Some Applications of Quarternions.

By

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(Received November 29, 1915.)

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1. Riccati's Equation and Cayley's Formula.

The general solution of Riccati's equation

$$\frac{dz}{dt} = Lz^2 + Mz + N,$$

where L, M and N are functions of t alone, has the form

$$z = \frac{\alpha z_0 + \beta}{\gamma z_0 + \delta}, \qquad \text{.........} \quad (1)$$

in which α, β, γ and δ are functions of t, and z_0 is an arbitrary constant. We can always assume $\alpha\delta - \beta\gamma = 1$. Conversely, given α, β, γ and δ, we can build an equation which z satisfies. For that purpose, solving the equation (1) with respect to z_0,

$$z_0 = \frac{\delta z - \beta}{-\gamma z + \alpha}.$$

Differentiating both sides with respect to t and noticing that

$$\frac{dz_0}{dt} = 0, \text{ and } \alpha\delta - \beta\gamma = 1,$$

we have the equation

$$\frac{dz}{dt} = (\gamma\delta' - \gamma'\delta)z^2 + \{(\delta\alpha' - \delta'\alpha) + (\beta\gamma' - \beta'\gamma)\}z + (\alpha\beta' - \alpha'\beta), \quad (2)$$

where the dashes mean the differentiation with respect to t. We now consider the form of the equation (2) for Cayley's formula,

$$z = \frac{(d+ic)z_0 - (b-ia)}{(b+ia)z_0 + (d-ic)}$$[1]

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1 Klein: Ikosaeder p. 34. (e.g.)

where a, b, c, d are real constants satisfying the condition

$$a^2+b^2+c^2+d^2 = 1.$$

Consider these constants as real functions of the real variable t; in this case the equation (2) becomes

$$\frac{dz}{dt} = -irz + \frac{q-ip}{2} + \frac{q+ip}{2} z^2, \tag{3}$$

where
$$\begin{aligned}\gamma\delta'-\gamma'\delta &= (b+ia)(d'-ic')-(b'+ia')(d-ic)\\ &= bd'-b'd+ac'-a'c\\ &\quad+ i\,(ad'-a'd+cb'-c'b)\\ &\equiv \frac{q+ip}{2},\\ \alpha\beta'-\alpha'\beta &= \bar{\delta}\,(-\bar{\gamma}')-\bar{\delta}'(-\bar{\gamma})\\ &= \overline{\gamma\delta'-\gamma'\delta}\\ &= \frac{q-ip}{2},\end{aligned}$$

lastly
$$\begin{aligned}\delta\alpha'-\delta'\alpha+\beta\gamma'-\beta'\gamma &= 2i\{(dc'-d'c)+(ab'-a'b)\}\\ &\equiv -ir.\end{aligned}$$

Or

$$\begin{aligned}\frac{p}{2} &= ad'-a'd+b'c-bc'\\ \frac{q}{2} &= ac'-a'c+bd'-b'd\\ \frac{r}{2} &= a'b-ab'+cd'-c'd\end{aligned} \tag{4}$$

Hence *the function*

$$z = \frac{(d+ic)z_0-(b-ia)}{(b+ia)\,z_0+(d-ic)}$$

is the general solution af such an equation as

$$\frac{dz}{dt} = -irz + \frac{q-ip}{2} + \frac{q+ip}{2} z^2,$$

where $$a^2+b^2+c^2+d^2 = 1.$$

2. Darboux-Riccati's Fquation.

Consider two systems of rectangular axes, having the same origin; the one space-fixed and the other rotating. Let α, β, γ be the direction-cosines of any one of the space-fixed axes referred to the moving system; p, q, r, three components of rotation of the moving axes referred to themselves; then the kinematical equations are

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \beta r - \gamma q \\ \frac{d\beta}{dt} &= \gamma p - \alpha r \end{aligned}\right\} \qquad (1)$$

$$\frac{d\gamma}{dt} = \alpha q - \beta p,$$

where t is the time.

