Graphical Solution of $\frac{dy}{dx} = f(x, y)$.

By

Takeo Wada.

(Received February 2, 1917.)

The problem of this paper is to investigate the integral curves defined by differential equations of the simplest form

$$\frac{dy}{dx} = f(x,y)$$

in which $f(x,y)$ is a rational function of x and y.

Equations of this form occur frequently in applied mathematics, and were first treated by Profs. Briot and Bouquet[1]. Subsequently Prof. Poincaré[2] gave a complete and elegant treatment. Both, however, being based upon the theory of analytic functions of complex variables, are a little complicated.

I have tried to consider the problem from the standpoint of real variables, and, by the help of the ideas of these professors, especially Prof. Poincaré's idea of consecutive point, to obtain a method of treatment. In principle the endeavour is to shew the outline of integral curves by the use of slope and concavity, just as is done in tracing curves defined by equations between x and y.

1. General Theory of $\frac{dy}{dx} = f(x,y)$.

By the famous theorem of Cauchy-Lipschitz it is known that, if $f(x,y)$ is continuous and $\frac{\partial f}{\partial y}$ is limited in the neighbourhood of a point x_0, y_0, there exists one integral of the differential equation

$$\frac{dy}{dx} = f(x,y), \tag{1}$$

which passes through the point x_0, y_0.

---

1 J. d l'Ecole Polytech. **36**, 133-198 (1856).

2 Sur les courbes définies par les équation differentielle (J. Liouville 1881, 1882 and 1885).

In particular, when $f(x,y)$ is a rational function (1) may be written in the form

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \tag{2}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials of x and y, which have no common factor; and we shall treat hereafter the differential equations of this form.

In equation (2), the existence-theorem applies at any point where $Q(x,y)$ does not vanish. Though at a point x_0, y_0 $f(x,y)$ vanishes, if $P(x_0, y_0) \neq 0$, considering y as the independent variable, the differential equation

$$\frac{dx}{dy} = \frac{Q(x,y)}{P(x,y)}$$

satisfies the Cauchy-Lipschitz's condition at x_0, y_0; and the existence of a single integral curve through the point is assured. Thus passing through any point where $P(x,y)$, $Q(x,y)$ do not vanish at the same time there is one integral curve of (2).

The points where $P(x,y)$ and $Q(x,y)$ do not vanish at the same time are called *ordinary points*; and the exceptional points are called *singular points* of the differential equation (2). The singular points are given by the simultaneous equations

$$P(x,y) = 0, \quad Q(x,y) = 0 \tag{3}$$

and there is a finite number of such points in the plane.

Denoting the integral curve which passes through an ordinary point by $C(x_0, y_0)$, it may be expressed analytically by

$$C(x_0, y_0): \begin{cases} x = \varphi(t; x_0, y_0), \\ y = \psi(t; x_0, y_0) \end{cases} \tag{4}$$

where φ and ψ are one-valued functions which are continuous with the derivatives φ_t', ψ_t' and satisfy the equation

$$\frac{\psi_t'}{\varphi_t'} = f(\varphi, \psi)$$

in a certain interval (T_1, T_2) of t and the initial conditions

$$\varphi(t_0; x_0, y_0) = x_0, \quad \psi(t_0; x_0, y_0) = y_0$$

t_0 being a fixed value in (T_1, T_2).

The functions φ, ψ are continuous not only of t but also of the parameters[1] x_0, y_0 in any domain $D(t, \xi, \eta)$ which contains $t=t_0, \xi=x_0, \eta=y_0$ and does not contain any singular point, provided that for any set of values t, ξ, η in D the functions $\varphi(t\,;\xi, \eta), \psi(t\,;\xi, \eta)$ do not represent any singular point. This means that the integral curve moves continuously with the initial point x_0, y_0.

From the continuity of φ_t' and ψ_t' it follows that the integral curve $C(x_0, y_0)$ is rectifiable. If we take the arc length s measured from x_0, y_0 on a direction as parameter, the equations to the curve will become

$$(4)_a \qquad C(x_0, y_0): \quad x=\varphi(s\,; x_0, y_0),\ y=\psi(s\,; x_0, y_0) \quad s_1 \leqq s \leqq s_2$$

of which we have

$$\frac{\psi_s'}{\varphi_s'} = f(\varphi, \psi), \quad (\varphi_s')^2+(\psi_s')^2 = 1$$

and

$$\varphi(0\,; x_0, y_0) = x_0, \quad \psi(0\,; x_0, y_0) = y_0.$$

As an integral curve can not stop at any ordinary point, it may be extended on both sides so long as it does not arrive at a singular point. The curve so extended as possible on both sides is called a *complete characteristic*. Of characteristics the following properties are reckoned:

1) They can not have any singular point in the interior, and their endpoints, if they exist, are singular;

2) They can not have any branch-point in the interior;

3) No two characteristics can have a point in common, which is not an endpoint of both;

4) The segment between any two interior points of a characteristic has a finite length.

That form of characteristic which is constructed through an ordinary point by possible extension on one side of the point is called a *semi-characteristic*. Assuming a direction, two semi-characteristics through an ordinary point are distinguished as progressive and regressive.

Of a semi-characteristic the following four forms are to be considered:

i) The continuation may finally arrive at a singular point and stop at that point;

ii) It may return again to the point of starting. In this case the curve is closed and is called a *cyclic characteristic.*;

---

[1] E. Picard, Traite d'analyse, II, p. 333. (1905).

iii) It may go to an infinite distance;

iv) The semi-characteristic may have no second endpoint, however we may continue the extension, the curve itself remaining in a finite domain. This form will be considered later.

2. Slope and Concavity defined by a Differential Equation.

In tracing a curve defined by an equation between x and y, the general principle is due to the investigation of 1) the existence of the curve, 2) $\frac{dy}{dx}$ i.e. the slope of the curve, 3) $\frac{d^2y}{dx^2}$ i.e. the concavity of the curve, 4) the nature of multiple points and 5) the forms of indefinitely distant branches. This consideration is directly applied also to the discussion of curves defined by a differential equation.

In the first place we shall consider the slope of characteristics defined by

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} = f(x,y). \tag{2}$$

Divide the points of the entire plane into five aggregates, according as the value of $f(x,y)$ is I positive, II negative, III zero, IV infinite or V indeterminate, then

at points of I the line-elements are increasing,
" " " II " " " " decreasing,
" " " III " " " " parallel to the x-axis,
" " " IV " " " " " " " y-axis

and the points of V are singular points of the differential equation. Aggregates I and II consist of points in one or more domains, III and IV of points on curves and isolated points and V of a finite number of points.

Next to find the concavity of the characteristics from the equation (2), deduce the second derivative;

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x,y) = \frac{Q^2P_x' + PQ\,(P_y' - Q_x') - P^2Q_y'}{Q^3} \tag{5}$$

and, denoting the expression on the right of (5) by $F(x,y)$, we have

$$\frac{d^2y}{dx^2} = F(x,y), \tag{5$_a$}$$

where the function $F(x,y)$ is evidently rational of x and y.

Again divide the points in the plane into aggregates, according as the value of $F(x,y)$ at x,y is VI positive, VII negative or VIII zero. Then

at points of VI the characteristics are concave upwards,
,, ,, ,, VII ,, ,, ,, ,, downwards
and ,, ,, ,, VIII ,, ,, have contact of higher

order than the first, with tangent lines at those points. For example, a point of VIII may be an inflexional point of the characteristic through that point.

By the above two steps the entire plane will have been divided into the following aggregates of points :

a) Four kinds of domains where

i) $f >$, $F > 0$, ii) $f > 0$, $F < 0$, iii) $f < 0, F < 0$, iv) $f < 0$, $F < 0$;

b) Three kinds of curves where

v) (6) $f(x,y) = 0$ or $P(x,y) = 0$,
vi) (7) $f(x,y) = \infty$ or $Q(x,y) = 0$,
vii) (8) $F(x,y) = 0$ or $Q^2P_x' + PQ(P_y' - Q_x') - P^2Q_y' = 0$;

c) viii) Finite number of singular points where P and Q vanish at the same time.

When the configuration of slope and concavity is known, we may trace the outline of characteristics excluded the neighbourhoods of the singular points.

In tracing the characteristics it is important to know how the curve (8) cuts the characteristics. Denoting by x_1, y_1 a point on (8), at that point the slope of this curve is $\left(-\frac{F_x'}{F_y'}\right)_{x_1, y_1}$ while that of the characteristic is $f(x_1, y_1)$. Hence if

$$(9) \qquad \chi_{F=0} = \frac{dy_1}{dx_1} - f(x_1, y_1) = -\left\{\frac{F_x' + F_y' f}{F_y'}\right\}_{x_1, y_1}$$

is positive the characteristic, in passing through the point (x_1, y_1) from the left to the right, cuts $F=0$ from the upper side to the lower; if it is negative, from the lower to the upper, and, finally, if it is zero, the curve (8) and the characteristic have the same line-elements at the point (x_1, y_1).

3. Direction Angles and Characteristic Angles.

Before going into the consideration of characteristics in the neighbourhood of a singular point, certain preliminary conceptions should be introduced.

Regular curve. By a regular (open) curve is meant an unclosed Jordan curve which has a tangent line everywhere, that moves continuously with the point on the curve.

The analytical expression of a regular curve is

$$x = \Phi(t), \qquad y = \Psi(t) \qquad T_1 \leqq t \leqq T_2 \tag{10}$$

in which $\Phi(t)$, $\Psi(t)$ aud their derivatives $\Phi'(t)$, $\Psi'(t)$ are continuous and satisfy the conditions

$$\{\Phi'(t)\}^2 + \{\Psi'(t)\}^2 > 0 \qquad T_1 \leqq t \leqq T_2,$$

and

$$\{\Phi(t_1) - \Phi(t_2)\}^2 + \{\Psi(t_1) - \Psi(t_2)\}^2 > 0$$

for any two different values t_1, t_2 in (T_1, T_2).

Directed line. A directed line is a straight line which has a fixed positive sense. Of a directed line two sides are distinguished, namely the right and the left sides.

