographic soma chain complexes:

$$(1)\quad a_i^2 = a_j^2,\ a_k^2 = a_s^2\,;$$
$$(2)\quad a_i^2 = a_j^2 = a_k^2 = a_s^2.$$

where i, j, k, s is a permutation of the figures 0, 1, 2, 3.

In the case (1)

$$a_0^2 = a_i^2 = a^2,\ a_j^2 = a_k^2 = b^2.$$

[1] See Coolidge's dissertation, 'Dual Projective Geometry,' p. 40.

We can write the equations to the homographic soma-chain complex in the forms

$$
\begin{aligned}
{}_lX_0' &= a\,{}_rX_0,\\
{}_lX_i &= a\,{}_rX_i,\\
{}_lX_j &= b\,{}_rX_j,\\
{}_lX_k &= b\,{}_rX_k.
\end{aligned}
$$

Let us now consider the case

$$i=1,\ j=2,\ k=3$$

and apply the equitranslation with respect to the X_1-axis, then the soma $(\mathscr{X})$ will be transformed into the soma $(\mathscr{X}')$. The relations between $(\mathscr{X})$ and $(\mathscr{X}')$ are

$$
\begin{aligned}
\rho\,{}_lX_0' &= a\,(\ \ p_0\,{}_rX_0 - p_1\,{}_rX_1), & \rho\,{}_rX_0' &= \ \ p_0\,{}_rX_0 - p_1\,{}_rX_0,\\
\rho\,{}_lX_1' &= a\,(-p_1\,{}_rX_0 - p_0\,{}_rX_1), & \rho\,{}_rX_1' &= -p_1\,{}_rX_0 - p_0\,{}_rX_1,\\
\rho\,{}_lX_2' &= b\,(-p_0\,{}_rX_2 - p_1\,{}_rX_3), & \rho\,{}_rX_3' &= -p_0\,{}_rX_2 - p_1\,{}_rX_3,\\
\rho\,{}_lX_3' &= b\,(\ \ p_1\,{}_rX_2 - p_0\,{}_rX_3), & \rho\,{}_rX_3' &= \ \ p_1\,{}_rX_2 + p_0\,{}_rX_3
\end{aligned}
$$

Thus

$$
\begin{aligned}
{}_lX_0' &= a\,{}_rX_0',\\
{}_lX_1' &= a\,{}_rX_1',\\
{}_lX_2' &= b\,{}_rX_2',\\
{}_lX_3' &= b\,{}_rX_3'.
\end{aligned}
$$

Therefore a soma of the soma-chain complex is transformed into another soma of the soma complex by the transformation. In the same manner, we can prove that the general case is true for the X_i-axis.

Case 3.

We may assume, by suitable choice of the bases of the homographic soma chain complex that

$$
\begin{aligned}
a_0 &> 0, & a_0 &= b,\\
a_1 &> 0, & a_1 &= b,\\
a_2 &> 0, & a_2 &= b,\\
a_3 &> 0, & a_3 &= b.
\end{aligned}
$$

But a rotation about the origin is defined by the equations

$$ {}_lX' = P_l X.\ ,{}_rX' = P_r X. $$

Or

$$\rho_l X_0' = p_0\,{}_lX_0 - p_1\,{}_lX_1 - p_2\,{}_lX_2 - p_3\,{}_lX_3,\ \rho_r X_0' = p_0\,{}_rX_0 - p_1\,{}_rX_1 - p_2\,{}_rX_2 - p_3\,{}_rX_3,$$
$$\rho_l X_1' = p_1\,{}_lX_0 + p_0\,{}_lX_1 - p_3\,{}_lX_2 + p_2\,{}_lX_3,\ \rho_r X_1' = p_1\,{}_rX_0 + p_0\,{}_rX_1 - p_3\,{}_rX_2 + p_2\,{}_rX_3,$$
$$\rho_l X_2' = p_2\,{}_lX_0 + p_3\,{}_lX_1 + p_0\,{}_lX_2 - p_1\,{}_lX_3,\ \rho_r X_2' = p_2\,{}_rX_0 + p_3\,{}_rX_1 + p_0\,{}_rX_2 - p_1\,{}_rX_3,$$
$$\rho_l X_3' = p_3\,{}_lX_0 - p_2\,{}_lX_1 + p_1\,{}_lX_2 + p_0\,{}_lX_3,\ \rho_r X_3' = p_3\,{}_rX_0 - p_2\,{}_rX_1 + p_1\,{}_rX_2 + p_0\,{}_rX_3.$$

...... (viii).

Therefore by the substitution of

$$\begin{aligned} {}_lX_0 &= b\,{}_rX_0, \\ {}_lX_1 &= b\,{}_rX_1, \\ {}_lX_2 &= b\,{}_rX_2, \\ {}_lX_3 &= b\,{}_rX_3. \end{aligned}$$

in (viii), we have

$$\rho\,{}_lX_0' = b(p_0\,{}_rX_0 - p_1\,{}_rX_1 - p_2\,{}_rX_2 - p_3\,{}_rX_3),\ \rho\,{}_rX_0' = p_0\,{}_rX_0 - p_1\,{}_rX_1 - p_2\,{}_rX_2 - p_3\,{}_rX_3,$$
$$\rho\,{}_lX_1' = b(p_1\,{}_rX_0 + p_0\,{}_rX_1 - p_3\,{}_rX_2 + p_2\,{}_rX_3),\ \rho\,{}_rX_1' = p_1\,{}_rX_0 + p_0\,{}_rX_1 - p_3\,{}_rX_2 + p_2\,{}_rX_3,$$
$$\rho\,{}_lX_2' = b(p_2\,{}_rX_0 + p_3\,{}_rX_1 + p_0\,{}_rX_2 - p_1\,{}_rX_3),\ \rho\,{}_rX_2' = p_0\,{}_rX_0 + p_3\,{}_rX_1 + p_0\,{}_rX_2 - p_1\,{}_rX_3,$$
$$\rho\,{}_lX_3' = b(p_3\,{}_rX_0 - p_2\,{}_rX_1 + p_1\,{}_rX_2 + p_0\,{}_rX_3),\ \rho\,{}_rX_3' = p_3\,{}_rX_0 - p_2\,{}_rX_1 + p_1\,{}_rX_2 + p_3\,{}_rX_3.$$

Thus

$$\begin{aligned} {}_lX_0' &= b\,{}_rX_0' \\ {}_lX_1' &= b\,{}_rX_1' \\ {}_lX_2' &= b\,{}_rX_2' \\ {}_lX_3' &= b\,{}_rX_3. \end{aligned}$$

Therefore such a soma-chain is transformed into itself by any rotation about the origin.

Case II.

$$|\,{}_lY_lZ_lT_lW\,| = 0,\ (\,|\,{}_rY_rZ_rT_rW\,| = 0\,)\,,\ |\,{}_rY_rZ_rT_rW\,| \neq 0,\ (\,|\,{}_lY_lZ_lT_lW\,| \neq 0).$$

In this case the coordinates $({}_lY), ({}_lZ), ({}_lT), ({}_lW), ({}_rY), ({}_rZ), ({}_rT), ({}_rW)$ of the given four somas are linearly dependent; thus the four points correspondent to $({}_lY), ({}_lZ), ({}_lT), ({}_lW)$ $[({}_rY)\ ({}_rZ), ({}_rT), ({}_rW)]$ in the left (right) representing space are coplanar, i.e.

$${}_lW_i = l_l Y_i + m_l Z_i + n_l T_i,\ ({}_rW_i = l_r Y_i + m_r Z_i + n_r T_i).$$

Therefore

$$\rho\, {}_lX_i = (a+ld)\, {}_lY_i + (b+md)\, {}_lZ_i + (c+nd)\, {}_lT_i,$$

$$(\rho\, {}_rX_i = (a+ld)\, {}_rY_i + (b+md)\, {}_rZ_i + (c+nd)\, {}_rT_i).$$

Let us now put

$$a+ld \equiv p,\ b+md \equiv q,\ c+nd \equiv s,$$

then we have

$$\rho_l X_i = p_l Y_i + q_l Z_i + s_l T_i,\ \rho_r X_i = p_r Y_i + q_r Z_i + s_r T_i - d(l_r Y_i + m_r Z_i + n_r T_i)$$

$$(\rho_r X_i = p_r Y_i + q_r Z_i + s_r T_i,\ \rho_l X_i = p_l Y_i + q_l Z_i + s_l T_i - d(l_l Y_i + m_l Z_i + n_l T_i)).$$

Therefore in this case, the system of somas can be grouped into a system of one dimensional soma strips. The canonical form of the equations of the soma chain will be

$$\begin{aligned} {}_lX_{i_0} &= a_{i_0}\, {}_rX_{i_0}, \\ {}_lX_{i_1} &= a_{i_1}\, {}_rX_{i_1}, \\ {}_lX_{i_2} &= a_{i_2}\, {}_rX_{i_2}, \\ {}_lX_{i_3} &= 0. \end{aligned}$$

$i_0, i_1\, i_2$ is a permutation of 0, 1, 2, 3 taken three at once.