To discuss the solutions of the equations Prof. Darboux[1] proceeded as follows:—

We notice the condition

$$\alpha^2 + \beta^2 + \gamma^2 = \text{constant}.$$

When the constant is not equal to zero, we may always assume

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \qquad (2)$$

which is consistent with our assumption that α, β, γ are the direction-cosines of an axis. Put

$$\left.\begin{aligned} \frac{\alpha + i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha - i\beta} \equiv x \\ \frac{\alpha - i\beta}{1-\gamma} &= \frac{1+\gamma}{\alpha + i\beta} \equiv -\frac{1}{y} \end{aligned}\right\} \qquad (3)$$

which give $\quad \alpha = \dfrac{1-xy}{x-y}, \ \beta = i\dfrac{1+xy}{x-y}, \ \gamma = \dfrac{x+y}{x-y};$

then the equatian (1) become

$$\frac{dx}{dt} = -irx + \frac{q-ip}{2} + \frac{q+ip}{2}x^2,$$

$$\frac{dy}{dt} = -iry + \frac{q-ip}{2} + \frac{q+ip}{2}y^2.$$

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1 Lecons sur la Théorie Génèrale des Surfaces, I (first edition) pp. 21–22.

Or writing σ for x or y,

$$\frac{d\sigma}{dt} = -\, ir\sigma + \frac{q-ip}{2} + \frac{q+ip}{2}\sigma^2, \tag{4}$$

which is the same equation as §1 (3). Hence the general solution is of the form

$$\sigma = \frac{(d+ic)\,\sigma_0-(b-ia)}{(b+ia)\,\sigma_0+(d-ic)},$$

and the relations between p, q, r and a, b, c, d are given by the equations §1 (4). Those equations may be transformed by some easy calculations into the following

$$\left.\begin{aligned} &\frac{da}{dt}+\frac{p}{2}d+\frac{q}{2}c-\frac{r}{2}b=0, \\ &\frac{db}{dt}-\frac{p}{2}c+\frac{q}{2}d+\frac{r}{2}a=0, \\ &\frac{dc}{dt}+\frac{p}{2}b-\frac{q}{2}a+\frac{r}{2}d=0, \\ &\frac{dd}{dt}-\frac{p}{2}a-\frac{q}{2}b-\frac{r}{2}c=0; \end{aligned}\right\} \tag{5}$$

noticing that $a^2+b^2+c^2+d^2=1$, therefore $a\frac{da}{dt}+b\frac{db}{dt}+c\frac{dc}{dt}+d\frac{dd}{dt}=0$.

3. Given p, q, r to find a, b, c, d, we have to solve the system of four differential equations (5) of the last section. If we put $d=0$, then the first three equations become as follows:

$$\frac{da}{dt}=b\frac{r}{2}-c\frac{q}{2},$$

$$\frac{db}{dt}=c\frac{p}{2}-a\frac{r}{2},$$

$$\frac{dc}{dt}=a\frac{q}{2}-b\frac{p}{2},$$

which are of the same type as the system of equations (1) of the last section. Therefore the system (5) may be considered as a more general case and consequently we attempt, by some method, to obtain Riccati's equation which is equivalent to the system of differential equations (5). For that purpose we use Hamilton's quarternions $1, i_1, i_2, i_3$ which satisfy the relations

$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1,$$

$$i_1 i_2 + i_2 i_1 = i_2 i_3 + i_3 i_2 = i_3 i_1 + i_1 i_3 = 0.$$

Now multiplying $1, i_1, i_2, i_3$ into the equations (5) and summing them up, we obtain after some easy calculations

$$\frac{dx}{dt} = x\left(\frac{p}{2} i_3 + \frac{q}{2} i_2 - \frac{r}{2} i_1\right), \qquad (1)$$

where $$x = a + bi_1 + ci_2 + di_3.$$

This equation may be considered a special Riccati's equation. In a space of two dimensions, the analogous system of equations obtained fiom (1) § 2 by the assumption $\gamma=0$, viz.,

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \beta r \\ \frac{d\beta}{dt} &= -\alpha r. \end{aligned}\right\} \qquad (2)$$

Multiplying 1, i and summing them up, we have

$$\frac{dx}{dt} = -irx, \qquad x = \alpha + i\beta. \qquad (3)$$

which is quite similar to (1).