Let there be two directed lines L_0 and L. Assuming the positive sense for measuring the angle counterclockwise, the angle between the two lines may be determined up to a multiple of 2π. This ambiguity is removed when multiples of 2π are left out of account; and then the measure is expressed by a value between 0 and 2π. The angle between L_0 and L lies in $(0, \pi)$ when L is directed leftwards; otherwise, the angle lies in $(\pi, 2\pi)$.

The angle between the positive x-axis and a directed line L is called the *direction angle* of the line L.

The conception of senses, sides and direction angles is also made use of in connection with regular curves and line-elements.

To apply these to the line-elements defined by a differential equation of the form (2), take an ordinary point $M_0(x_0, y_0)$, and for the direction angle at that point assign a fixed one, say α_0, of the angles which satisfy

$$\cos\alpha = \frac{\sigma\, Q(x_0, y_0)}{\sqrt{\{P(x_0, y_0)\}^2 + \{Q(x_0, y_0)\}^2}}, \quad \sin\alpha = \frac{\sigma\, P(x_0, y_0)}{\sqrt{\{P(x_0, y_0)\}^2 + \{Q(x_0, y_0)\}^2}}$$

in which for σ one of the values 1 or -1 is to be given at will. To define the direction angle at an arbitrary ordinary point $M_1(x_1, y_1)$, assume a curve γ_1 connecting M_0 and M_1, which does not contain any singular point of the differential equation. Then the angle α defined by

$$(11) \qquad \cos\alpha = \frac{\sigma\, Q(x,y)}{\sqrt{P^2+Q^2}}, \quad \sin\alpha = \frac{\sigma\, P(x,y)}{\sqrt{P^2+Q^2}}$$

varies continuously with the point $M(x,y)$ on the curve γ_1, starting from the initial value α_0. The final angle α_1 at M_1 is defined as the *complete direction angle* at M_1.

If instead of γ_1 another curve γ_2 which connects the same points M_0, M_1 is taken, the corresponding complete direction angle at M_1 will be determined. But since $\cos\alpha_1 = \cos\alpha_2$, $\sin\alpha_1 = \sin\alpha_2$ the two complete direction angles α_1, α_2 differ at most by a multiple of 2π. Thus the complete angle α at a point $M(x,y)$ depends not only upon the coordinates x, y but also upon the curve which combines the point with the initial point M_0.

To make the direction angle a one-valued function of the coordinates we have only to leave any integral multiple of 2π out of consideration. This measure of the direction angle is called the *simple direction angle*.

Generally if two ordinary points N_1, N_2, of which the complete angle at N_1 is given, be connected by two curves γ', γ'', two corresponding direction angles at N_2 are given immediately. The difference of these angles is an integral multiple of 2π; and is the change of the complete direction angle when the line-element makes a complete revolution moving along the closed curve γ formed by the two curves γ', γ'' (Fig. 1). It is clear that the difference depends upon the form of the closed curve γ and the sense of revolution but not upon the starting point. Assuming the sense of revolution counter-clockwise, the change of the complete direction angle, which arises from a complete revolution along γ is called the *characteristic angle of the closed curve* γ. It is an integral multiple of 2π.

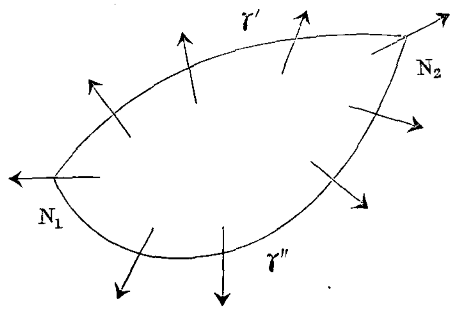


Fig. 1.

Suppose within and on a closed curve γ there is no singular point, then the complete direction angle α defined by (11) is one-valued and continuous within and on the closed curve γ. The characteristic angle of the cycle is therefore zero. Thus we have:

Theorem 1. If the characteristic angle of a closed curve is not zero, in the interior of the curve there exists at least one singular point.

As the characteristic angle of a cyclic characteristic is not zero, being 2π, it follows:

Cor. In the interior of a cyclic characteristic, there exists at least one singular point.

Let γ_1, γ_2 be any two closed curves which contain only one singular point S in the interior, then it is easily shewn that the characteristic angles of γ_1 and γ_2 are equal. Hence the characteristic angle of such a closed curve is proper to the point S, and is called the *characteristic angle of the singular point* S.

Without difficulty we may further conclude that the characteristic angle of any closed curve is the sum of the characteristic angles belonging to the singular points within the closed curve.

4. Consecutive and Contact Points.[1]

Let two regular curves γ_1 and γ_2 intersect at only two points M_1, M_2, then the two segments of the curves intercepted between the points, will form a closed curve which divides the entire plane into two parts, interior and exterior domains.

The form of the intersection of the curves is of two types: Type I, in which all the four extensions of the segments of γ_1, γ_2 at the ends M_1, M_2 lie in one and the same domain, exterior (Fig. 2) or interior (Fig. 3), Type II in which the two extensions of the segments at M_1 lie in one domain while those at the second endpoint M_2 lie in the other domain (Fig. 4).

In a plane where directed line-elements are defined by (11), consider a regular directed curve

$$\gamma: \quad x = \Phi(t),\ y = \Psi(t) \qquad T_1 \leqq t \leqq T_2$$

which passes none of the singular points of the differential equation $\dfrac{dy}{dx} = \dfrac{P(x,y)}{Q(x,y)}$.

1 J. Liouville, p. 409 (1881),

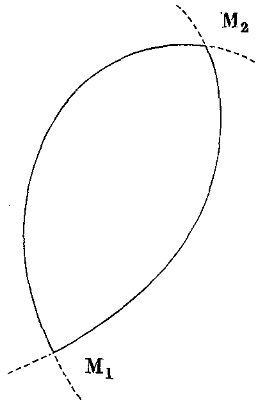


Fig. 2.

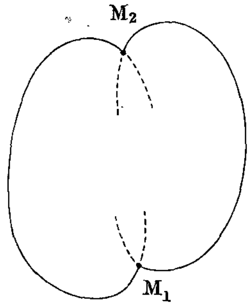


Fig. 3.

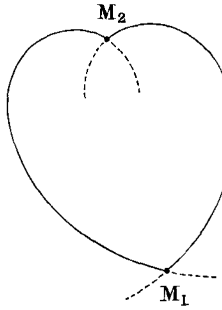


Fig. 4.

Take a point $M(t)$ on the curve γ and let $C(t)$ be the progressive semi-characteristic through the point M. This may be expressed by

$$C(t): \quad x = X(s, t), \; y = Y(s, t) \qquad\qquad s \geqq 0,$$

where s is the arc length measured from M on the positive direction and the functions X, Y satisfy the relations

$$(X_s')^2 + (Y_s')^2 = 1,$$
$$X(0, t) = \Phi(t), \quad Y(0, t) = \Psi(t).$$

When γ and $C(t)$ touch each other at M, the curve γ is said to have a *contact* at the point M. In particular the point M is called a contact point of the first kind when γ is cut by $C(t)$ at M, and of the second kind when, in the neighbourhood of M, $C(t)$ lies on one side of γ.

When these two curves have other common points besides M, if in a neighbourhood of M, γ and C(t) have no other common point, on the curve $C(t)$ there will be the first point N which is common to γ. The point N is called the *consecutive point* of M with regard to the curve γ. When, instead of the progressive semi-characteristic, the regressive semi-characteristic $C_2(t)$ through M is considered, the first point on $C_2(t)$ which is common with γ is called the *preceeding point* of M.

Suppose that a point M on the curve γ has the consecutive point N and the two curves γ and $C(t)$ do not touch at M and N, then the two segments of the curves will have an intersection of the type I or II. Accordingly the point N is called the consecutive point of the *first* or *second kind*.

We can now prove:

Theorem 2. If a point M on a regular curve γ which does not contain any singular point of a differential equation $\frac{dy}{dx}=f(x,y)$ have the consecutive point N of the first kind, then on the segment of γ between M, N there exists at least one contact point.

Proof. Let ds be the directed line-element defined by

$$\cos\alpha=\frac{Q}{\sqrt{P^2+Q^2}},\quad \sin\alpha=\frac{P}{\sqrt{P^2+Q^2}}.$$

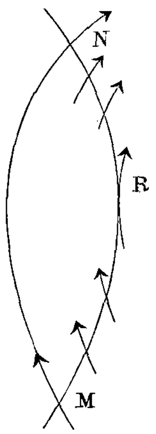


Fig. 5.

To make γ a directed curve assign a positive sense. Suppose at M ds is directed leftwards of γ, then as N is the consecutive point of the first kind, at that point ds is directed rightwards of γ. Denote by θ the simple measure of the angle between the curve γ and the line-element ds on γ, then

$$0<\theta_M<\pi, \text{ at } M \text{ and } \pi<\theta_N<2\pi \text{ at } N.$$

As the angle θ varies continuously from θ_M to θ_N when ds moves along γ, it must take the value 0 or π at a certain point R of the segment γ_{MN}. R is a point of contact of the curve γ.

5. Applications of the Theory of Contact Points.

There are numerous applications of the theory of contact points to the differential equation (2).

Theorem 3. Any algebraic curve which is not a characteristic of a differential equation (2) can not have indefinitely many contact points.

Proof. Let γ_A be the curve defined by an irreducible algebraic equation $A(x,y)=0$. The contact points of this curve are then given by the simultaneous equations

$$A=0,\quad Q\frac{\partial A}{\partial x}+P\frac{\partial A}{\partial y}=0.$$

As these two equations are algebraic there is a finite number of solutions. Specially if $A(x,y)$ is of the m^{th} degree and $f(x,y)$ is of the n^{th} degree, the curve γ_A may have at most $m\{(m-1)+n\}$ contact points.

Theorem 4. If a semi-characteristic of (2) has indefinitely many *common points with an algebraic curve which is not a characteristic, it is a spiral.*

Proof. Let C be a semi-characteristic which has indefinitely many points in common with an algebraic curve γ_A. As γ_A may have a finite number of branch points and singular points of the differential equation, we may determine such a segment γ_{NK}, that has 1) indefinitely many points, say $\{M\}$, belonging to C, of which N is a limiting point ; 2) no branch point, no singular point and no contact point in the interior.

