Construction of elements of a Lie group $G_2$
via spinor group $Spin(8)$

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1 Introduction

Let $\mathcal{C}$ be the division Cayley algebra over $\mathbb{R}$. It is known that the automorphism group $G_2 = \text{Aut}\mathcal{C}$ is a compact simply connected simple Lie group of type $G_2$ and $\mathfrak{g}_2 = \text{Der}\mathcal{C}$ is a compact simple Lie algebra of type $G_2$. In this paper, we give a concrete description of elements of the group $G_2$ by using elements of the Cayley algebra.

2 Spinor group

Let $T^+(\mathbb{R}^n)$ be the even tensor algebra of $\mathbb{R}^n$ and $U(\mathbb{R}^n)$ the two-sided ideal of $T^+(\mathbb{R}^n)$ generated by

$$x \otimes x + (x, x)1 \quad (x \in \mathbb{R}^n)$$

where $(\ , \ )$ is the canonical inner product of $\mathbb{R}^n$. Define the even Clifford algebra $C^+(\mathbb{R}^n)$ by

$$C^+(\mathbb{R}^n) := T^+(\mathbb{R}^n)/U(\mathbb{R}^n).$$

We denote the multiplication of $\alpha, \beta \in C^+(\mathbb{R}^n)$ by $\alpha \cdot \beta$. For $x, y \in \mathbb{R}^n$ we have

$$x \cdot y + y \cdot x = -2(x, y). \quad (1)$$

It is known that a spinor group $Spin(n)$ is defined by

$$Spin(n) := \left\{ \alpha = a_1 \cdot a_2 \cdots a_{2l} \in C^+(\mathbb{R}^n) \mid a_i \in \mathbb{R}^n, \prod_{i=1}^{2l} (a_i, a_i) = 1 \right\}.$$
The unit element of $Spin(n)$ is $1 = -a \cdot a$ \((a \in \mathbb{R}^n, (a, a) = 1)\) and an inverse element of
\[
\alpha = a_1 \cdot a_2 \cdots a_{2l-1} \cdot a_{2l} \in Spin(n)
\]
is
\[
\alpha^{-1} = a_{2l} \cdot a_{2l-1} \cdots a_2 \cdot a_1 \in Spin(n).
\]
The vector representation $p_1 : Spin(n) \to SO(n)$ is given by
\[
p_1(\alpha)x = \alpha \cdot x \cdot \alpha^{-1} \quad (x \in \mathbb{R}^n).
\]
It is known that $Spin(n)$ is a universal covering group of $SO(n)$ (double covering), and $Spin(n)$ \((n \geq 3)\) is simply connected.

3 Cayley algebra and spinor groups

In the division Cayley algebra $\mathcal{C}$, we denote the multiplication and the canonical conjugation by $xy$ and $\overline{x}$ \((x, y \in \mathcal{C})\) respectively. The inner product of $\mathcal{C}$ is defined by

\[
(x, y) := \frac{1}{2}(x\overline{y} + y\overline{x}).
\]

We describe here some formulas of the Cayley algebra (use in later). For $x, y, z \in \mathcal{C}$, we have

\[
\begin{align*}
    x(\overline{y}z) + y(\overline{x}z) &= 2(x, y)z = (zx)\overline{y} + (zy)\overline{x}, \\
    (xy)x &= x(yx) = yzx, \\
    (xy)(zx) &= x(yz)x, \\
    (xy, xz) &= (x, x)(y, z) = (yx, zx).
\end{align*}
\]

We identify $\mathcal{C}$ with $\mathbb{R}^8$ and $\text{ImC} = \{x \in \mathcal{C} \mid \overline{x} + x = 0\}$ with $\mathbb{R}^7$. Then we see

\[
Spin(8) = \left\{ \alpha = a_1 \cdot a_2 \cdots a_{2l} \mid a_i \in \mathcal{C}, \prod_{i=1}^{2l}(a_i, a_i) = 1 \right\},
\]

\[
Spin(7) = \left\{ \alpha = a_1 \cdot a_2 \cdots a_{2l} \mid a_i \in \text{ImC}, \prod_{i=1}^{2l}(a_i, a_i) = 1 \right\}.
\]

Lemma 3.1 For $\alpha = a_1 \cdot a_2 \cdots a_{2l} \in Spin(8)$ and $x \in \mathcal{C}$, we have

\[
p_1(\alpha) = a_1(\overline{a}_2(a_3(\cdots a_{2l-1}(\overline{a}_{2l}x\overline{a}_{2l})a_{2l-1} \cdots)_{a_3})\overline{a}_2)a_1.
\]