4. As a more general case than § 2 (5), we ccnsider the system of five equations

$$\left.\begin{aligned} \frac{d\alpha}{dt} &= \beta r - \gamma q - \delta p + \varepsilon q, \\ \frac{d\beta}{dt} &= \gamma p - \delta q + \varepsilon r - \alpha r, \\ \frac{d\gamma}{dt} &= -\delta r + \varepsilon r + \alpha q - \beta p, \\ \frac{d\delta}{dt} &= \varepsilon q + \alpha p + \beta q + \gamma r, \\ \frac{d\varepsilon}{dt} &= -\alpha q - \beta r - \gamma r - \delta q. \end{aligned}\right\} \qquad (1)$$

From these equations we have

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2 = \text{const.},$$

and this constant may be regarded as unity except in the case of zero; i.e.

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2 = 1.$$

We put, after Prof. Darboux, § 2 (3),

$$\left.\begin{aligned} \frac{\alpha+\beta i_1+\gamma i_2+\delta i_3}{1-\varepsilon} &= \frac{1+\varepsilon}{\alpha-\beta i_1-\gamma i_2-\delta i_3} \equiv x, \\ \frac{\alpha-\beta i_1-\gamma i_2-\delta i_3}{1-\varepsilon} &= \frac{1+\varepsilon}{\alpha+\beta i_1+\gamma i_2+\delta i_3} \equiv -y^{-1}, \end{aligned}\right\} \quad (2)$$

whence we have

$$\alpha+\beta i_1+\gamma i_2+\delta i_3 = (1-\varepsilon)\,x,$$

$$\alpha-\beta i_1-\gamma i_2-\delta i_3 = (1+\varepsilon)\,x^{-1}.$$

Therefore

$$\frac{d\alpha}{dt}+\frac{d\beta}{dt}i_1+\frac{d\gamma}{dt}i_2+\frac{d\delta}{dt}i_3 = (1-\varepsilon)\frac{dx}{dt}-\frac{d\varepsilon}{dt}x,$$

which, by the equations (1), is equal to the following sum

$$\beta r-\gamma q-\delta p+\varepsilon q+(\gamma p-\delta q+\varepsilon r-\alpha r)\,i_1$$

$$+(-\delta r+\varepsilon r+\alpha q-\beta p)\,i_2+(\varepsilon q+\alpha p+\beta q+\gamma r)\,i_3$$

$$= \varepsilon\,(q+ri_1+ri_2+qi_3)+(1-\varepsilon)\,x\,(-ri_1+qi_2+pi_3).$$

In the same way,

$$\frac{d\alpha}{dt}-\frac{d\beta}{dt}i_1-\frac{d\gamma}{dt}i_2-\frac{d\delta}{dt}i_3 = -(1+\varepsilon)\,x^{-1}\frac{dx}{dt}x^{-1}+\frac{d\varepsilon}{dt}x^{-1}$$

$$= \varepsilon\,(q-ri_1-ri_2-qi_3)+(1+\varepsilon)(ri_1-qi_2-pi_3)\,x^{-1}.$$

Hence we have

$$\frac{d\alpha}{dt}+\frac{d\beta}{dt}i_1+\frac{d\gamma}{dt}i_2+\frac{d\delta}{dt}i_3+x\left(\frac{d\alpha}{dt}-\frac{d\beta}{dt}i_1-\frac{d\gamma}{dt}i_2-\frac{d\delta}{dt}i_3\right)x$$

$$= -2\varepsilon\frac{dx}{dt}^{*}$$

$$= 2\varepsilon x\,(ri_1-qi_2-pi_3)+\varepsilon\,(q+ri_1+ri_2+qi_3)+\varepsilon x\,(q-ri_1-ri_2-qi_3)\,x,$$