Take a point M_1 of $\{M\}$, then on the arc γ_{M_1N} there is necessarily a consecutive or preceeding point of M_1. To fix the idea, suppose that there is a consecutive point of M_1, call it M_2, then since γ_{M_1N} contains no contact point, by theorem 2, M_2 is a consecutive point of the second kind. We know also that the arc $\gamma_{M_1M_2}$ besides M_1, M_2 contains no other point belonging to $\{M\}$. It is also clear that the point M_2 has a consecutive point of the second kind with regard to the curve γ_{NK}, call it M_3. M_3 lies in γ_{M_2N}. Similarly M_3 has a consecutive point M_4 on the arc γ_{M_3N}; and so on.

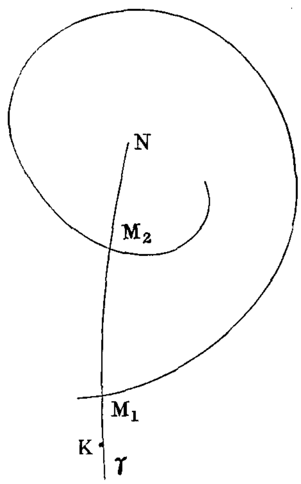


Fig. 6.

Thus proceeding we obtain an infinite sequence of points $M_1, M_2, \ldots M_i, M_{i+1}, \ldots$ converging to N, of which each is the consecutive point of the second kind of the just preceeding. Thus it is shewn that the characteristic C is really a spiral.

As each segment $C_{M_i M_{i+1}}$ of the characteristic C lies on the same sides of the preceeding segments $C_{M_j M_{j+1}}$ ($j=1, 2, \ldots i-1$), the arc $C_{M_i M_{i+1}}$ will converge to the point N or to a curve C_L through N.

In the first case the point N is a singular point, for each cycle λ_i which consists of the segments $C_{M_i M_{i+1}}$ and $\gamma_{M_i M_{i+1}}$ contains N and the characteristic angle of λ_i is 2π (§ 3).

In the second case the limiting curve C_L is a closed one. For, as the consecutive point of M_i is M_{i+1} and $\lim\limits_{i\to\infty} M_i = \lim\limits_{i\to\infty} M_{i+1} = N$, the curve C_L, which is the limiting position of $C_{M_i M_{i+1}}$, starting from the point N must return to that point. In the neighbourhood of a

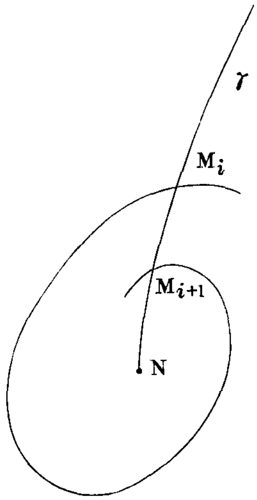


Fig. 7.

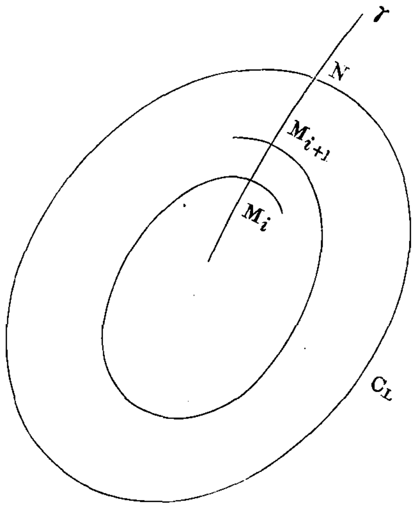


Fig. 8.

segment of C_L, which does not contain any singular point, the curves $CM_i\, M_{i+1}$ converge uniformly and therefore the line-element of C_L at any ordinary point satisfies the differential equation. This shews that C_L is an integral curve. We shall call the curve C_L, following Prof. Poincaré, a limiting cycle. When C_L does not pass through any singular point, it is a cyclic characteristic, but when it passes through singular points it may consist of a finite number of loops and a finite number of characteristics connecting two of the singular points as shewn in the figure.[1]

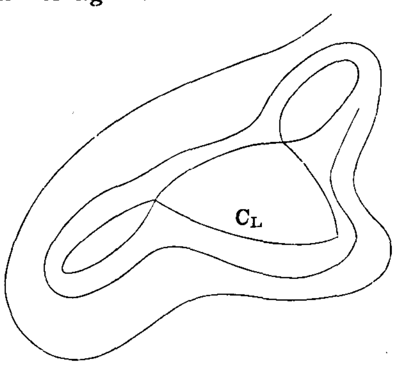


Fig. 9.

We have thus established:

Theorem 5. Any semi-characteristic of spiral form converges to a limiting cycle or to a singular point.

Theorem 6. Any semi-characteristic which has an infinite number of extreme values of x or y, or indefinitely many inflexional points is a spiral.

Proof. That a semi-characteristic C has an infinite number of extreme values of y means that the two curves C and $P(x, y)=0$ have

[1] See p. 180.

indefinitely many points in common. Hence, by theorem 4, C is a spiral. The other cases are proved similarly.

We can now determine the form of the semi-characteristic (iv) introduced in § 1.

Theorem 7. A semi-characteristic which does not go to an infinite distance or to any singular point is a spiral converging to a limiting cycle.

Proof. Had the semi-characteristic a limited arc-length, it must converge to a limiting point which is necessarily a singular point. As this case is rejected the curve must have an infinite arc-length. Hence it follows that the characteristic has an infinite number of extreme values of x or y; and thus it must be a spiral converging to a limiting cycle.

6. Possible Directions at Singular Points.

The first question which arises in the discussion of characteristics near a singular point S is whether there will be a characteristic which passes through the point S and has a definite direction at that point.

As the first step we investigate the directions along which there may exist an integral curve. Such directions are called *possible directions* at the singular point S.

For the purpose transform the origin to the singular point, then we have

$$\frac{dy}{dx} = \frac{P_1(x, y)}{Q_1(x, y)} \tag{12}$$

where P_1 and Q_1 are polynomials which satisfy

$$P_1(0, 0) = 0, \quad Q_1(0, 0) = 0.$$

Or, arranging the terms of P_1, Q_1 according to the degree of x, y, it is written in the form

$$\frac{dy}{dx} = \frac{H_m(x, y) + H_{m+1}(x, y) + \ldots}{K_n(x, y) + K_{n+1}(x, y) + \ldots} \tag{13}$$

where $H_\lambda(x, y)$ and $K_\lambda(x, y)$ are homogeneous integral expressions of λ^{th} degree of x and y.

Now suppose that there exists a characteristic which passes through the origin and has a determinate slope a at that point, then we have

$$\lim_{x \to 0} \frac{y}{x} = \left(\frac{dy}{dx}\right)_{x=0} = \lim_{x \to 0} \frac{P_1(x, y)}{Q_1(x, y)} = a. \tag{14}$$

Using this relation we may find the possible directions.

Case $m > n$. Write (13) in the form

$$\frac{dy}{dx} = \frac{x^{m-n}\{H_m(1, u)+xH_{m+1}(1, u)+\ldots\}}{K_n(1, u)+xK_{n+1}(1, u)+\ldots}$$

where $u = \frac{y}{x}$; then, by making x converge to zero, by (14) we have

$$a\, K_n(1, a) = 0$$

if a is finite.

Next, suppose that there exists a characteristic which has the slope ∞ at the origin. Then, writing (13) in the form

$$\frac{dx}{dy} = \frac{K_n(v, 1)+y\, K_{n+1}(v, 1)+\ldots}{y^{m-n}\{H_m(v, 1)+y\, H_{m+1}(v, 1)+\ldots\}}, \quad v = \frac{x}{y},$$

and making y converge to zero, we obtain

$$K_n(0, 1) = 0.$$

Thus, when $m > n$, the possible slopes are

1) $a=0$, 2) roots of $K_n(1, a)=0$ and 3) $a=\infty$ when $K_n(0, 1)=0$.

Case $m = n$. Again put $\frac{y}{x} = u$, then

$$\frac{dy}{dx} = \frac{H_m(1, u)+xH_{m+1}(1, u)+\ldots}{K_m(1, u)+xK_{m+1}(1, u)+\ldots}$$

whence, by making x converge to zero, we obtain

$$a\, K_m(1, a) = H_m(1, a)$$

if a is finite.

If the differential equation has any characteristic which touches the y-axis at origin, as in case $m > n$, we have

$$K_n(0, 1) = 0.$$

When $m=n$, the possible slopes at origin are 1) roots of $a\, K_m(1, a)-H_m(1, a)=0$, 2) $a=\infty$ when $K_m(0, 1)=0$.

Finally, the case $m<n$ may be reduced to the case $m>n$ considering the equation $\frac{dx}{dy} = \frac{Q_1}{P_1}$.

In short the possible directions of the equation (13) are the roots of

$$\sin\theta\, K_s(\cos\theta, \sin\theta)-\cos\theta\, H_s(\cos\theta, \sin\theta) = 0, \tag{15}$$

where s is the lowest degree of the polynomials P_1 and Q_1.

7. Criterions for Existence of Characteristics which have a Definite Possible Direction at a Singular Point.

We have shewn that a characteristic which has a determinate direction at a singular point S may exist only in a possible direction. We inquire conversely whether there will really exist a characteristic in a given possible direction.

To solve this question let ASB be an angle which contains a possible direction α and does not contain any other possible direction. [The case where all directions are possible is treated later].

Consider a sector damain D, bounded by the sides SA, SB of the angle and a circle described with the point S as center and radius R, and suppose that R is chosen so small that in D no singular point is contained and on the side SA no contact point lies.

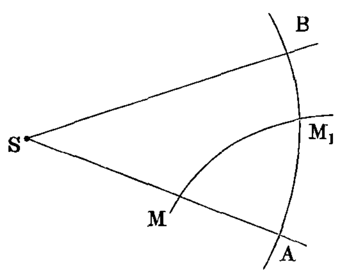


Fig. 10.