Let $({}_l\mathscr{X}^{(1)})({}_l\mathscr{X}^{(2)})({}_l\mathscr{X}^{(3)})({}_l\mathscr{X}^{(4)})$ be any four somas of the C_3, then

$$|{}_lX^{(1)}\, {}_lX^{(2)}\, {}_lX^{(3)}\, {}_lX^{(4)}| = \begin{vmatrix} a^{(1)} & b^{(1)} & c^{(1)} & d^{(1)} \\ a^{(2)} & b^{(2)} & c^{(2)} & d^{(2)} \\ a^{(3)} & b^{(3)} & c^{(3)} & d^{(3)} \\ a^{(4)} & b^{(4)} & c^{(4)} & d^{(4)} \end{vmatrix} \cdot |{}_lY_l Z_l T_l W|$$

$$= 0.$$

And

$$|\, {}_lX^{(1)}{}_lX^{(2)}{}_lX^{(3)}{}_lX^{(4)}\,| = a_0 a_1 a_2 a_3\, |\, {}_rX^{(1)}{}_rX^{(2)}{}_rX^{(3)}{}_rX^{(4)}\,|$$

Since by assumption

$$|\, {}_rX^{(1)}{}_rX^{(2)}{}_rX^{(3)}{}_rX^{(4)}| \neq 0,$$

one of a_0, a_1, a_2, a_3 must be zero. This is the fact which we have to prove.

When the rank of the determinant

$$|\, {}_lY_l Z_l T_l W|\ (\,|{}_rY_r Z_r T_r W|\,)$$

is one, we have a system of paratactic somas.

Case III.

If

$$|{}_lY_lZ_lT_lW| = 0, \quad |{}_rY_rZ_rT_rW| = 0,$$

without the vanishing of the first minors of either determinant, we have C_3' whose somas are all orthogonal to all the somas of the C_3. C_3' may be represented by the equations (in the lower layer)

$$\rho u_i = \lambda \begin{vmatrix} \mathcal{Y}_j & \mathcal{Y}_k & \mathcal{Y}_s \\ \mathcal{Z}_j & \mathcal{Z}_k & \mathcal{Z}_s \\ \mathcal{T}_j & \mathcal{T}_k & \mathcal{T}_s \end{vmatrix} + \mu \begin{vmatrix} \mathcal{Z}_j & \mathcal{Z}_k & \mathcal{Z}_s \\ \mathcal{T}_j & \mathcal{T}_k & \mathcal{T}_s \\ w_j & w_k & w \end{vmatrix} + \nu \begin{vmatrix} \mathcal{T}_j & \mathcal{T}_k & \mathcal{T}_s \\ w_j & w_k & w_s \\ \mathcal{Y}_j & \mathcal{Y}_k & \mathcal{Y}_s \end{vmatrix} + \sigma \begin{vmatrix} w_j & w_k & w_s \\ \mathcal{Y}_j & \mathcal{Y}_k & \mathcal{Y}_s \\ \mathcal{Z}_j & \mathcal{Z}_k & \mathcal{Z}_s \end{vmatrix}$$

A system of somas whose dual coordinates ($\mathcal{X}$) are functions of three essential real parameters u, v, w shall be called 'soma-system Ω.' We mean by the 'general position' such a position where the determinant

$$\left| \mathfrak{U} \frac{\partial \mathfrak{U}}{\partial u} \frac{\partial \mathfrak{U}}{\partial v} \frac{\partial \mathfrak{U}}{\partial w} \right|$$

is a proper dual number, i.e.

$$\left| {}_lU \frac{\partial_l U}{\partial u} \frac{\partial_l U}{\partial v} \frac{\partial_l U}{\partial w} \right| \not\equiv 0, \quad \left| {}_rU \frac{\partial_r U}{\partial u} \frac{\partial_r U}{\partial v} \frac{\partial_r U}{\partial w} \right| \not\equiv 0.$$

In this case, we shall have as the coordinates of the common orthogonal soma to a soma $(u), (u+du_1), (u+du_2)$ (taken in the lower layer) of such a system Ω the following ratio:

$$\mathcal{X}_0 : \mathcal{X}_1 : \mathcal{X}_2 : \mathcal{X}_3 = \begin{vmatrix} \mathfrak{U}_0 & \mathfrak{U}_1 & \mathfrak{U}_2 & \mathfrak{U}_3 \\ \frac{\partial \mathfrak{U}_0}{\partial u} du_1 + \frac{\partial \mathfrak{U}_0}{\partial v} dv_1 + \frac{\partial \mathfrak{U}_0}{\partial w} dw_1 & ,, & ,, & ,, \\ \frac{\partial \mathfrak{U}_0}{\partial u} du_2 + \frac{\partial \mathfrak{U}_0}{\partial v} dv_2 + \frac{\partial \mathfrak{U}_0}{\partial w} dw_2 & ,, & ,, & ,, \end{vmatrix}$$

$$= V_0^{(1)} du_2 + V_1^{(1)} dv_2 + V_2^{(1)} dw_2 : V_0^{(2)} du_2 + V_1^{(2)} dv_2 + V_2^{(2)} dw_2$$

$$: V_0^{(3)} du_2 + V_1^{(3)} dw_2 + V_2^{(3)} dw_2 : V_0^{(4)} du_2 + V_1^{(4)} dw_2 + V_2^{(4)} dw_2.$$

If the soma $(\mathfrak{U}), (\mathfrak{U}+d\mathfrak{U})$ are fixed, then the totalisy of the orthogonal somas to the soma $(\mathfrak{U}), (\mathfrak{U}+d\mathfrak{U}_1), (\mathfrak{U}+d\mathfrak{U}_2)$ forms a soma-chain complex

Specially in the case,

$$\left| \mathfrak{U} \frac{\partial \mathfrak{U}}{\partial u} \frac{\partial \mathfrak{U}}{\partial v} \frac{\partial \mathfrak{U}}{\partial w} \right| \equiv 0, \text{ i e.}$$

$$\left| {}_lU \frac{\partial_l U}{\partial u} \frac{\partial_l U}{\partial v} \frac{\partial_l U}{\partial w} \right| \equiv 0, \left| {}_rU \frac{\partial_r U}{\partial u} \frac{\partial_r U}{\partial v} \frac{\partial_r U}{\partial w} \right| \equiv 0.$$

the soma, whose dual coordinates are

$$\rho \mathfrak{X}_\nu = \begin{vmatrix} \frac{\partial \mathfrak{U}_i}{\partial u} & \frac{\partial \mathfrak{U}_k}{\partial u} & \frac{\partial \mathfrak{U}_s}{\partial u} \\ \frac{\partial \mathfrak{U}_i}{\partial v} & \frac{\partial \mathfrak{U}_k}{\partial v} & \frac{\partial \mathfrak{U}_s}{\partial v} \\ \frac{\partial \mathfrak{U}_i}{\partial w} & \frac{\partial \mathfrak{U}_k}{\partial w} & \frac{\partial \mathfrak{U}_s}{\partial w} \end{vmatrix},$$

is the common orthogonal soma to a soma $(\mathfrak{U})$ and every one of the somas adjacent to it.

CHAPTER II.

The Geometry of Somas in the Hyperbolic Space.

SECTION I.

Soma and Its Transformations.

§ 9.

Soma and Its Coordinates.

Let A represents a quaternion

$$a_0 + ia_1 + ja_2 + ka_3 \qquad (a_0 a_1 a_2 a_3 \neq 0),$$

and $\dot{A}$

$$a_0 - ia_1 - ja_2 + ka_3,$$

where

$$i^2 + 1 = j^2 + 1 = k^2 + 1 = ijk + 1$$
$$= ij + ji = jk + kj = ki + ik = 0.$$

Taking the coordinate in the hyperbolic space as in the elliptic space, we can write the equation to the absolute in this space in the form

$$\mathfrak{x}_0^2 + \mathfrak{x}_1^2 + \mathfrak{x}_2^2 + x_3^2 = 0.$$

For the sake of brevity, we put the measure of the curvature of the space $\frac{1}{K^2}$ equal to -1, and take as the coordinates of a point $(\mathfrak{x})$ the ratio (x) such that

$$ix_0 : x_1 : x_2 : x_3 = \mathfrak{x}_0 . \mathfrak{x}_1 : \mathfrak{x}_2 : \mathfrak{x}_3.$$

Thus we see that (x) is equal to the ratio of four real numbers.