* *The product of a scalar and a quarternion is permutable.*

whence we have

$$\frac{dx}{dt} = -x(ri_1 - qi_2 - pi_3) - \frac{q + ri_1 + ri_2 + qi_3}{2} - x\frac{q - ri_1 - ri_2 - qi_3}{2}x. \quad (3)$$

Starting from the equations

$$\alpha + \beta i_1 + \gamma i_2 + \delta i_3 = -(1+\varepsilon)y,$$

$$\alpha - \beta i_1 - \gamma i_2 - \delta i_3 = -(1-\varepsilon)y^{-1},$$

we arrive at the same equation for y; hence x and y are the solutions of the following equation

$$\frac{d\sigma}{dt} = -\sigma(ri_1 - qi_2 - pi_3) - \frac{q + ri_1 + ri_2 + qi_3}{2} - \sigma\frac{q - ri_1 - ri_2 - qi_3}{2}\sigma. \quad (4)$$

This equation is quite analogical with § 2 (4)

$$\frac{d\sigma}{dt} = -ir\sigma + \frac{q - ip}{2} + \frac{q + ip}{2}\sigma^2.$$

5. Serret-Frenet's Formula.

Serret-Frenet's formulas of a space curve are discussed as a special case of the kinematical equations § 2 (1). Formulas for hyperspace[1] e.g., of five dimensions are

$$\left.\begin{aligned} \frac{d\alpha}{ds} &= \frac{\beta}{R_1}, \\ \frac{d\beta}{ds} &= \frac{\gamma}{R_2} - \frac{\alpha}{R_1}, \\ \frac{d\gamma}{ds} &= \frac{\delta}{R_3} - \frac{\beta}{R_2}, \\ \frac{d\delta}{ds} &= \frac{\varepsilon}{R_4} - \frac{\gamma}{R_3}, \\ \frac{d\varepsilon}{ds} &= \qquad - \frac{\delta}{R_4}, \end{aligned}\right\} \quad (1)$$

where s is the arc length, and $\frac{1}{R}$'s are curvatures. This system of equations does not admit of the application of quarternions. For that

[1] G. E. A. Brunnel, Math. Ann., 19 (1882). F. Meyer, Jahresbericht d. Deut. Math. Ver., 19 (1910); &c.

purpose, we must lead these equations to those § 4 (1). But the equations (1) are a special case of a kinematical equations for hyper-space; and we rather consider the reduction of the latter equations.

The kinematical equations for n dimensions are given by the following system of n equations

$$\frac{d\alpha_i}{dt}=\sum_{h=1}^{n} p_{ih}\,\alpha_h \qquad i=1,2,\ldots\ldots n. \tag{2}$$

where $\qquad p_{ii}=0, \qquad p_{ih}=-p_{hi},$

so that the coefficients p make a skew symmetric determinant. Now we transform these equations by the following orthogonal transformations.

$$\left.\begin{aligned}\beta_j&=\sum_{k=1}^{n} a_{jk}\,\alpha_k, & |a_{jk}|&=+1,\\ & & j&=1,2,\ldots n,\\ \text{or}\quad \alpha_k&=\sum_{j=1}^{n} a_{jk}\,\beta_j, & k&=1,2,\ldots n.\end{aligned}\right\} \tag{3}$$

Differentiating with respect to t,

$$\frac{d\beta_j}{dt}=\sum_{k=1}^{n}\left(a_{jk}\frac{d\alpha_k}{dt}+\alpha_k\frac{da_{jk}}{dt}\right),$$

by (2) $$=\sum_{k=1}^{n}\left(a_{jk}\sum_{h=1}^{n} p_{kh}\,\alpha_h+\alpha_k\frac{da_{jk}}{dt}\right)$$

by (3) $$=\sum_{k=1}^{n}\left(a_{jk}\sum_{h=1}^{n} p_{kh}\sum_{r=1}^{n} a_{rh}\,\beta_r+\sum_{r=1}^{n} a_{rk}\,\beta_r\frac{da_{jk}}{dt}\right)$$

$$=\sum_{r=1}^{n}\left(\sum_{h,k=1}^{n} p_{kh}\,a_{jk}\,a_{rh}+\sum_{k=1}^{n} a_{rk}\frac{da_{jk}}{dt}\right)\beta_r.$$