Let M be a point on SA and let $C(M)$ be the semi-characteristic through M directed leftwards of SA. Then $C(M)$ has another intersection with the boundary of D. For if $C(M)$ lay entirely in the interior of D, it would be a spiral converging to a limiting cycle (theorem 7). Consequently, by theorem 1, in the interior of D there must exist a singular point. But this is contrary to the supposition. Thus there exists a point M_1 at which $C(M)$ cuts the boundary of D for the first time. We shall use again the term consecutive point for M_1.

As the side SA contains no contact point, the consecutive point M_1 lies on the broken line SBA.

1° Suppose on SA there exists a point M' whose consecutive point is the singular point S. Then $C(M')$ is surely a characteristic which has the slope α at S. For, since $C(M')$ can not make indefinitely many oscillations (theorem 6), it must have a definite direction at S; and in D, as there is only one possible direction α, $C(M')$ must have the slope α at that point. In this case any characteristic $C(N)$ of which N is a point on the segment SM' has the slope α at S (Fig. 11).

2° Suppose, for a sufficiently small R, all the consecutive points of $M's$ lie on the segment AB, then evidently $\lim\limits_{M \to S} C(M)$ is a characteristic in question (Fig. 12).

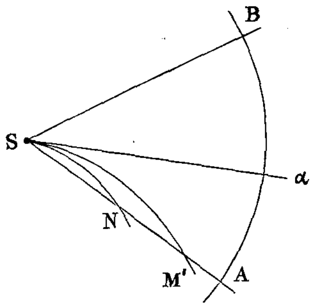


Fig. 11.

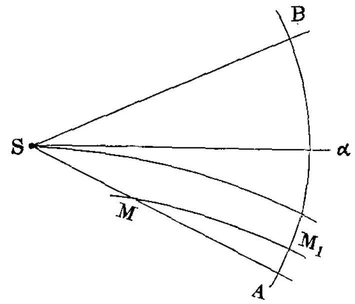


Fig. 12.

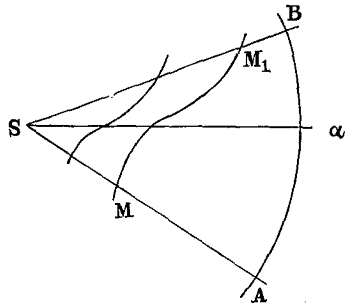


Fig. 13.

3° Suppose, however small R may be taken, on SA there is a point M whose consecutive point M_1 lies on the side SB, then in D there is no characteristic in question (Fig. 13).

Here we remark that the segment of the circle AB and the sides SA, SB may be replaced by suitable curves.

The above criterion may be expressed in a more convenient form as follow:

Criterion 1. Let α be a possible direction at the singular point $S(0, 0)$ of the differential equation (12). Let $D[(\alpha-\delta)x \leqq y \leqq (\alpha+\delta)x,$ $0 < x \leqq X]$ be a domain where $f(x,y)$ is continuous and in which no other possible direction is contained. For simplicity let the expression $\dfrac{d\,\Phi(x)}{dx} - f(x, \Phi(x))$ be denoted by $\chi(x, \Phi(x))$.

Then a) if $\chi(x, (\alpha-\delta)x) < 0$, $\chi(x, (\alpha+\delta)x) > 0$ for all x, $0 < x \leqq X_1$ $(\leqq X)$, there exists at least one characteristic that has the slope α at S; b) if $\chi(x, (\alpha-\delta)x) > 0$, $\chi(x, (\alpha+\delta)x) < 0$ for all x, $0 < x \leqq X_1$, all the characteristics in $[(\alpha-\delta)x \leqq y \leqq (\alpha+\delta)\,x,\ 0 < x \leqq X_1]$ pass through S and have the slope α at that point; c) let $\chi(x, (\alpha-\delta)x)$, $\chi(x, (\alpha+\delta)x)$ have the same sign for all x, $0 < x \leqq X_1$, say positive. If there exists a system of curves

$$\gamma(c): \quad y = \Phi(x, c), \ 0 < c \leqq c_0$$

ending at the lines $y = (\alpha-\delta)\,x$, $y = (\alpha+\delta)\,x$, such that the curve converges to S when c converges to zero, and $\chi(x, \Phi\,(x, c)) > 0$ $(0 < c \leqq c_1)$, in D there is no characteristic through S.

Proof. Consider the domain $D_1[(a-\delta)\,x \leqq y \leqq (a+\delta)x,\ 0 \leqq x \leqq X_1]$ and let $\varphi(x\,;\xi,(a-\delta)\,\xi)$ be the upper semi-characteristic through the point $\xi,(a-\delta)\,\xi\ (0<\xi<X_1)$. By the supposition a), it is clear that the consecutive point of $x=\xi, y=(a-\delta)\,\xi$, with regard to the boundary of D_1, lies on the side $x=X_1, (a-\delta)\,X_1 \leqq y \leqq (a+\delta)\,X_1$; hence the characteristic $\lim\limits_{\xi\to 0} \varphi(x\,;\xi,(a-\delta)\,\xi)$ is one in question.

The other two propositions are proved similarly.

Criterion 2. Let $y_1=\psi_1(x), y_2=\psi_2(x)$ be any two different curves through the same singular point $S(x_0,y_0)$ of $\dfrac{dy}{dx}=f(x,y)$, and suppose $\psi_1(x)<\psi_2(x),\ x_0 \leqq x \leqq X$. Then, if in $D[\psi_1(x)\leqq y \leqq \psi_2(x),\ x_0<x\leqq X]$ $\dfrac{\partial f}{\partial y}$ is negative, in D there can not exist two characteristics which pass through S.

Proof. Suppose, if possible, there were two such characteristics $y=\varphi_1(x)$ and $y=\varphi_2(x)$ and let

$$\varphi_1<\varphi_2,\quad x_0<x\leqq X_1\,(\leqq X).$$

Then $$\varphi_2'-\varphi_1'=f(x,\varphi_2)-f(x,\varphi_1)=(\varphi_2-\varphi_1)f_y'(x,\eta),$$

where η is a certain value between φ_1 and φ_2.
Therefore $\varphi_2'-\varphi_1'<0$ when $x_0<x\leqq X_1$. and consequently by the condition $\varphi_1(0)=\varphi_2(0)=y_0$ we should have

$$\varphi_2(x)<\varphi_1(x),\quad x_0<x\leqq X_1$$

which is contradictory.

Criterion 3. Let $y_1=\psi_1(x),\ y_2=\psi_2(x)$ be any two curves through the same singular point $S(x_0,y_0)$ of $\dfrac{dy}{dx}=f(x,y)$ and let $\psi_1(x)<\psi_2(x)$, $x_0<x\leqq X$. Then, if in $D\,[\psi_1(x)\leqq y\leqq \psi_2(x),\ x_0<x\leqq X]\,f(x,y)$ is continuous and $\dfrac{\partial F}{\partial y}$ is negative, where $F(x,y)$ is the concavity function of the characteristics (§ 2), there can not exist two different characteristics which pass through S and have the same slope at S.

Proof. Were there any two such characteristics, let them be called $y=\varphi_1(x),\ y=\varphi_2(x)$ and suppose

$$\varphi_1(x)<\varphi_2(x),\quad x_0<x\leqq X_1\,(\leqq X).$$

Then $\varphi_2''-\varphi_1''=F(x,\varphi_2)-F(x,\varphi_1)=(\varphi_2-\varphi_1)\,F_y'(x,\eta),\ \varphi_1<\eta<\varphi_2$.
Therefore $\varphi_2''(x)<\varphi_1''(x),\ x_0<x\leqq X_1$; hence by

$\varphi_2'(x_0)=\varphi_1'(x_0)$, we should have

$\varphi_2'(x)<\varphi_1'(x),\ x_0<x\leqq X_1$, and consequently, since $\varphi_1(x_0)=\varphi_2(x_0)$

$=y_0$, the relation $\varphi_2(x) < \varphi_1(x)$, $x_0 < x \leqq X_1$ which is contradictory.

Criterion 4. Let $S(x_0, y_0)$ be a singular point of $\frac{dy}{dx}=f(x,y)$ and let $y=\varphi(x)$ be a characteristic which satisfies $\varphi(x_0)=y_0$, $\varphi'(x_0)=\alpha$. Let $y=\psi(x)$ be a curve of which $\psi(x_0)=y_0$, $\psi'(x_0)=\alpha$ and $\varphi(x) < \psi(x)$, $x_0 < x \leqq X$. Then, if in $D[\varphi \leqq y \leqq \psi,\ x_0 < x \leqq X]\, f(x,y)$ is continuous and $\chi(x, \psi(x)) \equiv \psi'(x)-f(x, \psi(x)) < 0$, all the characteristics $y=\varphi(x;\xi, \psi(\xi))$, $(x_0 < \xi \leqq X)$, pass through the singular point S and have the slope α at that point.

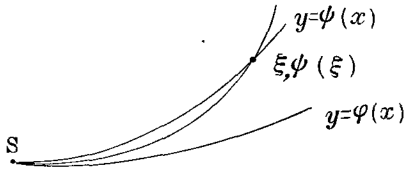


Fig. 14.

Proof. The left semi-characteristic $y=\varphi(x;\xi, \psi(\xi))(x \leqq \xi)$ through $\xi, \psi(\xi)$ lies below the curve and as it can not cut either of the two curves, $y=\varphi(x)$, $y=\psi(x)$ in $(x_0 < x \leqq \xi)$, it must go to the singular point S. From $\varphi(x) \leqq \varphi(x;\xi, \psi(\xi)) \leqq \psi(x)$, $x_0 \leqq x \leqq \xi$ and $\varphi'(x_0)=\alpha$, $\psi'(x_0)=\alpha$ it follows $\varphi'(0;\xi, \varphi(\xi))=\alpha$.

8. Discussion of Particular Possible Directions.

We shall now consider separately each case of possible directions introduced in § 6.

$$\text{I.}\quad m > n.\quad \frac{dy}{dx} = \frac{x^{m-n}\{H_m(1, u)+x H_{m+1}(1, u)+\ldots\}}{K_n(1, u)+x\, K_{n+1}(1, u)+\ldots},\quad u=\frac{y}{x}.$$

$1°$ $\alpha=0$. When $K_n(1,0) \neq 0$ the equation has at least one integral on each side of the origin, which satisfies the conditions $y(0)=0$, $y'(0)=0$.