The equation to the absolute may be written in the following form by adopting the new coordinates:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0.$$

Let us now put

$$\xi \equiv \frac{x_1 + ix_1 + jx_2 + kx_3}{x_0},$$

then the rotation about the coordinate-origin is represented by the equation

$$\xi' = A\xi(\dot{A})^{-1} \; (A \equiv a_0 + a_1 i + a_2 j + a_3 k).$$

The translation by which the origin is brought to the point

$$1 : \frac{2u_0u_1}{(uu)} : \frac{2u_0u_2}{(uu)} : \frac{2u_0u_3}{(uu)},$$

where

$$(uu) \equiv u_0^2 + u_1^2 + u_2^2 + u_3^2,$$

is given by the equation

$$\xi' = \frac{\xi + U}{U\xi + 1} \left(U \equiv \frac{u_1 + iu_2 + ju_2}{u_0} \right)$$

By the combination of the two kinds of motions, any motion in the hyperbolic space can be obtained. Thus any motion in the space may be represented by the transformation equation

$$\xi' = \frac{A\xi + B}{\dot{B}\xi + \dot{A}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where

$$(B \equiv b_0 + b_1 i + b_2 j + b_3 k).$$

The transformation may also be represented by the bi-quaternion

$$A + B\sigma,$$

where σ is a unit such that

$$i\sigma + \sigma i = j\sigma + \sigma j = 0.$$

The compound motion of the two motions $A_1 + B_1\sigma$ and $A_2 + B_2\sigma$ is given by the motion $A_2 + B_3\sigma$ such that

$$A_3 + B_3\sigma = (A_2 + B_2\sigma)(A_1 + B_1).$$

Every biquaternion

$$Q \equiv a_0+ia_1+ja_1+ka_3+(b+ib+jb+kb)\,\sigma$$

represents a motion when

$$AU = B.$$

Thus, for a motion

$$A+B\sigma,$$

$$(a_0-ia_1-ja_2-ka_3)(b_0+ib_1+jb_2+kb_3)$$

must have the form

$$p_1+ip_2+jp_3, \text{ i.e.}$$

$$a_0b_3-a_1b_2+a_2b_1-a_3b_0 = 0.$$

Let us put

$$\sigma ij = \varepsilon\,(\varepsilon^2=-\sigma^2)$$

and substitute

$$\begin{aligned} a_0 &= a_3, & b_0 &= b_0, \\ a_1 &= -a_2, & b_1 &= b_1, \\ a_2 &= a_1, & b_2 &= b_2, \\ a_3 &= -a_0, & b_3 &= b_3, \end{aligned}$$

then the biquaternion $(A+B\sigma)$ may be written in the form

$$(A+A\sigma)\sigma \equiv \varepsilon A^{(1)}+B^{(1)} = \varepsilon(a_0+ia_1+ja_2+ka_3)+b_0+ib_1+jb_2+kb_3$$

with the condition

$$(ab) \equiv a_0b_0+a_1b_1+a_2b_2+a_3b_3 = 0.$$

Now let

$$A_1+B_1\sigma = (C+D\sigma)(A+B\sigma),$$

then

$$A_1+B_1\sigma = CA+D\dot{B}+(CB+D\dot{A})\sigma \ldots\ldots\ldots\ldots \text{(ii)}.$$

The entire preceeding part of this paragraph is abstracted from Vahlen's paper, 'Ueber Bewegungen und complex Zahlen,' Math. Ann. Vol. 55, p. 585 and Vahlen's book, 'Abstruckte Geometrie,' pp. 279–283.

We shall define a soma and the protosoma as in the case of the elliptic space by a position of a rigid body and a fixed position respectively. A soma is represented by three oriented lines cutting orthogonally each other and fixed in the body, thus a soma can be brought from the protosoma whose representing oriented lines coincide with the coordinate-axes. Thus a soma may be represented by the biquaternion $A+B\sigma$.

A motion represented by $A+B\sigma$ may be expressed by the equations

$$x_0'=(a_0^0+a_1^2+a_2^2+a_3^2+b_0^2+b_1^2+b_2^2+b_3^2)\,x_0+2\,(a_0b_0+a_1b_1+a_2b_2+a_3b_3)\,x,$$

$$+2\,(a_0b_1-a_0b_1-a_2b_3+b_2a_3)\,x_2+2\,(a_0b_2+a_1b_3-b_0a_2-a_3b_1)\,x_3,$$

$$x_1'=2\,(a_0b_0-a_1b_1-a_2b_2+a_3b_3)\,x_0+(a_0^2-{}_2a_1-a_2^2+a_3^2+b_0^2-b_1^2-b_2^2+b_3^2)x_1,$$

$$+2\,(-a_0a_1-a_2a_3+b_0b_1+b_2b_3)\,x_2+2\,(-a_0a_2+a_1a_3+b_0b_2-b_1b_3)\,x_3,$$

$$x_2'=2\,(a_0b_1+a_1b_0-a_2b_3-a_3b_2)\,x_0+2\,(a_0a_1-a_2a_3+b_0b_1-b_2b_3)\,x_1$$

$$+(a_0^2-a_1^2+a_2^2-a_3^2-b_0^2+b_1^2-b_2^2+b_3^2)x_2+2(a-{}_0a_3-a_1a_2+b_1b_2+b_0b_3)x_3,$$

$$x_3'=2\,(a_0b_2+a_1b_3+b_0a_2+b_1a_3)\,x_0+2\,(a_0a_2+b_0b_2+a_1a_3+b_1b_3)\,x_1$$

$$+2(a_0a_3-a_1a_2+b_1b_2-b_0b_3)\,x_2+2(a_0^2+a_1^2-a_2^2-a_3^2-b_0^2-b_1^2-b_2^2+b_3^2)\,x_3.$$

If we put

$$a_0+\sqrt{-1}\;b_0\equiv\alpha_0,\quad c_0+\sqrt{-1}\;d_0\equiv\beta_0,\quad a_0'+\sqrt{-2}\;b_0'\equiv\alpha_0',$$

$$a_1+\sqrt{-1}\;b_1\equiv\alpha_1.\quad c_1+\sqrt{-1}\;d_1\equiv\beta_1,\quad a_1'+\sqrt{-1}\;b_1'\equiv\alpha_1,$$

$$a_2+\sqrt{-1}\;b_2\equiv\alpha_2,\quad c_2+\sqrt{-1}\;d_2\equiv\beta_2,\quad a_2'+\sqrt{-1}\;b_2'\equiv\alpha_2',$$

$$a_3+\sqrt{-2}\;b_3\equiv\alpha_3,\quad c_3+\sqrt{-1}\;d_3\equiv\beta_3,\quad a_3'+\sqrt{-1}\;b_3'\equiv\alpha_3',$$

then the relation (ii) may be represented by another set of equations. Since

$$\alpha_0'=-a_0c_3-a_1c_2+a_2c_1-a_3c_0+b_0d_3+b_1d_2+b_2d_1+b_3d_0$$

$$+\sqrt{-1}\,(a_0d_3+a_1d_2-a_2d_1+a_3d_0+b_0c_3+b_1c_2-b_2c_1+b_3c_0),$$

$$\alpha_1'=a_0c_2-a_1c_3-a_2c_0+a_3c_1-b_0d_2+b_1d_3+b_2d_0+b_3d_1$$

$$+\sqrt{-1}\,(-a_0d_2+a_1d_3+a_2d_0+a_3d_1-b_0c_2+b_1c_3+b_2c_0+b_3c_1),$$

$$\alpha_2'=-a_0c_1+a_1c_0-a_2c_3-a_3c_2+b_0d_1-b_1d_0+b_2d_3+b_3d_2$$

$$+\sqrt{-2}\,(a_0d_1-a_1d_0+a_2d_3+a_3d_2+b_0c_1-b_1c_0+b_2c_3+b_3c_3),$$

$$\alpha_3'=a_0c_0+a_1c_1+a_2c_2-a_3c_3-b_0d_0-b_1d_1-b_2d_2+b_3d_3$$

$$+\sqrt{-1}\,(-a_0d_0-a_1d_1-a_2d_2+a_3d_2-b_0c_0-b_1c_1-b_2c+b_3c_3),$$

$$-\alpha_3'=\tilde\beta_0\tilde\alpha_0-\tilde\beta_1\tilde\alpha_1-\tilde\beta_2\tilde\alpha_2-\tilde\beta_3\tilde\alpha_3,$$

$$-\alpha_2'=-\tilde\beta_1\tilde\alpha_0+\tilde\beta_0\tilde\alpha_1+\tilde\beta_3\tilde\alpha_2+\tilde\beta_2\tilde\alpha_2,$$

$$-\alpha_1' = -\tilde{\beta}_2\tilde{\sigma}_0+\tilde{\beta}_3\tilde{\sigma}_1+\tilde{\beta}_0\tilde{\sigma}_2+\tilde{\beta}_1\tilde{\sigma}_3,$$
$$-\alpha_0' = \tilde{\beta}_3\tilde{\sigma}_0+\tilde{\beta}_2\tilde{\alpha}_1-\tilde{\beta}_1\tilde{\sigma}_2+\tilde{\beta}_0\tilde{\sigma}_3.$$