Put $$P_{jr}'\equiv\sum_{h,k=1}^{n} p_{kh}\,a_{jk}\,a_{rh},$$

* T. Craig, *Displacements depending on One, Two and Three Parameters in a space of Four Dimensions*, Amer. J. Math., **20** (1898); for n dimensions, N. J. Hatzidakis, *Ibid.*, **22** (1900).

$$P_{jr}'' \equiv \sum_{k=1}^{n} a_{rk} \frac{da_{jk}}{dt},$$

and $$P_{jr} \equiv P_{jr}' + P_{jr}''.$$

Then we have

$$P_{jr}' = -\sum_{k,h=1}^{n} p_{hk}\, a_{rh}\, a_{jk} = -P_{rj}',$$

$$P_{jr}'' = -\sum_{k=1}^{n} a_{jk} \frac{da_{rk}}{dt} = -P_{rj}'';$$

hence $$P_{jr} = -P_{rj}, \text{ and } P_{jj} = 0.$$

Therefore the transformed equations are

$$\left.\begin{aligned} &\frac{d\beta_i}{dt} = \sum_{r=1}^{n} P_{jr}\, \beta_r, \\ &P_{jr} = -P_{rj}, \qquad j = 1, 2, \ldots n, \\ \text{where}\quad &P_{jj} = 0, \qquad r = 1, 2, \ldots n. \end{aligned}\right\} \tag{4}$$

When the transformed equations are required to be

$$\left.\begin{aligned} &\frac{d\beta_j}{dt} = \sum_{r=1}^{n} q_{jr}\, \beta_r, \\ &q_{rj} = -q_{rj}, \qquad j = 1, 2, \ldots n, \\ &q_{jj} = 0, \qquad r = 1, 2, \ldots n. \end{aligned}\right\} \tag{5}$$

Then considering a's as unknown, we have to solve the system of n^2 differential equations

$$P_{jr} = q_{jr}, \qquad j = 1, 2, \ldots n,$$
$$r = 1, 2, \ldots n;$$

or more completely

$$\sum_{k=1}^{n} a_{rk} \frac{d\bar{a}_{jk}}{dt} + \sum_{h\ k=1}^{n} p_{kh}\, a_{jk}\, a_{rh} = q_{jr}, \qquad j, r = 1, 2, \ldots n.$$

Or multiplying a_{rs} into both sides and summing with respect to r

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{js}\, a_{rk}\, \frac{da_{jk}}{dt} + \sum_{r=1}^{n}\sum_{h,k=1}^{n} p_{kh}\, a_{rs}\, a_{fk}\, a_{rh} = \sum_{r=1}^{n} a_{rs}\, q_{jr},$$

or

$$\frac{da_{js}}{dt} + \sum_{k=1}^{n} p_{ks}\, a_{jk} = \sum_{j=1}^{n} a_{js}\, q_{jr},$$

or

$$\frac{da_{js}}{dt} = \sum_{k=1}^{n} (p_{sk}\, a_{jk} + q_{jk}\, a_{ks}), \qquad j, s = 1, 2, \dots n. \tag{6}$$

On conditions that

$$\left.\begin{aligned} &\sum_{r=1}^{n} a_{jr}^2 = 1, && i = 1, 2, \dots n, \\ &\sum_{r=1}^{n} a_{ir}\, a_{jr} = 0, && j = 1, 2, \dots n;\ i \neq j. \end{aligned}\right\} \tag{7}$$

By these conditions the number of the differential equations is reduced to $\frac{n(n-1)}{2}$. After Cayley these n^2 quantities *a's* can be represented by $\frac{n(n-1)}{2}$ essential parameters; considering these parameters as new unknowns, the equations (6) will be transformed into a system of $\frac{n(n-1)}{2}$ differential equations with $\frac{n(n-1)}{2}$ unknowns. Any particular solution will suffice to obtain *a's* and consequently the orthogonal transformations (3).