Proof. By the conditions $m-n \geqq 1$ and $K_n(1,0) \neq 0$, we may determine positive numbers δ and X such that in $D[-\delta x \leqq y \leqq \delta x$, $0 < x \leqq X]\ f(x,y)$ is continuous and

$$\chi(x, \delta x) = \delta - f(x, \delta x) > 0,\ \chi(x, -\delta x) = -\delta - f(x, -\delta x) < 0$$

for all values of x, $0 < x \leqq X$. Hence there exists a characteristic in question. The existence of such integral on the left of the origin is proved similarly.

In the exceptional case $K_n(1,0)=0$, the equation may or may not have the characteristic in question as shewn in the example.

Ex. The equation $\frac{dy}{dx} = -\frac{x^3}{y}$ has no integral through the origin.

but $\frac{dy}{dx} = \frac{x^3}{y}$ has two integrals $y = \frac{x^2}{\sqrt{2}}$, $y = -\frac{x^2}{\sqrt{2}}$ subject to the conditions $y(0)=0$, $y'(0)=0$.

2° α is a root of $K_n(1, u)=0$. The differential equation may or may not have an integral subject to the conditions $y(0) = 0$, $y'(0) = \alpha$.

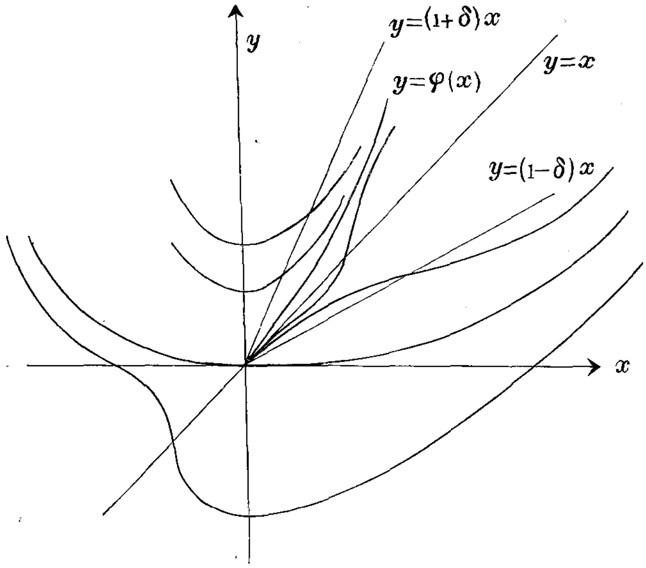


Fig. 15.

Ex. $$\frac{dy}{dx} = \frac{x^3}{(y-x)^2}, \quad \text{(Fig. 15)}.$$

Here $f(x,y) = \frac{x^3}{(y-x)^2}$ and it is zero when $x=0$, ∞ when $y=x$ and at any other point it is positive or negative according as x is positive or negative. The possible slopes at the origin are $\alpha=0$, $\alpha=1$, and we consider the latter.

Now $\chi(x, (1+\delta)x) = 1+\delta-\frac{x}{\delta^2}$, $\chi(x, (1-\delta)x)=1-\delta-\frac{x}{\delta^2}$, in which δ is a positive constant less than 1, are positive when x is sufficiently small. Also we have

$$\chi(x, x) = -\infty \text{ when } x > 0 \text{ and } \chi(x, x) = +\infty \text{ when } x < 0.$$

As $\chi(x, (1+\delta)x)$ and $\chi(x, x)$ have different signs, in $D[x \leqq y \leqq (1+\delta)x, 0 < x \leqq X]$ X being a sufficiently small positive value, by criterion 1 a) the equation has an integral $y=\varphi(x)$ which satisfies the conditions $\varphi(0)=0$, $\varphi'(0)=1$.

Since $\chi(x, (1-\delta)x) > 0$ when $0 < x \leqq X_1$, where X_1 is a positive value, all the characteristics in $[(1-\delta)x \leqq y \leqq \varphi(x), 0 < x \leqq X_1]$ pass through the origin having the slope $+1$ at that point.

When $x < 0$, $f(x,y)$ is negative and there is no characteristic which passes through the origin in the third quadrant.

3° $\qquad a = \infty, \; K_n(0, 1) = 0.$

Ex. $\dfrac{dy}{dx} = \dfrac{y^3}{x^2 - y^3}$. The upper semi-characteristics $\varphi(x; \xi, \xi^{\frac{2}{3}})$, $\xi < 0$, lie between the positive y-axis and the curve $y = x^{\frac{2}{3}}$, $x < 0$, hence $\lim_{\xi \to 0} \varphi(x; \xi, \xi^{\frac{2}{3}})$ is a characteristic which touches the positive y-axis at the origin. Below the x-axis there is no characteristic which touches the y-axis at the origin (Fig. 16).

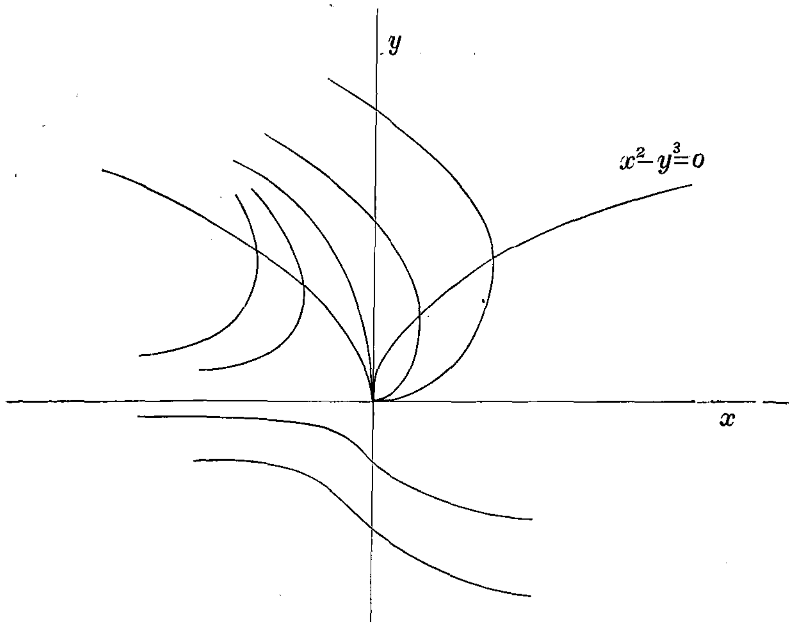


Fig. 16.

$$\text{II.} \quad m = n. \quad \frac{dy}{dx} = \frac{H_m(x,y) + H_{m+1}(x,y) + \dots}{K_m(x,y) + H_{m+1}(x,y) + \dots}.$$

The possible directions of this equation are 1) roots of

$$(15)_a \qquad \pi(u) \equiv uK_m(1, u) - H_m(1, u) = 0$$

and 2) $a = \infty$ if $K_m(0, 1) = 0$.

In case 2), if we interchange the variables x and y, it will be reduced to case 1) corresponding to the slope $\alpha=0$. Hence we have only to consider the existence of integrals of the form

$$y = x(\alpha+\varepsilon(x))$$

where α is a root of $(15)_\alpha$ and $\varepsilon(x)$ is a variable which vanishes with x.

1° α is a common root of $u\,K_m(1,u)=0$ and $H_m(1,u)=0$.

Ex. $\dfrac{dy}{dx} = \dfrac{y-x^2+2x^3}{y}$. $\alpha=0$ is a common root of $H_1(1,u)=u=0$ and $u\,K_1(1,u)=u^2=0$ and there is the integral $y=x^2$ of the required form.

$\dfrac{dy}{dx} = \dfrac{y^2+x^4}{y^2}$ has no integral which satisfies $y(0)=0$ and $y'(0)=0$, since $\dfrac{dy}{dx}$ is always greater than 1.

2° α is a root of $\pi(u)=0$ but not of $K_m(1,u)=0$. Put $y=(\alpha+y_1)x$, then

$$x\frac{dy_1}{dx}+\alpha+y_1 = \frac{H_m(1,\alpha+y_1)+x\,H_{m+1}(1,\alpha+y_1)+\dots}{K_m(1,\alpha+y_1)+x\,K_{m+1}(1,\alpha+y_1)+\dots}$$

or since $K_m(1,\alpha)\neq 0$, $\pi(\alpha) = \alpha\,K_m(1,\alpha) - H_m(1,\alpha) = 0$, it may be written in the form

$$(16) \qquad x\frac{dy_1}{dx} = \frac{ax+by_1+\dots}{1+Ax+By_1+\dots} \equiv R(x,y_1)$$

where $R(x,y_1)$ is a rational function. The differential equation of form (16) will be considered in subsequent article.

3° Case where $\pi(u)=0$ is an identity. In this case all directions are possible.

Put $y = xy_1$, then, remembering $y_1 K_m(1,y_1)-H_m(1,y_1) = 0$, the differential equation will be transformed into

$$\frac{dy_1}{dx} = \frac{L_m(y_1)+x\,L_{m+1}(y_1)+\dots}{K_m(1,y_1)+xK_{m+1}(1,y_1)+\dots} \equiv R(x,y_1)$$

where $L_i(y_1)$ are polynomials of y_1 and $R(x,y_1)$ is a rational function.

Let c be any value for which $K_m(1,c)\neq 0$, then R and $\dfrac{\partial R}{\partial y_1}$ are continuous functions of x and y_1 in the neighbourhood of $x=0, y_1=c$. Therefore, by Cauchy's theorem, there exists one integral of $\dfrac{dy_1}{dx} = R(x,y_1)$, which assumes the value c at $x=0$. The original equation has therefore the integral of the form

$$y = \{c+\varepsilon(x,c)\}x$$

where $\lim\limits_{x\to 0}\varepsilon=0$ and c is an arbitrary constant for which $K_m(1,c)\neq 0$.

To know if the given differential equation will have an integral which has the direction of the y-axis at origin, put $\frac{x}{y}=v$ and consider y as the independent variable. Then in the same way we may prove the existence of the integrals

$$x = \{c'+\varepsilon_1(y, c')\} y ,$$

where $\lim\limits_{y\to 0} \varepsilon_1=0$ and c' is an arbitrary constant for which $K_m(c', 1)\neq 0$. The condition $y'_{x=0}=\infty$ corresponds to $c'=0$.