Hence, we have

$$-A' = \widetilde{B}.\widetilde{A}, \quad \text{.........................} \quad \text{(iii)},$$

where

$$A \equiv \alpha_3+\alpha_2 i+\alpha_1 j+\alpha_0 k,$$
$$A' \equiv \alpha_3'+\alpha_2' i+\alpha_1' j+\alpha_0' k,$$
$$B \equiv \beta_3+\beta_2 i+\beta_1 j+\beta_0 k.$$

Therefore the motion B of the soma A to the soma A' may be given by the equation (iii).

The biquaternion $A+B\sigma$ which represents a soma may also written in the form

$$P \equiv A+B\sigma = -(a_0+b_0\varepsilon)+(a_1+b_1\varepsilon)\,i-(a_2+b_2\varepsilon)\,j+(a_3+b_3\varepsilon)\,k.$$

Therefore the composition of the two motions P, Q may be expressed by the equation

$$P' = Q.P,$$

where

$$Q \equiv -(c_0+d_0\varepsilon)+(c_1+d_1\varepsilon)\,i-(c_2+d_2\varepsilon)\,j+(c_3+d_3\varepsilon)\,k,$$
$$P' \equiv -(a_0'+b_0'\varepsilon)+a_1'+b_1'\varepsilon)\,i-(a_2'+b_2'\varepsilon)\,j+(a_3'+b_3'\varepsilon)\,k.$$

Or at full length, it may be written in the form

$$\begin{aligned}
&-(a_0'+b_0\varepsilon)+(a_1'+b_1\varepsilon)\,i-(a_2'+b_2\varepsilon)\,j+(a_3'+b_3\varepsilon)\,k \\
&=(c_0+d_0\varepsilon)(a_0+b_0\varepsilon)-(c_1+d_1\varepsilon)(a_1+b_1\varepsilon)-(c_2+d_2\varepsilon)(a_2+b_2\varepsilon) \\
&\quad -(a_3+b_3\varepsilon)(a_3+b_3\varepsilon) \\
&+[-(c_1+d_1\varepsilon)(a_0+b_0\varepsilon)-(c_0+d_0\varepsilon)(a_2+b_2\varepsilon)+(c_3+d_3\varepsilon)(a_2+b_2\varepsilon) \\
&\quad -(c_2+d_2\varepsilon)(a_3+b_3\varepsilon)]\,i \\
&+[(c_2+d_2\varepsilon)(a_0+b_0\varepsilon)+(c_3+d_3\varepsilon)(a_1+b_2\varepsilon)+(c_0+d_0\varepsilon)(a_2+b_2\varepsilon) \\
&\quad +(c_1+d_1\varepsilon)(a_3+b_3\varepsilon)]\,j \\
&+[-(c_3+d_3\varepsilon)(a_0+b_0\varepsilon)+(c_2+d_2\varepsilon)(a_1+b_1\varepsilon)-(c_1+d_1\varepsilon)(a_2+b_2\varepsilon) \\
&\quad -(c_0+d_0\varepsilon)(a_3+b_3\varepsilon)]\,k.
\end{aligned}$$

Thus

$$\begin{aligned}
a_0'+b_0'\varepsilon = &-(c_0+d_0\varepsilon)(a_0+b_0\varepsilon)+(c_1+d_1\varepsilon)(a_1+b_1\varepsilon)+(c_2+d_2\varepsilon)(a_2+b_2\varepsilon) \\
&+(c_3+d_3\varepsilon)(a_3+b_3\varepsilon),
\end{aligned}$$

$$a_1'+b_1'\varepsilon=-(c_1+d_1\varepsilon)(a_0+b_0\varepsilon)+(c_0+d_0\varepsilon)(a_1+b_1\varepsilon)+(c_3+d_3\varepsilon)(a_2+b_2\varepsilon)$$
$$-(c_2+d_2\varepsilon)(a_3+b_3\varepsilon),$$
$$a_2'+b_2'\varepsilon=-(c_2+d_2\varepsilon)(a_0+b_0\varepsilon)-(c_3+d_3\varepsilon)(a_1+b_1\varepsilon)-c_0+d_0\varepsilon)(a_2+b_2\varepsilon)$$
$$+(c_1+d_1\varepsilon)(a_3+b_3\varepsilon),$$
$$a_3'+b_3'\varepsilon=-(c_3+d_3\varepsilon)(a_0+b_0\varepsilon)+(c_2+d_2\varepsilon)(a_1+b_1\varepsilon)-(a_1+b_1\varepsilon)(a_2+b_2\varepsilon)$$
$$-(c_0+d_0\varepsilon)(a_3+b_1\varepsilon) \quad \ldots\ldots\ldots\ldots \quad \text{(iv)}.$$

We will discuss certain special positions of two somas.

Now

$$CA=c_0a_0-c_1a_1-c_2a_2-c_3a_3+(c_1a_0+c_0a_1-c_3a_2+c_2a_3)\,i$$
$$+(c_2a_3+a_1c_3+c_0a_2-c_1a_3)\,j+(c_3a_0-a_1c_2+c_1a_2+c_0a_0)\,k,$$
$$DB=d_0b_0+d_1b_1+d_2b_3-d_3b_3+(d_1b_0-d_0b_1+d_3b_2+d_2b_3)\,i$$
$$+(d_2b\ -d_3b_1-d_0b_2-d_1b_3)\,j+(d_3b_0+d_2b_1-d_1b_2+d_0b_3)\,k.$$
$$DA=d_0a_0+d_1a_1+d_2a_2-d_3a_3+(d_1a_0-d_0a_1+d_3a_2+d_2a_3)\,i$$
$$+(d_2a_0-d_3a_1-d_2a_2-d_1a_3)\,j+(d_3a_0+d_2a_1-d_1a_2+d_0a_3)\,k,$$
$$CB=c_0b_0-c_1b_1-c_2b_2-c_3b_3+(c_1b_0+c_0b_1-c_3b_2+c_2b_3)\,i$$
$$+c_2b_0+c_3b_1+c_0b_2-c_1b_3)\,j+(c_3b_0-c_2b_1+c_1b_2+c_0b_3)\,k.$$

Therefore if $d_3=0$, then the following relation is true:

$$a_0'b_3-a_1'b_2+a_2'b_1-a_3,b_0-b_0'a_3+b_1'a_2-b_2'a_1+b_3'a_0$$
$$=\{b_0a_0-c_2a_2-c_3a_4+d_3b_0+d_0b_0+d_1b_1+d_2b_2\}\ b_3$$
$$-c_1a_0+c_0a_1-c_3a_2+c_2a_3+d_1b_0-d_0b_1+d_2b_3\}\ b_2$$
$$+\{c_2a_0+a_1c_3+c_0a_2-c_1a_3+d_2b_0-d_0b_2-d_1b_3\}\ b_1$$
$$-\{c_3a_0-a_1c_2+c_1a_2+c_0a_3-d_2b_1-d_1b_2+d_0b_3\}\ b_0$$
$$-\{c_0b_0-c_1b_1-c_2b_2-c_3b_3+d_0a_0-d_1a_1+d_2a_2\}\ a_3$$
$$+\{c_1b_0+c_0b_1-c_3b_2+c_2b_3+d_1a_0-d_0a_1+d_2a_3\}\ a_2$$
$$-\{c_2b_0+c_3b_1+c_0b_2-c_1b_3+d_2a_0-d_0a_2+d_2a_3\}\ a_1$$
$$+\{c_3b_0-c_2b_1+c_1b_2+c_0b_3+d_2a_1-d_1a_2+d_0a_3\}\ a_0$$
$$=0.$$

The converse of this is also true.

But the relation

$$d_3 \equiv \frac{c_1u_3}{u_0} - \frac{c_2u_2}{u_0} + \frac{c_3u_1}{u_0} = 0$$

means that the point (u) lies on the plane whose plane-coordinates are $(o, c_3. -c_2, c_1)$.