Specially for $n = 5$, we have

$$q_{12} = q_{25} = q_{35} = q_{43} = r,$$

$$q_{15} = q_{31} = q_{42} = q_{15} = q,$$

$$q_{23} = q_{41} = p,$$

then the system of differential equations

$$\frac{da_i}{dt} = \sum_{h=1}^{5} p_{ih}\, a_h, \qquad i = 1, 2, \dots 5,$$

may be transformed into the system § 4 (1) and hence into § 4 (4). (For the case $n=4$, we may treat similarly.) Serret-Frenets' formulas (1) are a special type of kinematical equations for which

$$p_{12}=\frac{1}{R_1}, \qquad p_{13}=p_{14}=p_{15}=0,$$

$$p_{23}=\frac{1}{R_2}, \qquad p_{24}=p_{25}=0,$$

$$p_{34}=\frac{1}{R_3}, \qquad P_{35}=0,$$

$$p_{45}=\frac{1}{R_4}.$$

Hence, by the above theory, this system of equations may be transformed into § 4 (4). Thus Prof. Darbaux's method[1] admits theoietically of a similar extension[2]

6. Line Element on Hypersphere.

If ds_n denote the line element on the hypersphere of n dimensions, then for the hypersphere

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=1, \qquad (1)$$

$$ds_5^2 = d\alpha^2+d\beta^2+d\gamma^2+d\delta^2+d\varepsilon^2. \qquad (2)$$

On the other hand, from the equations § 4 (2)

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = -(1-\varepsilon)^2xy^{-1} = 1-\varepsilon^2,$$

or dividing by $(1-\varepsilon)$, and rearranging we have

$$\left.\begin{aligned} \varepsilon &= (x+y)(x-y)^{-1} \\ \text{and} \qquad 1-\varepsilon &= -2y\,(x-y)^{-1} = -2\,(x-y)^{-1}y.^{3} \end{aligned}\right\} \qquad (2)$$

Now $\quad d\alpha+d\beta i_1+d\gamma i_2+d\delta i_3 = (1-\varepsilon)\,dx-xd\varepsilon$

$$= -2\,dx\,(x-y)^{-1}y-xd\varepsilon,$$

also $\quad d\alpha-d\beta i_1-d\gamma i_2-d\delta i_3=(1-\varepsilon)\,y^{-1}\,dy\,y^{-1}+y^{-1}\,d\varepsilon\,;$

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1 *Loc. cit.*

2 Mr. Eiesland treated the same problem of integration of kinematical equations, from another point of view, in Amer. J. Math., 20 (1898); 28 (1906).

3 *When the product of two quarternions is equal to a scalar, then the factors are permutable.*

hence $d\alpha^2+d\beta^2+d\gamma^2+d\delta^2 = 4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1}-xy^{-1}\,d\varepsilon^2$
$$+2\,(xy^{-1}\,dy-dx)(x-y)^{-1}\,d\varepsilon.$$

But $$d\varepsilon\,(x-y) = (\varepsilon-1)\left(\frac{1+\varepsilon}{\varepsilon-1}\,dy-dx\right) = (\varepsilon-1)(xy^{-1}\,dy-dx),$$

hence $$xy^{-1}\,dy-dx = (x-y)\,\frac{d\varepsilon}{\varepsilon-1}.$$

Therefore $$d\alpha^2+d\beta^2+d\gamma^2+d\delta^2=4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1}$$
$$-\frac{1+\varepsilon}{\varepsilon-1}\,d\varepsilon^2+2\,\frac{d\varepsilon^2}{\varepsilon-1},$$

or $$ds_5^2=d\alpha^2+d\beta^2+d\gamma^2+d\delta^2+d\varepsilon^2=4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1}, \tag{4}$$

which is quite analogical with
$$ds_3^2 = d\alpha^2+d\beta^2+d\gamma^2 = \frac{4\,dx\,dy}{(x-y)^2}$$[1]