Thus, when $\pi(u)=0$ is an identity, there exists one integral on each side of the origin, which has an arbitrarily given slope c at the origin, provided that c is not the slope of the lines $\frac{K_m(x,y)}{x}=0$.

In the exceptional directions there may or may not exist integrals

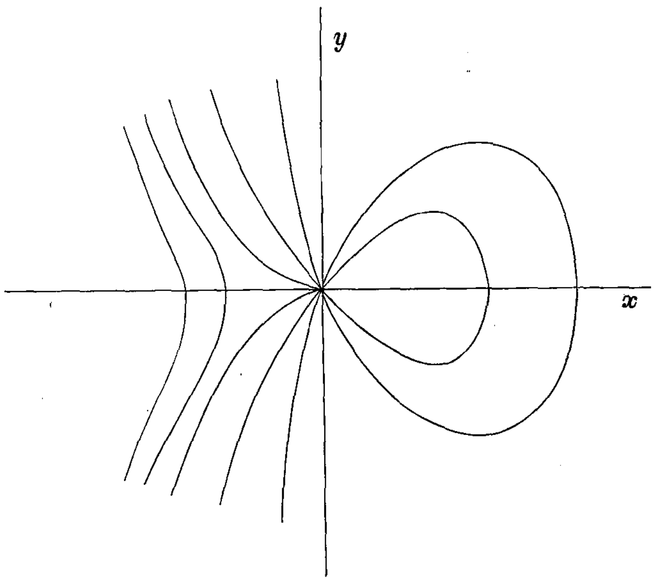


Fig. 17.

as illustrated by the example.

Ex. $$\frac{dy}{dx}=\frac{y^2-x^3}{xy} \quad \text{(Fig. 17)}.$$

The exceptional direction is $\theta=0$. On the right of the origin there is no integral which satisfies $y(0)=0$, $y'(0)=0$; but on the left there are two such integrals $y=\sqrt{2}\,(-x)^{\frac{3}{2}}$ and $x=-\sqrt{2}(-x)^{\frac{3}{2}}$. The general integral of the equation is $y^2=cx^2-2x^3$.

9. Form of Characteristics in the Neighbourhood of a Singular Point.

To know the nature of characteristics near a singular point S, consider a closed curve γ which contains only the singular point S in the interior. Such a curve is called a *cycle about* S. The simplest cycle about S is a circle of the center S, which contains no other singular point. We shall call it a circle about S. We here consider the form of characteristics which lie in a cycle K about S.

With respct to K the characteristics are divided into five types:

i) Both sides may end at K (characteristic of *parabolic* form);

ii) One side may end at K and the other side at the singular point S having a definite direction at that point (characteristic of *ray* form);

iii) Both sides may end at S (characteristic of *loop* form);

iv) The characteristic may be a spiral;

v) It may be a cycle about S.

Of course these types are not essential of the characteristics but depend upon the cycle of reference, e.g. a characteristic of ray form with regard to one cycle, may be a loop with regard to a larger cycle.

We consider first a singular point where no characteristic passes, which has a definite direction at that point.

Let l be a straight line through S, then as this line is not a characteristic we may determine a segment SM which contains no contact point. Let K be a cycle about S which passes through M and has no other intersection with the segment SM.

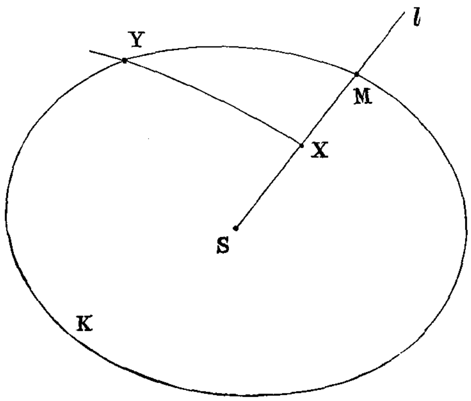


Fig. 18

Let $C(X)$ be the left semi-charateristic through a point X on l_{SM} and consider the curve K_1 which consists of l_{SM} and the cycle K. With respect to K_1, the point X has the consecutive point Y which is different from S. Further we know that the consecutive points $Y's$ corresponding to $X's$ $(S < X \leqq M)$ can not all lie on K. For were this the case, denoting $\lim\limits_{X \to S} Y$ by Y_S, the characteristic $C(Y_S)$ would be a ray contrary to supposition. Hence on l_{SM} there must be a point, call it X_1, whose consecutive point Y_1 lies on l_{SM}.

When $X_1 = Y_1$ the characteristic $C(X_1)$ is a cycle about S, and all the characteristics within it must be cycles or spirals.

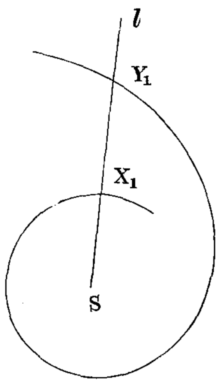


Fig. 19.

When X_1 is different from Y_1, Y_1 is a consecutive point of the second kind, and the segment $C(X_1 Y_1)$ of the characteristic $C(X_1)$, intercepted between X_1, Y_1 with the segment $l_{X_1 Y_1}$, will form a cycle K_2 about S. Within K_2 there is no characteristic of ray or loop or parabolic form and all the characteristics must be cycles or spirals.

Theorem 8. If through a singular point S there is not any characteristic which has a definite direction at S, then a neighbourhood of S can be constructed such that all the characteristics within it are spirals or cycles.

Suppose the consecutive point Y_1 of X_1 lies between S and X_1. Then $C(X_1)$ is a spiral and it will converge to the point S or to a limiting cycle about S. In the first case all the characteristics in the cycle K_2 are spirals converging to the singular point S. In the second case the limiting cycle C_L and the cycle K_2 will form a ring domain R; and all the characteristics in R will be spirals converging to the same limiting cycle C_L.

If all the characteristics in a neighbourhood of a singular point S are cycles, the point S is called a *center*. On the contrary, if in any cycle about S there is a characteristic of spiral form, the point S is called a *focus*.

Using the terms we may say: if through a singular point S no characteristic which has a determinate direction at that point passes, the point S is a focus or a center.

Remark 1. If at a singular point S no possible direction exists, S is a focus or a center.

Remarks 2. The characteristic angle belonging to a focus or to a center is 2π.

Remark 3.[1] Prof. Poincaré has shewn that if in a cycle K about S there exists an infinite number of cyclic characteristics which do not have S as a limiting point, then all the characteristics in the neighbourhood of S are cycles and the point S is a center.

1 J. Liouville pp. 256–257 (1882).

10. Form of Characteristics in the Neighbourhood of a Singular Point which is neither a Focus nor a Center.

Let S be a singular point which is neither a focus nor a center, then through S at least one characteristic passes which has a definite direction at S. Consider a circle K about S which cuts the characteristic. Then, since the circle K is an algebraic curve and is not a characteristic, it contains a finite number of contact points. Of these contact points if there are some at which the semi-characteristics have parabolic form these semi-characteristics will divide the domain bounded by K into a finite number of parts.

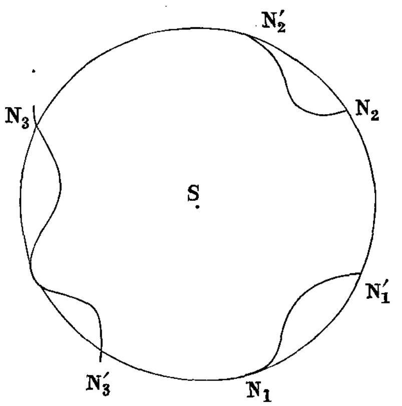


Fig. 20.

One of the subdomains contains S in the interior, call it D_1. It will generally be bounded by a finite number of segments of the circle K, $KN_1'N_2$, $KN_2'N_3$, ... $KN_n'N_1$ and a finite number of characteristics $C(N_1N_1')$, $C(N_2N_2')$ ... $C(N_n N_n')$ as shewn in Fig. 20. The boundary B_1 of D_1 may also contain a finite number of contact points M_1, M_2, ... M_m not belonging to the segments $\{(NN')\}$, and we shall obtain an aggregate $\{M, (NN')\}$ of contact points and segments of characteristics.

Let $C_1(M)$, $C_2(M)$ be the two semi-characteristics at a contact point M and let $C_1(N)$, $C_2(N')$ be the semi-characteristics at N and N', which are the extensions of the characteristic $C(N, N')$. The aggregate $\{M, (NN')\}$ is divided into the following three classes:

a) Those points or segments at which both the semi-characteristics C_1 and C_2 lie out of D_1 (Fig. 21);

b) Those at which both lie in D_1 (Fig. 22);

c) Those at which one lies out of D_1 and the other lies in D_1 (Fig. 23).

It is to be noticed that when the semi-characteristic at a point M or N enters into the interior of D_1, it must be a ray, as it can not be parabolic.

Let $C(N, N')$ be a segment belonging to a) and let L be the ad-

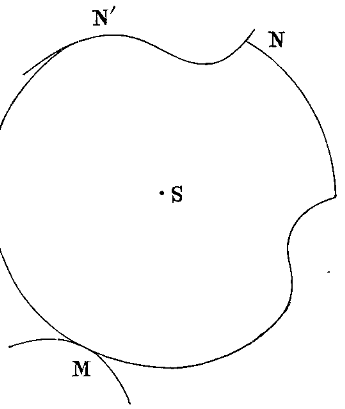


Fig. 21.

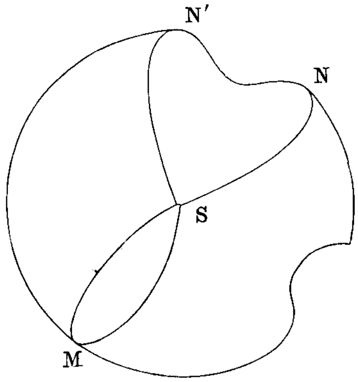


Fig. 22.