Now a rotation represented by the trans formation-equation

$$\xi' = C\xi\,(\dot{C})^{-1} \qquad (C\text{: a quaternion})$$

is a rotation about a line as its inner axis whose equations are

$$\left.\begin{aligned} c_1x_2 + c_2x_3 &= 0 \\ c_1x_1 - c_3x_3 &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{(v)}.$$

The axis passes through the origin of coordinates. The absolute pole of the plane whose plane-coordinates are $(o, c, -c_2, c_1)$ is the point whose point-coordinates are $(o, c_3, -c_3, c_1)$. But the point $(o.\ c_3, -c_3, c_1)$ lies on the axis. Therefore the plane $(o, c_3, -c_3, c_1)$ intersects orthogonally the axis at the origin of coordinates. Thus the motion defined by (1) must be composed of the translation by which the plane $(o, c_2, -c_2, c_1)$ remains in itself and of the rotation about the axis perpendicular to the plane $(o, c_3, -c_2, c_1)$ and passing through the origin. When a soma can be brought to another soma by such a motion, the somas shall be called 'semisymmetrical' to each other.

If $d_3 = 0$ and moreover $c_0 = 0$, the following relation is true:

$$\begin{aligned}
& a_0'a_0 + a_1'a_1 + a_2'a_2 + a_3'a_3 - (b_0b'_0 + b_1'b_1 + b_2'b_2 + b_3'b_3) \\
&= (-c_0a_3 - c_1a_1 - c_2a_2 - c_3a_3)\,a_0 + (d_0b_0 + d_1b_1 + d_2b_2 - d_3b_3)a_0 \\
&+ (c_1a_0 + c_0a_1 - c_3a_2 + c_2a_3)\,a_1 + (d_1b_0 - d_0b_1 + d_3b_2 + \dot{d}_0b_2)\,a_1 \\
&+ (c_2a_0 + a_1c_3 \dot{+} c_0a_2 - c_1a_3)\,a_2 + (d_2b_0 - d_3b_1 - d_0b_2 - d_1b_3)\,a_2 \\
&\quad (c_0b_0 - c_1b_1 - c_2b_2 - c_3b_3)\,b_0 + (d_0a_0 + d_1a_1 + d_2a_2 - d_3a_3)\,b_0 \\
&\quad (c_1b_0 + c_0b_1 - c_3b_2 + c_2b_3)\,b_1 + (d_1a_3 - d_0a_1 + d_3a_2 + d_2a_3)\,b_1 \\
&\quad (c_2b_0 + c_3b_1 + c_0b_2 - c_1b_3)\,b_2 + (d_2a_0 - d_3a_1 - d_0a_2 - d_1a_3)\,b_2 \\
&\quad (c_3b_0 - c_2b_1 + c_1b_2 + c_0b_3)\,b_3 + (d_0a_0 + d_2a_1 - d_1a_2 + d_0a_3)\,b_3 \\
&= c_0^2\,(a_0^2 + a_1^2 + a_2^2 + a_3^2 + b_0^2 + b_1^2 + b_2^2 + b_3^2) \\
&= 0
\end{aligned}$$

In our case, since $c_0 = 0$, the rotation

$$\xi' = C\xi(\dot{U})^{-1}$$

must be a half rotation about the line (v) as its inner axis. For the joining line of the initial and final points of the motion cuts the axis of rotation.

When Q represents a half rotation

$$c_3 + d_3\,\varepsilon = 0.$$

Therefore, in this case, the equation (iii) becomes as follows :

$$\begin{aligned} a_0' + b_0'\varepsilon = &-(c_0+d_0\varepsilon)(a_0+b_0\varepsilon)+(c_1+d_1\varepsilon)(a_1+b_1\varepsilon)+(c_2+d_2\varepsilon)(a_2+b_2\varepsilon) \\ &+(c_3+d_3\varepsilon)(a_3+b_3\varepsilon), \\ a_1' + b_1'\varepsilon = &-(c_1+d_1\varepsilon)(a_0+b_0\varepsilon)-(c_0+d_0\varepsilon)(a_1+b_1\varepsilon)+(c_3+d_3\varepsilon)(a_2+b_2\varepsilon) \\ &-(c_2+d_2\varepsilon)(a_3+b_3\varepsilon), \\ a_2' + b_2'\varepsilon = &-(c_2+d_2\varepsilon)(a_0+b_0\varepsilon)-(c_3+d_3\varepsilon)(a_1+b_1\varepsilon)-(c_0+d_0\varepsilon)(a_2+b_2\varepsilon) \\ &+(c_1+d_1\varepsilon)(a_3+b_3\varepsilon), \\ a_3' + b_3'\varepsilon = &-(c_3+d_3\varepsilon)(a_0+b_0\varepsilon)+(c_2+d_2\varepsilon)(a_1+b_1\varepsilon)-(c_1+d_1\varepsilon)(a_2+b_2\varepsilon) \\ &-(c_0+d_0\varepsilon)(a_3+b_3\varepsilon). \end{aligned}$$

When a soma P can be brought to a soma Q by a half rotation, these somas shall be called ' symmetrical ' to each other.

The totality of the somas which are symmetrical to a soma shall be called a ' normal net ' of somas.

Now put

$$\begin{aligned} a_0 + b_0\varepsilon &= \lambda X_0, \\ a_1 + b_1\varepsilon &= \lambda X_1, \\ a_2 + b_2\varepsilon &= \lambda X_2, \\ a_3 + b_3\varepsilon &= \lambda X_3, \end{aligned}$$

where $\varepsilon^2 = -1$. We may take the ratio

$$(X) \equiv (X_0 : X_1 : X_2 : X_3)$$

as the coordinates of the soma represented by the biquaternion

$$\varepsilon A^{(1)} + B^{(1)}.$$

Conversely the coordinates $(a), (b)$ of the soma can be obtained from (X) by giving to λ such a value that the coordinates shall satisfy the fundamental identity

$$(ab) = 0.$$

For this, it is necessary and sufficient that the imaginary part of $\lambda^2(XX)$ should be zero, i.e.

$$\lambda a_i = \frac{X_i}{\sqrt{(XX)}} + \frac{X_i}{\sqrt{(XX)}},$$

$$\lambda b_i = \left(\frac{X_i}{\sqrt{(XX)}} + \frac{X_i}{\sqrt{(XX)}}\right)\varepsilon : i=0, 1, 2, 3,$$

with the condition

$$(XX) \neq 0.$$

When two somas (X) and (X') are symmetrical

$$(XX')=0.$$

But in the case

$$(XX) = 0,$$

the equation determining a_i and b_i becomes illusory since

$$(XX) = (a+b\varepsilon.\, a+b\varepsilon)$$

$$= (aa)\left(1-\frac{u_1^2}{u_0^2}-\frac{u_2^2}{u_0^2}-\frac{u_3^2}{u_0^2}\right),$$

the relation

$$(XX) = 0$$

indicates that

$$u_0^2-u_1^2-u_2^2-u_3^2 = 0,$$

which occurs when the point (u) lies on the absolute in the hyperbolic space. In this case, the point $\left(1, \frac{2u_0u_1}{(uu)}, \frac{2u_0u_2}{(uu)}, \frac{2u_0u_3}{(uu)}\right)$ to which the origin is translated in the motion

$$\xi' = \frac{\xi+U}{U\xi+1}$$

becomes a point at infinity which has the coordinates (u). Thus such a kind of soma is obtained from the protosoma by a translation of an infinite distance from it and by a rotation about axis (inner axis) passing through the origin of coordinates. We shall define a soma whose coordinates satisfy the equation

$$(XX) = 0$$

an improper soma in the hyperbolic space.