7. Parametric Representation.

By aid of x and y, the five variables $\alpha, \beta, \gamma, \delta, \varepsilon$ can be represented as in the case of three variables. Now add the equations
$$\left.\begin{aligned} \alpha+\beta i_1+\gamma i_2+\delta i_3 &= (1-\varepsilon)\,x, \\ \alpha-\beta i_1-\gamma i_2-\delta i_3 &= -(1-\varepsilon)\,y^{-1}. \end{aligned}\right\} \tag{1}$$

Then we have $$2\,\alpha = (x-y^{-1})(1-\varepsilon),$$

by § 6 (3) $$= -2\,(x-y^{-1})\,y\,(xy-)^{-1}.$$

Hence $$\alpha = (1-xy)(x-y)^{-1}. \tag{2}$$

Next from (1)
$$\alpha i_1-\beta-\gamma i_3+\delta i_2=(1-\varepsilon)\,xi_1,$$
$$\alpha i_1+\beta-\gamma i_3+\delta i_2 = -(1-\varepsilon)\,i_1 y^{-1}.$$

Adding we have $$2\,\beta = -(1-\varepsilon)(xi_1+i_1\,y^{-1}),$$
$$= 2\,(xi_1+i_1 y^{-1})\,y\,(x-y)^{-1}.$$

Hence $$\beta = i_1\,(1-i_1\,xi_1 y)(x-y)^{-1}. \tag{3}$$

γ and δ may be obtained in the same way; hence, adding the first formula § 6 (3), we have

[1] Darboux: *Loc. cit.* p. 30.

$$\left.\begin{aligned} \alpha &= (1-xy)(x-y)^{-1}, \\ \beta &= i_1(1-i_1 x i_1 y)(x-y)^{-1}, \\ \gamma &= i_2(1-i_2 x i_2 y)(x-y)^{-1}, \\ \delta &= i_3(1-i_3 x i_3 y)(x-y)^{-1}, \\ \varepsilon &= (x+y)(x-y)^{-1}. \end{aligned}\right\} \qquad (4)$$

For the hypersphere of four dimensions, we have to put $\varepsilon=0$, and hence $x=-y$,

$$\left.\begin{aligned} \alpha &= \frac{1}{2}(x^{-1}+x), \\ \beta &= \frac{1}{2}(i_1 x^{-1}-x i_1), \\ \gamma &= \frac{1}{2}(i_2 x^{-1}-x i_2), \\ \delta &= \frac{1}{2}(i_3 x^{-1}-x i_3)\ ; \end{aligned}\right\} \qquad (5)$$

for the sphere we have to put $i_2=i_3=0$, $i_1 \equiv i$,

$$\left.\begin{aligned} \alpha &= (1-xy)(x-y)^{-1}, \\ \beta &= i\,(1+xy)(x-y)^{-1}, \\ \varepsilon &= (x+y)(x-y)^{-1}\ ; \end{aligned}\right\} \qquad (6)$$

for the circle, $i_2 = i_3 = \varepsilon = 0$, $\quad i_1 \equiv i$ and $x = -y$,

$$\left.\begin{aligned} \alpha &= \frac{1}{2}(x^{-1}+x), \\ \beta &= \frac{i}{2}(x^{-1}-x). \end{aligned}\right\} \qquad (7)$$