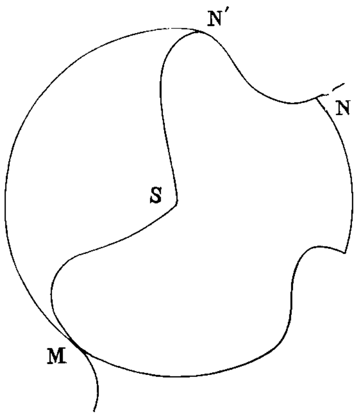


Fig. 23.

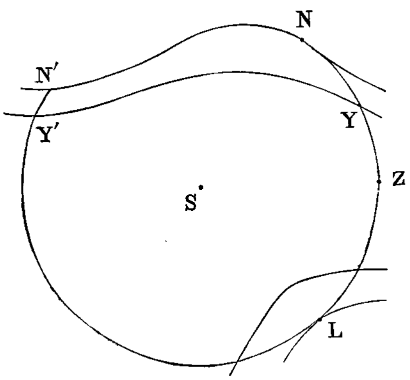


Fig. 24.

jascent point of N, which is different from N' and belongs to the aggregate $\{M, N, N'\}$. L is a contact point or an endpoint of another characteristic $C(\bar{N}, \bar{N}')$, and in $K(L, N)$ there is no contact point (Fig 24).

Let Y be a point of $K(L, N)$, then, remembering that the characteristic through a point P moves continuously with the point P, we see that when Y is sufficiently near to N the characteristic $C(Y)$ is parabolic and separates the characteristic $C(N, N')$ from the singular point S. The point Y has therefore its consecutive point Y_1 with respect to the boundary B_1.

Now divide all the points on $K(L, N)$ into two groups A, B by the following rule: Assume the order LNN' and let Z be any point on $K(L, N)$. 1° If all the characteristics $C(Y)$, $Z < Y < N$, be parabolic and separate $C(N, N')$ from S, we put the point Z into group B. 2° If the point Z does not satisfy the condition 1°, we put it into group A.

Then the partition A, B will determine a point E, which may coincide with L when the characteristic $C(L)$ is a ray.

Let $\lim_{Y\to E+0} C(Y, Y')=C(E, E')$, then $C(E, E')$ is a characteristic which passes through the point S. For, were $C(E, E')$ a characteristic of parabolic form, suppose it lay on the same side of S with $C(N, N')$, in $K(L, E)$ there would be a point Z through which a characteristic passes that has the same property as $C(E, E')$ and the point $Z(<E)$ would belong to group B. This is impossible. Similarly $C(E, E')$ can not be a parabolic characteristic which lies on the opposite side of S with $C(N, N')$. Thus the curve $C(E, E')$ must pass through S, and it will generally consist of two rays and a number of loops.

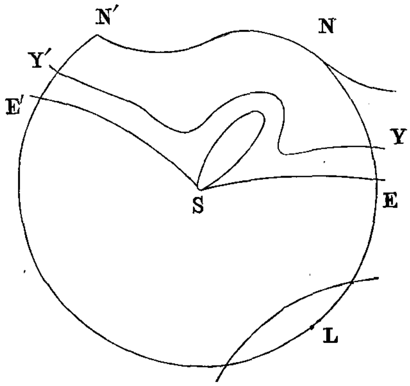


Fig. 25.

It is generally shewn that the number of loops which have the common vertex S and of which any lies out of the others must be finite. For, supposing there be l such loops, draw a circle about S, which cuts all the loops, then, in each segment of the circle intercepted by one loop there exists at least one contact point and in the circle there are at least l contact points. Now, by theorem IV, when the degree of $f(x,y)=\dfrac{P}{Q}$ is m, any circle can not have more than $2(m+1)$ contact points. This shews that through the point S there can not be more than $2(m+1)$ loops of the said property.

Thus, when $C(NN')$ be a segment of a) class, there exist two ray characteristics SE, SE' and finite loops which with the segment $ENN'E'$ (of B_1) form a domain where all the characteristics are parabolic (Fig. 25).

The same may be said of a contact point of class a) i.e. when M is a point of a) there exist two ray characteristics SF, SF' and finite loops such that in the domain, bounded by the curves and the segment FMF' of B_1, all characteristics are parabolic. We shall call these domains *parabolic domains*.

We consider now the elements of class b).

Let a point M or an arc $C(NN')$ be an element of class b). Then two ray characteristics $C_1(M)$, $C_2(M)$ or three characteristics $C_1(N)$, $C_2(N')$ and $C(NN')$ will form a loop, in the interior of which all characteristics are loops. The same may be said of any loop which forms the boundary of a parabolic domain. These domains are called *loop domains*.

From D_1 when all parabolic and loop domains are removed, there will remain a finite number of sector domains in which the characteristics are rays. We call these *ray-domains*.

Theorem 9. When S is a singular point which is neither a focus nor a center, there exists a neighbourhood D_1 of S and a finite number of characteristics of ray and loop forms, which divide D_1 into subdomains of the three kinds: parabolic, ray and loop.

A singular point is called a *node* when all the characteristics in a neighbourhood of the point are rays.

From theorems 8, 9 it follows:

Cor. If there exists a sequence of cycles about a singular point S, which converges to S and has no contact point, then the singular point S is a focus or a node.

We can now determine what form a limiting cycle may have.

Let C_L be a limiting cycle of a spiral C. If C_L does not pass through any singular point it is a regular closed curve.

When it passes through singular points it will consist of characteristics whose extremities are singular points. Let S_1 be a singular point on C_L and $C_1(S_1)$ be a branch of C_L through S_1. If $C_1(S_1)$ is not a loop it is a ray which goes to another singular point. That the number of characteristics belonging to C_L, which pass through the point S_1, is finite, may be proved as in theorem 9.

It will be further shewn that the number of branches of C_L at S_1 is even. If there were an odd number of branches of C_L through S_1, there must one, say $C_i(S_1)$, on both sides of which the spiral C would lie. Let M_1MM_2 be a

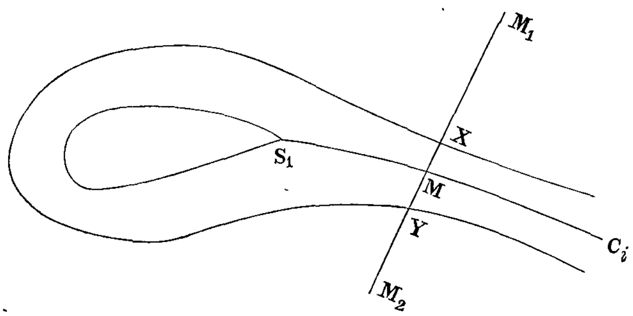


Fig. 26.

segment of a straight line which contains a point M on $C_i(S)$ and does not contain any contact point. Then on each segment MM_1, MM_2 there would be indefinitely many points on C and we might determine a point X on $M_1 M$, which would be common to C, such that the consecutive point Y with respect to the segment M_1MM_2 would be on MM_2 and be of the first kind. This would bring a contact point on M_1M_2, which is contradictory.

Hence we may conclude: any limiting cycle may consist of a finite number of loops and a finite number of ray characteristics which connect different singular points, and at each singular point meet an even number of branches (Fig. 9).

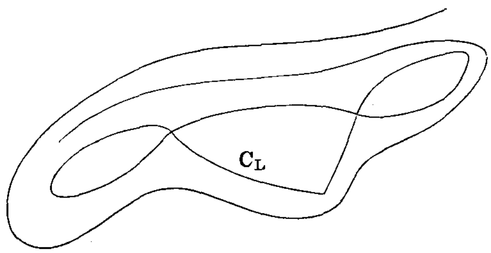


Fig. 9.

11. Form of Characteristics in an Infinite Distance.

If there be any characteristic $y=\varphi(x)$ of a differential equation

$$\frac{dy}{dx}=f(x,y)=\frac{P(x,y)}{Q(x,y)}$$

which has a definite direction at infinite distance, then the three limiting values

$$\lim_{x\to\infty}\frac{dy}{dx},\quad \lim_{x\to\infty}\frac{y}{x}\quad\text{and}\quad \lim_{x\to\infty}f(x,y)$$

will have the same value. Using these conditions we can determine the possible directions of characteristics at infinite distance.

For the purpose, arrange the terms of $P(x,y)$ and $Q(x,y)$ in descending powers of x and y, thus

$$\frac{dy}{dx}=\frac{H_m(x,y)+H_{m-1}(x,y)+\dots}{K_n(x,y)+K_{n-1}(x,y)+\dots},$$

where $H_i(x,y)$, $K_i(x,y)$ are homogeneous expressions of i^{th} degree. Treating this in a similar manner as we did a singular point, we find that the possible directions at infinite distance are:

i) $m > n$, $\cos\theta\, H_m(\cos\theta, \sin\theta) = 0$;
ii) $m = n$, $\sin\theta\, K_m(\cos\theta, \sin\theta) - \cos\theta\, H_m(\cos\theta, \sin\theta) = 0$;
iii) $m < n$, $\sin\theta\, K_n(\cos\theta, \sin\theta) = 0$.

To examine the existence of characteristics in a possible slope α we use the projective transformation

$$x = \frac{1}{X}, \quad y = \frac{Y}{X} \tag{17}$$

if $\alpha \neq \infty$ (i.e. $\theta \neq \frac{\pi}{2}$). Then

$$\frac{dY}{dX} = \frac{Y - f\left(\frac{1}{X}, \frac{Y}{X}\right)}{X} \tag{18}$$

$$= \frac{YX^{m-n}\{K_n(1,Y)+XK_{n-1}(1,Y)+\ldots\} - \{H_m(1,Y)+XH_{m-1}(1,Y)+\ldots\}}{X^{m-n+1}\{K_n(1,Y)+XK_{n-1}(1,Y)+\ldots\}}$$

and we have to consider the characteristics at $X = 0$, $Y = \alpha$ of the new equation. If (18) has a characteristic of the form

$$Y = \alpha + X(\beta + \varepsilon(X)), \quad |X| < \delta$$

where $\lim_{X\to 0} \varepsilon(X) = 0$, then the original equation has a characteristic of the form

$$y = \alpha x + \beta + \varepsilon\left(\frac{1}{x}\right), \quad |x| > \frac{1}{\delta}.$$

When $\alpha = \infty$ or α is a possible slope which is not zero, put

$$X = \frac{1}{y}, \quad Y = \frac{x}{y} \tag{19}$$

then the equation (2) is transformed into

$$\frac{dY}{dX} = \frac{f\left(\frac{Y}{X}, \frac{1}{X}\right)Y - 1}{Xf\left(\frac{Y}{X}, \frac{1}{X}\right)} \tag{20}$$

$$= \frac{Y\{H_m(Y,1)+XH_{m-1}(Y,1)+\ldots\} - X^{m-n}\{K_n(Y,1)+XK_{n-1}(Y,1)+\ldots\}}{X\{H_m(Y,1)+XH_{m-1}(Y,1)+\ldots\}}$$

and we have to consider the characteristics through the point $X=0$, $Y = \frac{1}{\alpha}$, when α is finite, or the new origin, when α is infinite.