The figure formed by the totality of the improper somas shall be called the soma-absolute in the hyperbolic space. This is a manifoldness ∞^5 and the equation to the soma-absolute is

$$(XX) = 0.$$

If the improper somas defined by the biquaternions $A+B\sigma$ and $A'+B'\sigma$ be obtained by the same translation and by the different rotations about the axis passing through the origin, then these somas are symmetrical. In fact, in the case (X) and (X') corresponding to $A+B\sigma, A'+B'\sigma$ are

$$X_0 = -a_3 + \varepsilon\left(\frac{a_0u_1}{u_0} - \frac{a_1u^2}{u_0} - \frac{a_2u_3}{u_0}\right),$$

$$X_1 = \quad a_2 + \varepsilon\left(\frac{a_1u_1}{u_0} + \frac{a_0n_2}{u_0} - \frac{a_3u_3}{a_0}\right),$$

$$X_2 = -a_1 + \varepsilon\left(\frac{a_2u_1}{u_0} + \frac{a_3u_2}{u_0} + \frac{a_0u_3}{u_0}\right),$$

$$X_3 = \quad a_0 + \varepsilon\left(\frac{u_3u_1}{u_0} - \frac{a_2u_2}{u_0} + \frac{a_1u_3}{u_0}\right),$$

$$X_0' = -a_3' + \varepsilon\left(\frac{a_0'u_1}{u_0} - \frac{a_1'u_2}{u_0} - \frac{a_2'u_3}{u_0}\right),$$

$$X_1' = \quad a_2' + \varepsilon\left(\frac{a_1'u_1}{u_2} + \frac{a_0'u_2}{u_0} - \frac{a_3'u_3}{u_0}\right),$$

$$X_2' = -a_1' + \varepsilon\left(\frac{a_2'u_1}{u_0} + \frac{a_3'u_2}{u_0} + \frac{a_0'u_3}{u_0}\right),$$

$$X_3' = \quad a_0' + \varepsilon\left(\frac{a_3'u_1}{u_0} - \frac{a_2'u_3}{u_0} + \frac{a_1'u_3}{u_0}\right),$$

respectively.

Thus

$$(XX) = (aa') - \sum\left(\frac{a_0u_1}{u_0} - \frac{a_1u_2}{u_0} - \frac{a_2u_3}{u_0}\right)\left(\frac{a_0'u_1}{u_0} - \frac{a_1'u_2}{u_0} - \frac{a_2'u_3}{u_0}\right)$$

$$+\varepsilon\left[\sum\left\{-a_3'\left(\frac{a_0u_1}{u_0} - \frac{a_1u_2}{u_0} - \frac{a_2u_3}{u_0}\right)\right\} + \sum\left\{-a_3\left(\frac{a_0'u_1}{u_0} - \frac{a_1'u_2}{u_0} - \frac{a_2'u_3}{u_0}\right)\right\}\right]$$

$$=(aa')\left(1-\frac{u_1^2}{u_0^2}-\frac{u_2^2}{u_0^2}-\frac{u_3^2}{u_0^2}\right).$$

But in our case

$$1-\frac{u_1^2}{u_0^2}-\frac{u_2^2}{u_0^2}-\frac{u_3^2}{u_0^2}=0.$$

Hence, we have

$$(XX')=0.$$

The common symmetrical soma (U) to the three improper somas $(X), (Y), (Z)$ is given by the equations

$$(UX)=0,$$
$$(UY)=0,$$
$$(UZ)=0.$$

The solution of these equations is

$$U_0 : U_1 : U_2 : U_3 = |XYZ|.$$

Thus

$$(UU)=\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}^2 = |XYZ|^2$$

$$=\begin{vmatrix} (XX) & (YX) & (ZX) \\ (XY) & (YY) & (ZY) \\ (XZ) & (YZ) & (ZZ) \end{vmatrix}$$

$$=2(ZY(XZ)(YX).$$

When the soma (U) is proper

$$(UU) \neq 0,$$

hence non of the expression $(ZY), (XY), (YX)$ can vanish. Thus any two of the somas $(X), (Y), (Z)$ can not be obtained by the same translation.

We can establish one to one correspondence between the totality of all somas in the hyperbolic space and the totality of all complex points (or planes) in the elliptic space. For the sake of clearness we shall assume that our soma-space is doubly overlaid. We shall say that a soma belongs to the 'upper layer' when it is represented by a complex plane, we shall call a soma of the 'lower layer. In this representation improper somas correspond to the complex points on the absolute in the representing elliptic space. The soma absolute will be represented by the absolute in the representing elliptic space. The necessary and sufficient condition that two somas of the different layers should be symmetrical is that the corresponding complex point and plane should be in the united position. Therefore the two somas of the different layers symmetrical to each other may be represented by a complex plane-element (system of a point and a plane through this point).

We shall add that $W_1 : W_2 : W_3$ are the coordinates of the invariant-cross in the half rotation (W) by which a soma is transformed into another soma. When the motion (W) is defined by a biquaternion $A+B\sigma$, we see easily that the point (o, a_3, a_2, a_1) is invariable for the motion

$$\xi'' = A\xi'(\dot{A})^{-1} \qquad \text{(i)},$$

as well as for the motion

$$\xi' = \frac{\xi + U}{\dot{U}\xi + 1} \qquad \text{(ii)},$$

consequently for the motion

$$\xi'' = \frac{A\xi + B}{\dot{A}\xi + \dot{B}} \qquad \text{(iii)},$$

Further the point $(u_0, -u_1, -u_2, -u_3)$ goes to the point $(u_0\, u_1\, u_2\, u_3)$ in the motion (ii) and the point (u) is transformed into the point $(u_0, -u_1, -u_2, -u_3)$ in the motion (i). Thus, the cross formed by the joining of the points $(u_0, -u_1, -u_2, -u_3)$ and $(u_0\, u_1\, u_2\, u_3)$ with its absolute polar is an invariant figure in the motion (iii). But the coordinates of the cross defined by the joining of the points $(u_0, -u_1, -u_2, -u_3)$ and $(o, a_3, -a_2, -a_1)$ and its absolute polar is equal to the expressions

$$a_3 + \varepsilon b_0,$$

$$-a_2 + \varepsilon b_1,$$

$$a_1 + \varepsilon b_2, \qquad (\varepsilon^2 = -1)$$

which are equivalent to

$$W_1,$$

$$W_2,$$

$$W_3.$$

§ 10.

The Fundamental Somas.

If we consider three mutually orthogonally cutting oriented lines $\overrightarrow{OX_1}$, $\overrightarrow{OX_2}$, $\overrightarrow{OX_3}$ which represent the protosoma, then we see that the dual coordinates of the protosoma are equal to the ratio (1000). We can obtain the three somas represented by the systems of the oriented lines

$$(\overrightarrow{OX_1}, \overleftarrow{O_2X}, \overleftarrow{OX_3}),$$
$$(\overleftarrow{O_1X}, \overrightarrow{O_2X}, \overleftarrow{OX_3}),$$
$$(\overleftarrow{OX_1}, \overleftarrow{OX_2}, \overrightarrow{OX_3}),$$

by the half rotations about the axes OX_1, O_2X, OX_3 respectively. The dual coordinates of the somas thus obtained are equal to the ratios

$$(0100),$$
$$(0011),$$
$$(0001)$$

respectively. These somas aud the protosoma are symmetrical to each other. We shall define the somas

$$(1000),$$
$$(0100),$$
$$(0010),$$
$$(0001)$$

the 'fundamental somas' as in the case of the elliptic space.

The fundamental somas are analogous to the edge crosses of a fundamental orthogonal tetrahedron in the hyperbolic space.

§ 11.

The Transformation Groups of Soma-space in the Hyperbolic Space.

The group of the dual projective geometry contains 30 essential parameters and is isomorphic with the 30-parametric group of the complex space: it is made by two parts $\mathfrak{G}_{30}$, $\mathfrak{H}_{30}$.

The transformations of $\mathfrak{H}_{30}$ form no group as a succession of two of them makes a transformation of $\mathfrak{G}_{30}$ The general transformation of $\mathfrak{G}_{30}$ is

$$X_i' = (A_i X), \quad i=0, \quad 1, 2. 3;$$
$$A_{ij} = a_{ij} + a'_{ij} z, \quad |A_{ij}| \neq 0.$$

Let $(X), (Y), (Z), (T)$ be the dual coordinates of any four somas of a one dimensional soma chain defined by the equation of the form

$$X_i = aP_i + bQ_i \qquad (a, b; \text{ real numbers})$$

and $(R), (S)$ be those of any two somas which do not belong to any normal net of somas with any two of them. And consider the ratio

$$(XYZT) \equiv \frac{|RSXZ|}{|RSYZ|} : \frac{|RSXT|}{|RSYT|}.$$

This shall be called the dual double ratio of the four somas (X), $(Y), (Z), (T)$. The dual double ratio of any four somas of a one-dimensional soma chain is an invariant for the transformations of $\mathfrak{G}_{30}$.

The transformation of $\mathfrak{H}_{30}$ has the form

$$X_i' = (A_i \overline{X}).$$

This may evidently be obtained from the general transformation of $\mathfrak{G}_{30}$, combined with the single transformation

$$X_i' = \overline{X}_i,$$

where $\overline{X}_i$ means the conjugate complex number to X_i. The soma represented by $(\overline{X})$ is obtained from the protosoma by the translation represented, and by the rotation same with that of the soma represented by (X).