8. Transformation of Line-element.

The square of the line element ds^2 does not change by the substitution

$$x = Ax_1 B + C, \qquad y = Ay_1 B + C, \qquad (1)$$

where A, B, C are constant quarternions. For

$$ds^2 = 4\, dx\,(x-y)^{-1}\, dy\,(x-y)^{-1}.$$

$$= x\, Adx_1\, B\{A\,(x_1-y_1)\, B\}^{-1}\, Ady_1\, B\{A\,(x_1-y_1)\, B\}^{-1},$$

$$= 4\,Adx_1\,(x_1-y_1)^{-1}\,dy_1\,(x_1-y_1)^{-1}\,A_1.$$

Since ds^2 is a scalar

$$ds^2 = 4\,dx_1\,(x_1-y_1)^{-1}\,dy_1\,(x_1-y_1)^{-1}.$$

Next ds^2 does not change by the substitutions

$$x = x_1^{-1}, \qquad y = y_1^{-1}. \tag{2}$$

For since $(x-y)^{-1} = x_1\,(y_1-x_1)^{-1}\,y_1 = y_1\,(y_1-x_1)^{-1}\,x_1$,

$$ds^2 = dx_1\,(x_1-y_1)^{-1}\,dy_1\,(x_1-y_1)^{-1}.$$

Lastly ds_5^2 does not change by the substitutions

$$\left.\begin{aligned} x &= (Ax_1+B)(Cx_1+D)^{-1}, \\ y &= (Ay_1+B)(Cy_1+D)^{-1} \end{aligned}\right\} \tag{3}$$

where A, B, C, D are constant quarternions.

Since $$x = \{AC^{-1}(Cx_1+D)+B-AC^{-1}D\}(Cx_1+D)^{-1}$$
$$= AC^{-1}+(B-AC^{-1}D)(Cx_1+D)^{-1}.$$

Putting $\quad x_2 \equiv Cx_1+D, \qquad x_3 \equiv x_2^{-1},$

then $\quad x \equiv AC^{-1}+(B-AC^{-1}D)\ x_3,$

in the same way

$$y_2 \equiv Cy_1+D, \qquad y_3 \equiv y_2^{-1},$$

then $$y = AC^{-1}+(B-AC^{-1}D)\,y_3.$$

Therefore by (1) and (2), we may prove the invariance of ds_5^2. We remark that since

$$ds_5^2 = 4\,dx\,(x-y)^{-1}\,dy\,(x-y)^{-1},$$
$$= 4\,dy\,(x-y)^{-1}\,dx\,(x-y)^{-1},$$

ds_5^2 is also invariant by the transformations

$$\left.\begin{aligned} x &= (Ay_1+B)(Cy_1+D)^{-1}, \\ y &= (Ax_1+B)(Cx_1+D)^{-1}. \end{aligned}\right\} \tag{4}$$

9. Meaning of the Parameter x.

When $(\omega, \xi, \eta, \zeta)$ denote the coordinates of the projection of a point on the hypersphere

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2 = 1,$$

from its pole on the equatorial hyperspace of four dimensions, then by analogy, the equations of projection are

$$\frac{\alpha}{\omega} = \frac{\beta}{\xi} = \frac{\gamma}{\eta} = \frac{\delta}{\zeta} = 1-\varepsilon,$$

whence by easy calculations

$$\alpha = \frac{2\omega}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$\beta = \frac{2\xi}{\omega^2+\zeta^2+\eta^2+\zeta^2+1},$$

$$\gamma = \frac{2\eta}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$\delta = \frac{2\zeta}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$\varepsilon = \frac{\omega^2+\xi^2+\eta^2+\zeta^2-1}{\omega^1+\xi^2+\eta^2+\zeta^2+1}.$$

Now $$\alpha+\beta i_1+\gamma i_2+\delta i_3 = \frac{2(\omega+\xi i_1+\eta i_2+\zeta i_3)}{\omega^2+\xi^2+\eta^2+\zeta^2+1},$$

$$= (1-\varepsilon)\,x,$$

hence $$= \frac{2}{\omega^2+\xi^2+\eta^2+\zeta^2+1}\,x.$$

Therefore $$x = \omega+\xi i_1+\eta i_2+\zeta i_3.$$

For a sphere, x is the complex variable on the Gauss' plane.