Proof. It is sufficient to prove the claim for $\alpha = a \cdot b \in Spin(8)$. From (1) we see $b \cdot x \cdot b = (b, b)x - 2(x, b)b$ and

\[
p_1(\alpha)x = a \cdot b \cdot x \cdot b \cdot a
\]
\[
= (a, a)(b, b)x - 2(x, a)(b, b)a - 2(x, b)(a, a)b + 4(x, b)(a, b)a.
\]
On the other hand, from (2) we see $\bar{b}x\bar{b} = 2(x, b)\bar{b} - (b, b)x$ and
\[
a(\bar{b}x\bar{b}) = (a, a)(b, b)x - 2(x, a)(b, b)a - 2(x, b)(a, a)b + 4(x, b)(a, b)a.
\]

A linear transformation
\[
\mathcal{C} \otimes \mathcal{C} \to \text{End}(\mathcal{C}), \quad a \otimes b \mapsto -L_a L_{\bar{b}}
\]
(where $L_a x = ax$) can be extended to a representation
\[
p_2 : T^+(\mathcal{C}) \to \text{End}(\mathcal{C}), \quad p_2(a \otimes b) \mapsto -L_a L_{\bar{b}}.
\]
From (2) we see
\[
p_2(a \otimes a + (a, a)1) = -L_a L_{\bar{a}} + (a, a)1 = 0,
\]
then we have $p_2(U(\mathcal{C})) = \{0\}$ and $p_2$ is a representation of $C^+(\mathcal{C}) := T^+(\mathcal{C})/U(\mathcal{C})$. Then we have a representation $p_2$ of $Spin(8)$ on $\mathcal{C}$
\[
p_2(a_1 \cdot a_2 \cdots a_{2q}) = (-1)^q L_{a_1} L_{\overline{a}_2} L_{a_3} \cdots L_{\overline{a}_{2q}}.
\]
In a similar way we define a representation $p_3$ of $Spin(8)$ on $\mathcal{C}$ by
\[
p_3(a_1 \cdot a_2 \cdots a_{2q}) = (-1)^q R_{a_1} R_{\overline{a}_2} R_{a_3} \cdots R_{\overline{a}_{2q}}
\]
where $R_a x = xa$. Then we have following lemma from (4) and lemma 3.1.

**Lemma 3.2** For $\alpha \in Spin(8)$ and $x, y \in \mathcal{C}$, we have
\[
p_1(\alpha)(xy) = (p_2(\alpha)x)(p_3(\alpha)y).
\]

From (5), we see
\[
p_2(\alpha), p_3(\alpha) \in SO(8) = \{A \in GL(\mathcal{C}) \mid (Ax, Ay) = (x, y), \quad x, y \in \mathcal{C}\}.
\]
If $p_2(\alpha)1 = 1$, since
\[
p_1(\alpha)x = p_1(\alpha)(1x) = (p_2(\alpha)1)(p_3(\alpha)x) = p_3(\alpha)x,
\]
we see $p_1(\alpha) = p_3(\alpha)$. Similarly, if $p_3(\alpha)1 = 1$ we see $p_1(\alpha) = p_2(\alpha)$. Hence we have

**Proposition 3.3** For $\alpha \in Spin(8)$, the following two conditions are equivalent.

(i) $p_2(\alpha)1 = p_3(\alpha)1 = 1$,

(ii) $p_1(\alpha) = p_2(\alpha) = p_3(\alpha) \in G_2$. 

4 Main theorem

It is known that the group $G_2$ is a subgroup of $SO(7) = p_1(Spin(7))$.

**Lemma 4.1** For $\alpha = a_1 \cdot a_2 \cdots a_{2l} \in Spin(7)$, 2 conditions $p_2(\alpha)1 = 1$ and $p_3(\alpha)1 = 1$ are equivalent.