By the general transformation of the group $\mathfrak{G}_{30}$, every normal net of somas is transformed into a normal net of somas.

There is another sort of transformation called 'dual correlation,' where by the dual coordinates of a soma of one layer are expressed as linear homogeneous functions of those of a soma of the other layer the determinant of the transformation being different from zero. Such a transformation may clearly be obtained from the combination of the group $\mathfrak{G}_{30}$ and

$$X_i' = U_i.$$

This is merely to replace each soma of one layer by the soma of the other layer coincident with it. In the representing space, the transformation is the replacing of the absolute pole and polar plane with respect to the absolute.

There is another kind of transformation called " dual anticollineation expressed by the equation

$$X_i' = (A_i \overline{U}).$$

This is composed by a transformation of the dual collineation together with a transformation

$$U' = \overline{U}.$$

$\mathfrak{G}_{30}$ has a group $\mathfrak{G}_{15}$ which are formed from ths transformation of the form

$$X_i' = (A_i X), \qquad i = 0, 1, 2, 3,$$

where $\qquad A_{ij}$: real.

By this transformation, a soma obtained from the protosoma by the rotation about the origin is transformed into another soma. For, in the case of such somas whose coordinates (X) are all real, the corresponding coordinates (X') are also all real.

Next, let us consider the orthogonal transformations of somas by which the improper somas are transformed into other improper somas. These form a mixed groups $\mathfrak{G}_{12}$, $\mathfrak{H}_{12}$ with 12 essential parameters. The transformations of $\mathfrak{G}_{12}$ have the form

$$X' = AXB.$$

And the transformations of $\mathfrak{H}_{12}$ have the form

$$X' = AXB.$$

The group $\mathfrak{G}_{12}$ can be decomposed into two permutable groups $\mathfrak{G}_6$, $\mathfrak{G}'_6$, which are defined by the equations:

$$X' = AZ,$$

and

$$X' = ZB.$$

$\mathfrak{G}_6$ is the group of motion in the hyperbolic space.

A hyperbolic motion may be represented by the biquaternion

$$Q = -(a_0+b_0\varepsilon)+(a_1+b_1\varepsilon)\,i-(a_2+b_2\varepsilon)\,j+(a_3+b_3\varepsilon)\,k$$
$$= -X_0+X_1\,i-X_2\,j+X_3\,k,$$

as it was already noticed.

We shall call the special points on the absolute whose coordinates are

$$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (1, 0, 0)$$

and

$$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (-1, 0, 0)$$

as its north and south poles.

If we consider the special case in which

$$(a_1+b_1\varepsilon)\,\varepsilon = (a_2+b_2\varepsilon),$$

or

$$X_1\,\varepsilon = X_2,$$

then we see that

$$a_1+b_1 = 0, \quad a_2+b_2 = 0.$$

Thus the motion

$$\xi' = \frac{A\xi+B}{\dot{A}\xi+\dot{B}}$$

is that by which the north pole remains unchanged. Such motions form a group $\mathfrak{G}_4$ with four essential parameters. The transformation of $\mathfrak{G}_4$ takes the form:

$$Q = -(a_0+b_0\varepsilon)+(a_1+b_1\varepsilon)\,i-\varepsilon(a_1+b_1\varepsilon)\,j+(a_3+b_3\varepsilon)\,k$$
$$= -X_0+X_1\,i-\varepsilon\,X_1\,j+X_3\,k$$

$\mathfrak{G}_4$ has a subgroup $\mathfrak{G}_3$ whose transformation is represented by the

$$Q = -a_0+a_1\,i-\varepsilon\,a_1\,j+\varepsilon\,b_3\,k.$$

$\mathfrak{G}_4$ contains also a subgroup $\mathcal{G}_3$ whose transformation is

$$Q = -a_0+a_1\,i+b_1\,j+a_3\,k.$$

$\mathfrak{G}_4$ contains a subgroup $\mathfrak{G}_2$ with two essential parameters. The transformation of the group is expressed by the biquaternion

$$Q = -(a_0+b_0\,\varepsilon)+(a_3+b_3\,\varepsilon)\,k$$
$$= -X_0+X_3\,k.$$

By this motion, the north and south poles remains fixed.

$\mathfrak{G}_2$ contains a continuous group $\mathfrak{G}_1$ and a mixed group $\mathscr{G}_1$ with one essential parameter.

A motion Included in $\mathfrak{G}_1$ shall be called thc rotation about the axis of the space.

A motion included in $\mathscr{G}_1$ shall be called the sliding along the axis of the space.

A translation of $\mathfrak{G}_1$ may be represented by the quaternion

$$Q = -a_0 + a_3 k$$
$$= -{}_rX_0 + {}_iX_0\, k.$$

A transfoımation of $\mathscr{G}_1$ is represented by the biquaternion

$$Q = -a_0 + b_3 \varepsilon\, k$$
$$= -{}_rX_0 + {}_iX_3\, k.$$

SECTION II.

The Soma-manifoldness in the Hyperbolic Space.

The soma manifoldness in the hyperbolic space may be discussed. as in the case of the elliptic space. Consider a som-manifoldness whose coordinates (X) are linearly dependent on that $n+1$ different fixed somas $(X^{(0)}), (X^{(1)}) \dots (X^{(n)})$, i.e.

$$\rho X_i = a_0 X^{(0)} + a_1 X_1^{(1)} + \dots + a_n X_i^{(n)},\quad i=0,1,2,3;$$
$$a_i : \text{all real.}$$

We shall define the soma-manifoldness an n-dimensional soma-chain.' The somas $(X^{(0)}) \dots (X^{(n)})$ shall be called the 'bases' of the soma-chain. The fundamental soma-chains are one-, two-, three-dimensional soma-chains. We shall use the notation C_n to denote an n-dimensional soma-chain. Any $n+1$ somas of C_n

$$\rho X_i = a_0^{(1)} X_i^{(0)} + \dots\dots + a_n^{(1)} X_i^{(n)},$$
$$\rho Y_i = a_0^{(2)} X_i^{(0)} + \qquad + a_n^{(1)} X_i^{(n)},$$
$$\rho T_i = a_0^{(n+1)} X_i^{(0)} + \qquad + a^{(n+1)} X_i^{(n)},$$

may be taken as its bases, provided that

$$\begin{vmatrix} a_0^{(1)} & a_1^{(1)} & \ldots\ldots & a_n^{(1)} \\ a^{(n+1)} & a^{(n+1)} & \ldots\ldots & a^{(n+1)} \end{vmatrix} \neq 0.$$

If of two families of somas of different layers each consists of all the somas which are symmetrical to all the somas of the other, then the both families are soma-chains. We shall call the soma-chain arranged to pair 'reciprocal soma chain' to each other.

§ 12.

The One-dimensional Soma-chain.

The one dimensional soma-chain has the equations

$$\rho\, X_i = aY_i + bZ_i \qquad (i=0,\ 1,\ 2,\ 3)$$
$$a_1\, b: \text{ all real.}$$

Every C_1 has at least one pair of real somas which are symmetrical to each other. These somas shall be called 'principal somas' of C_1 Now we will prove the existence of such somas.

If the somas $(ay+bz)$ and $(a'y+b'z)$ of C_1 be symmetrical, then they must satisfy the equation

$$(ay+bz,\ a'y+b'z) = 0.$$

Or

$$aa'\{({}_ry_ry)-({}_\varepsilon y_\varepsilon y)\}+(ab'+a'b)\{({}_ry_rz)-({}_\varepsilon y_\varepsilon z)\}+bb'\{({}_rz_rz)-({}_\varepsilon z_\varepsilon z)\}=0,$$

$$aa'\ ({}_ry_\varepsilon y) \qquad +(a'b+a'b)\{({}_ry_\varepsilon z)+({}_\varepsilon z_ry)\}+bb'\ ({}_rz_\varepsilon z) \qquad =0,$$

where ${}_ry_i, {}_\varepsilon y_i$ are the coefficients of the units 1 and ε respectively.

Let $\overline{Y_i}$ and $\overline{Z_i}$ be the conjugate imaginary numbers to Y_i and Z_i respectively, then the equations take the forms

$$aa'(YY)+(aa'+a'b)(YZ)+bb'(ZZ) = 0,$$
$$aa'(\overline{Y}\overline{Y})+(ab+'a'b)(YZ)+bb'(ZZ) = 0 \quad \ldots\ldots\ldots\ldots \quad (1).$$

We see Y_i, $\overline{Y_i}$ correspond to ${}_rY_i$, ${}_rY_i$ in the equation for

$$Z_i,\ \overline{Z_i} \qquad\qquad {}_iZ_i,\ {}_rZ_i$$

the determination of the principal somas of C_1 in the elliptic space. Thus we see that the equation which is obtained by the elimination of $a':b'$ from the equation (i) has real roots and they are distinct.