Proof. Since $\bar{a} = -a$ for $a \in \text{Im}\mathbb{C}$, we see

$$p_2(\alpha) = L_{a_1}L_{a_2} \cdots L_{a_{2l}}, \quad p_3(\alpha) = R_{a_1}R_{a_2} \cdots R_{a_{2l}}.$$  

If $p_2(\alpha)1 = 1$, we see

$$1 = \overline{1} = p_2(\alpha)1 = a_1(a_2(\cdots a_{2l-2}(a_{2l-1}a_{2l}))).$$

$$= (((\bar{a}_{2l}\bar{a}_{2l-1})\bar{a}_{2l-2} \cdots)\bar{a}_2)a_1 = (((a_{2l}a_{2l-1})a_{2l-2} \cdots)a_2)a_1 = p_3(\alpha)1.$$  

If $p_3(\alpha)1 = 1$, similarly we have $p_2(\alpha)1 = 1$. 

\[\square \]

From proposition 3.3 and lemma 4.1, we have the following.

**Proposition 4.2** For $a_1, a_2, \cdots, a_{2l} \in \text{Im}\mathbb{C}$, let us put $g := L_{a_1}L_{a_2} \cdots L_{a_{2l}}$. If $g1 = 1$, $g \in G_2$.

Let us put

$$K := \{g = L_{a_1}L_{a_2} \cdots L_{a_{2l}} \mid a_k \in \text{Im}\mathbb{C}, g1 = 1\}.$$  

Then $K$ is a subgroup of $G_2$. Since $\alpha L_{a} \alpha^{-1} = L_{\alpha(a)}$ ($\alpha \in G_2$) the subgroup $K$ is normal. In next section we show $K \neq \{e\}$. Then we have

**Theorem 4.3**

$$G_2 = \{g = L_{a_1}L_{a_2} \cdots L_{a_{2l}} \mid a_k \in \text{Im}\mathbb{C}, g1 = 1\}.$$  

5 Example of elements

Let $\{e_0 = 1, e_1, e_2, \cdots, e_7\}$ be a basis of $\mathbb{C}$ with following conditions.

$e_0 e_k = e_k e_0 = e_k$ ($0 \leq k \leq 7$),  \hspace{0.5cm} (e_0 = 1, \text{ the unit element}),

$e_k^2 = -e_0$ ($k \neq 0$),  \hspace{0.5cm} $e_k e_l = -e_l e_k$ ($1 \leq k \neq l \leq 7$),

$e_1 = e_2 e_3 = e_4 e_5 = e_6 e_7$,  \hspace{0.5cm} $e_2 = e_3 e_1 = e_6 e_4 = e_5 e_7$,

$e_3 = e_1 e_2 = e_4 e_7 = e_5 e_6$,  \hspace{0.5cm} $e_4 = e_5 e_1 = e_2 e_6 = e_7 e_3$,

$e_5 = e_1 e_4 = e_7 e_2 = e_6 e_3$,  \hspace{0.5cm} $e_6 = e_7 e_1 = e_4 e_2 = e_3 e_5$,

$e_7 = e_1 e_6 = e_2 e_5 = e_3 e_4$. 


Define an element $G_{ij}$ ($0 \leq i \neq j \leq 7$) of $\text{so}(8) = \{X \in \mathfrak{gl}(\mathbb{C}) \mid (Xx, y) + (x, Xy) = 0, \ x, y \in \mathbb{C}\}$ by

$$G_{ij}e_k = \delta_{jk}e_i - \delta_{ik}e_j.$$ 

In [F], Freudenthal proved if $e_i e_j = e_k e_l$ (for example $e_2 e_3 = e_4 e_5, e_3 e_1 = e_6 e_4, \cdots$ etc.), $G_{ji} - G_{lk} \in \mathfrak{g}_2$. Let us put

$$a_1 = \cos \frac{\theta}{2}e_i + \sin \frac{\theta}{2}e_j, a_2 = e_i, a_3 = e_k, a_4 = \cos \frac{\theta}{2}e_k + \sin \frac{\theta}{2}e_l, h = h_{ijkl}(\theta) = L_{a_1}L_{a_2}L_{a_3}L_{a_4}.$$ 

By a straightforward calculation, we have

$$h_{1} = 1 \quad \text{and} \quad he_p = \begin{cases} 
\cos \theta e_i + \sin \theta e_j & (p = i), \\
-\sin \theta e_i + \cos \theta e_j & (p = j), \\
\cos \theta e_k - \sin \theta e_l & (p = k), \\
\sin \theta e_k + \cos \theta e_l & (p = l), \\
e_p & (\text{others}).
\end{cases}$$

This show

$$\frac{d}{d\theta}h_{ijkl}(\theta)\bigg|_{\theta=0} = G_{ji} - G_{lk}.$$ 

References