In general, one of the principal somas is not the somas which is obtained from the other by the half rotation about one of the three oriented lines which represents the latter soma. In this special case, by taking the principal somas of the C_1 as the fundamental somas (1000), (0100), the equation of the C_1 may be reduced into its canonical form

$$\begin{aligned} X_0 &= 0, \\ X_1 &= a(p+q\varepsilon), \\ X_2 &= b(r+s\varepsilon), \\ X_3 &= 0. \end{aligned}$$

Such C_1 may be generated by a cross-chain whose equations are

$$\begin{aligned} X_1 &= 0, \\ X_2 &= a(p+q\varepsilon), \\ X_3 &= b(r+s\varepsilon). \end{aligned}$$

The half rotations of the soma defined by the fundamental orthogonal tetrahedron about every cross of this C_1 give a soma-chain

$$\begin{aligned} X_0 &= 0, \\ X_1 &= a(p+q\varepsilon), \\ X_2 &= b(r+s\varepsilon), \\ X_3 &= 0. \end{aligned}$$

§ 13.

The Two-dimensional Soma-chain.

The simplest two dimensional system of somas is the two-dimensional soma-chain. This is made of all the somas which have the coordinates linearly dependent with real coefficients on those of the three given somas, i.e.

$$\rho X_i = aY_i + bZ_i + cW_i \qquad (i=0, 1, 2, 3).$$

In the case

$$|YZW| \neq 0,$$

the soma (in the lower layer) represented by the equation

$$U_1 = \begin{vmatrix} Y_j & Y_k & Y_s \\ Z_j & Z_k & Z_s \\ W_j & W_k & Z_s \end{vmatrix}$$

is the soma which is symmetrical to all the somas of the soma-chain. This soma (U) shall be called the 'nucleus' of the C_2.

The soma-chain will be represented by all the complex points of the representing complex space, and the nucleus the complex plane.

§ 14.

The Three-dimensional Soma-chain.

The soma-chain of three dimensions is the system of somas whose coordinates (X) are linearly dependent on that of the four somas (Y), (Z), (T), (W) with real coefficients a, b, c, d, i.e.

$$X_i = aY_i + bZ_i + cT_i + dW_i \qquad (i = 0, 1, 2, 3) \ldots\ldots (1).$$

Case 1.

$$|YZTW| \neq 0.$$

This system is represented by all the complex points of the iep-resenting space.

The somas which are represented by the totality of the points of the real domain of the representing space will generate a soma-chain complex. The reciprocal soma-chain in the lower layer will have the equations

$$\rho U_i = p \begin{vmatrix} Y_j & Y_k & Y \\ Z_j & Z_k & Z_s \\ W_j & W_k & W_s \end{vmatrix} + q \begin{vmatrix} Z_j & Z_k & Z_s \\ W_j & W_k & W_s \\ T_j & T_k & T_s \end{vmatrix} + r \begin{vmatrix} W_j & W_k & W_s \\ T_j & T_k & T_s \\ Y_j & Y_k & Y_s \end{vmatrix} + s \begin{vmatrix} T_j & T_k & T_s \\ Y_j & Y_k & Y_s \\ W_j & W_k & W_s \end{vmatrix}$$

We can piove that there are four somas in a soma-chain which coincide with those of its reciprocal soma-chain as in the case of the elliptic space. When they are obtained from one of these somas by the half rotations about every one of the three mutually orthogonally interesting oriented lines which present this soma, by taking these somas as the fundamental somas the equations to the soma-chain may be reduced to its canonical forms

$$\begin{aligned}
\rho X_0 &= a(t_0+s_0\,\varepsilon),\\
\rho X_1 &= b(t_1+s_1\,\varepsilon),\\
\rho X_2 &= c(t_2+s_2\,\varepsilon),\\
\rho X_3 &= d(t_3+s_3\,\varepsilon) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(i)},
\end{aligned}$$

where a, b, c, d are real numbers.

There are two specially interesting cases.

(I) $\quad t_0 : s_0 = t_j : s_j,\ t_k : s_j = t_l : s_l\ ;$

(II) $\quad t_0 : s_0 = t_1 : s_1 = t_2 : s_2 = t_4 : s_3$

In the case (I), the soma of the soma-chain (i) is transformed into another soma of the same soma-chain by any rotation about X_j-axis.

In fact, if

$$t_0 : s_0 = t_1 : s_1,\ t_2 : s_2 = t_3 : s_3,$$

then the soma

$$\begin{aligned}
\rho X_0 &= a(t_0+s_0\,\varepsilon),\\
\rho X_1 &= b(t_1+s_1\,\varepsilon),\\
\rho X_2 &= c(t_2+s_2\,\varepsilon),\\
\rho X_3 &= d(t_3+s_3\,\varepsilon).
\end{aligned}$$

is transformed by a rotation about x_1—axis (as its inner axis) to the soma

$$\begin{aligned}
\tau X_0' &= -c_0(t_0+\varepsilon\,s_0)+c_1(t_1+\varepsilon\,s_1),\\
\tau X_1' &= -c_1(t_0+\varepsilon\,s_0)-c_0(t_1+\varepsilon\,s_1),\\
\tau X_2' &= -c_0(t_2+\varepsilon\,s_2)-c_1(t_3+\varepsilon\,s_3),\\
\tau X_3' &= -c_1(t_2+\varepsilon\,s_2)-c_0(t_3+\varepsilon\,s_3).
\end{aligned}$$

Or

$$\begin{aligned}
\tau X_0' &= (-c_0+c_1)(t+\varepsilon\,s),\\
\tau X_1' &= (-c_1-c_0)(t-\varepsilon\,s),\\
\tau X_2' &\doteq (-c_0-c_1)(t+\varepsilon\,s),\\
\tau X_3' &= (-c_0-c_2)(t+\varepsilon\,s).
\end{aligned}$$

In this case (II), the soma-chain remains unchanged by the rotation about the origin. Let

$$\xi' = C\xi\,(\dot{C})^{-1}$$

represents the rotation about the origin, then the soma

$$\rho X_0 = a(t_0+\varepsilon s_0),$$
$$\rho X_1 = b(t_1+\varepsilon s_1),$$
$$\rho X_2 = c(t_2+\varepsilon s_2)$$
$$\rho X_3 = d(t_3+\varepsilon s_3).$$

is tiansformed by this motion into the soma

$$\lambda X_0' = -C_0 X_0 + C_1 X_1 + C_2 X_2 + C_3 X_3,$$
$$\lambda X_1' = -C_1 X_0 - C_0 X_1 + C_3 X_2 - C_2 X_3,$$
$$\lambda X_2' = -C_2 X_0 - C_3 X_1 - C_0 X_2 + C_1 X_3,$$
$$\lambda X_3' = -C_3 X_1 - C_2 X_1 - C_2 X_2 - C_0 X_3$$
$$\lambda X_0' = (c_0+c_1+c_2+c_3)(t+\varepsilon s),$$
$$\lambda X_1' = (-c_1-c_0+c_3-c_2)(t+\varepsilon s),$$
$$\lambda X_2' = (-c_2-c_3-c_0+c_1)(t+\varepsilon s),$$
$$\lambda X_3' = (-c_3+c_2-c_1-c_0)(t+\varepsilon s).$$

The soma-chain (X') is the same with the soma-chain (X).

Case II.

$$|XZWT| = 0.$$

But $(Y), (Z), (W), (T)$ are not linearly dependent with real coefficients k, l, m, n, i.e

$$kY_i + lZ_i + mW_i + nT_i \neq 0,$$

In this case, we have a soma which is symmetrical to all the somas of the soma-chain, i.e.

$$U_i = \begin{vmatrix} Y_j & Y_s & Y_p \\ Z_j & Z_s & Z_p \\ W_j & W_s & W_p \end{vmatrix},$$

wheie $ijsp$ is a permutation of the figures 0, 1, 2, 3. Conversely, we can prove that every soma which is symmetrical to the soma (U) may be expressed in this form. Therefore if we take (U) as the protosoma, then the equations of the chain may be written in the foim

$$\begin{aligned}
\rho X_0 &= 0 \\
\rho X_1 &= aY_1 + bZ_1 + cT_1 + dW_1, \\
\rho X_2 &= aY_2 + bZ_2 + cT_2 + dW_2, \\
\rho X_3 &= aY_3 + bZ_3 + cT_3 + dW_3.
\end{aligned}$$