By the transformation (17) or (19) the infinity line in the xy-plane is transformed into the Y-axis of the new plane; and by the equation

(18) or (20) we see that, with the exception of the case where $m=n$ and $\sin\theta\, K_m(\cos\theta, \sin\theta)-\cos\theta\, H_m(\cos\theta, \sin\theta)=0$ is an identity, the Y-axis ($X=0$) is a characteristic and any possible direction at infinite distance is transformed into a singular point which lies on the Y-axis of the new plane.

12. Singular Points of the First Order. Focus and Center.

Suppose a singular point S of a differential equation (2) is taken for the origin, then we have

$$\frac{dy}{dx}=\frac{ax+by+\ldots}{Ax+By+\ldots}. \tag{21}$$

If in this equation $D\equiv aB-Ab \neq 0$, the origin is called a *singular point of the first order*. In the following we shall consider the characteristics of (21) in the neighbourhood of the origin.

The equation for possible slopes of (21) is

$$Bt^2+(A-b)t-a=0 \tag{22}$$

and there are two real different, or real coincident, or no real possible slopes, according as the discriminant

$$E\equiv(A-b)^2+4aB$$

is positive, zero or negative. We consider these cases separately.

When E is negative, the origin is a focus or a center as there is no possible direction at that point (§ 9, Remark 1).

To find the form of characteristics near the origin more exactly, put

$$y=r\sin\theta, \quad x=r\cos\theta,$$

then (21) becomes

$$\frac{r'}{r}=\frac{A\cos^2\theta+(B+a)\cos\theta\sin\theta+b\sin^2\theta+r(\quad)}{a\cos^2\theta+(b-A)\cos\theta\sin\theta-B\sin^2\theta+r(\quad)}, \tag{23}$$

where the expresssions in the brackets are polynomials of r, $\cos\theta$ and $\sin\theta$, and, by the form of the equation, we know that, for sufficiently small r, $\dfrac{dr}{d\theta}$ is continuous of r and θ.

Neglecting higer infinitesimals the last equation becomes

$$\frac{R'}{R}=\frac{A\cos^2\theta+(B+a)\cos\theta\sin\theta+b\sin^2\theta}{a\cos^2\theta+(b-A)\cos\theta\sin\theta-B\sin^2\theta};$$

and, integrating this, we obtain

$$R = C\frac{e^{M\, arctg\, G}}{\sqrt{a\cos^2\theta+(b-A)\cos\theta\sin\theta-B\sin^2\theta}}$$

where $$M=\frac{b+A}{\sqrt{-E}},\quad G=\frac{-2B\tan\theta+b-A}{\sqrt{-E}}$$

and C is an arbitrary constant. The integrals are spirals if $M \neq 0$, since

$$R(\theta+2\pi)=e^{\sigma 2M\pi}R(\theta)$$

where σ is 1 or -1 according as B is negative or positive.

As the integral r differs from R by an infinitesimal, we may say that the characteristics of (23) in the neighbourhood of origin are spirals converging to the origin when $E<0$ and $A+b\neq 0$.

When $A+b=0$ the integrals R are cycles since $R(\theta+2\pi)=R(\theta)$. It can not be inferred that the same holds for the integrals r. We may so far only conclude that the origin may be a center when $E<0$ and $A+b=0$.[1]

13. Singular Points of the First Order which is neither a Focus nor a Center.

Though equation (21), of which $E=(A-b)^2+4aB$ is not negative may be treated directly, it is more convenient to consider the reduced equation.

Put

(24) $$x=\xi+\eta,\quad y=\alpha\xi+\beta\eta$$

then

(25) $$\frac{d\eta}{d\xi}=\frac{a_1\xi+b_1\eta+\dots}{A_1\xi+B_1\eta+\dots}$$

where

(26) $$\begin{cases} a_1=-\{B\alpha^2+(A-b)\alpha-a\}, & b_1=a+b\beta-A\alpha-B\alpha\beta, \\ A_1=-a+A\beta-b\alpha+B\alpha\beta, & B_1=B\beta^2+(A-b)\beta-a. \end{cases}$$

When $D>0$ the two roots of (22) are real and different, call them t_1, t_2, and in the equations of transformation put $\alpha=t_1$, $\beta=t_2$, then we have

$$a_1=0,\quad b_1=(t_2-t_1)(Bt_2+A),\quad A_1=(t_2-t_1)(Bt_1+A),\quad B_1=0$$

and (25) becomes

1 A rigorous criterion to distinguish center and focus was given by Prof. Poincaré in J. Liouville, pp. 173–196 (1885).

(27) $$\frac{d\eta}{d\xi}=\frac{\lambda\eta+\dots}{\xi+\dots}$$

where $\lambda=\dfrac{Bt_1+A}{Bt_2+A}$. The value of λ is the ratio of the two roots of the equation

$$\rho^2-(A+b)\,\rho+Ab-aB_3=0;$$

and it is positive or negative according as D is negative or positive.

When $D=0$ the two roots of (22) are equal, say t_1, and put t_1 for β in (23), then

$$a_1=-B(t_1-a)^2,\quad A_1=b_1=-Bt_1^2+t_1(b+Ba)-ba,\quad B_1=0$$

and (25) becomes

(28) $$\frac{d\eta}{d\xi}=\frac{\eta+\lambda\xi+\dots}{\xi+\dots}$$

where $$\lambda=\frac{B\,(t_1-a)^2}{B\,t_1^2-t_1(b-Ba)+ba},\quad t_1=\frac{A-b}{2B}$$

and a is a value which is not equal to t_1.

As the transformation (24) is linear homogeneous, to a cycle γ about the origin of xy-plane, a cycle γ_1 about the origin of $\xi\eta$-plane corresponds and to a characteristic C in the old, a characteristic C_1 in the new plane. Also it is evident that the type of C with regard to γ is the same as that of C_1 with regard to γ_1. Thus to find the form of characteristics of (21) we have to investigate the characteristics of the transformed equation. It is also to be noticed that in the transformed equation (27) the possible directions are $\tau_1=0$, $\tau_2=\infty$ and that these correspond to t_1, t_2 of the original equation.

1° $$\frac{d\eta}{d\xi}=\frac{\lambda\eta+\dots}{\xi+\dots}=f(\xi,\eta),\quad \lambda<0.$$

To apply criterion 1 § 7, take two lines $y=lx$ $y=-lx$ where l is a positive number. Since $\lim\limits_{x\to0} f(x,ux)=\lambda u$, we may determine a positive value X such that, in $D[-lx\leqq y\leqq+lx, 0<x\leqq X]$ $f(x,y)$ is continuous and

$$f(x,-lx)>-l,\quad f(x,lx)<l\quad (0<x\leqq X).$$

Hence in D there exists a characteristic $y=\varphi_0(x)$ which satisfies $\varphi_0(0)=0$, $\varphi_0'(0)=0$. Besides $y=\varphi_0(x)$, in D there is no characteristic which passes through the origin, as

$$\frac{\partial f}{\partial y}=\frac{\lambda x+\dots}{(x+\dots)^2}$$

is negative in $[-lx \leqq y \leqq +lx, 0 < x \leqq X_1]$, when X_1 is sufficiently small.

Similarly, it is shewn that in the direction $\theta=\pi$ there exists a single integral.

To find the existence of characteristics in the directions $\theta=\frac{\pi}{2}$, $\theta=\frac{3\pi}{2}$, write the equation in the form

$$\frac{d\xi}{d\eta}=\frac{\frac{1}{\lambda}\xi+\dots}{\eta+\dots},\quad \frac{1}{\lambda}<0,$$

and, considering η as the independent variable, we may conclude that there exists a single characteristic in each direction of $\theta=\frac{\pi}{2}$ and $\theta=\frac{3\pi}{2}$.

Thus the equation (27) has four ray characteristics in the directions $\theta=0$, $\theta=\frac{\pi}{2}$, $\theta=\pi$ and $\theta=\frac{3\pi}{2}$; and all the other characteristics near the origin are parabolic.

A singular point of this sort is called a *col*.[1]

Remark. At a *col* the characteristic angle is -2π.

$2°$
$$\frac{d\eta}{d\xi}=\frac{\lambda\eta+\dots}{\xi+\dots}\equiv f(\xi,\eta),\quad 0<\lambda.$$

We treat first the case $0<\lambda<1$. To find the existence of characteristics in the direction $\theta=0$, again apply criterion 1 § 7. We have $\lim_{x\to 0} f(x, ux)=\lambda u$ and there exists a positive number X such that in $[lx \leqq y \leqq +lx, 0 < x \leqq X]$, $f(x, y)$ is continuous and

$$-l-f(x,-lx)<0,\quad l-f(x,lx)>0\quad (0<x\leqq X)$$

where l is a positive number. This shews that in the direction $\theta=0$ there exists a characteristic. Now by calculation we find

$$\frac{\partial}{\partial\eta}\left(\frac{d^2\eta}{d\xi^2}\right)=\frac{\lambda(\lambda-1)\xi^2+\dots}{(\xi+\dots)^4}\quad (0<\lambda<1)$$

and, as this value is negative in $[-lx \leqq y \leqq +lx, 0 < x \leqq X_1]$ for

[1] As a proper definition is not found, following Prof. Poincaré, the french term 'col' is adopted.

sufficiently small positive X_1, in the direction $\theta=0$ there is not any other characteristic. Similarly, in the direction $\theta=\pi$, there exists only one characteristic.

Next consider a circle of radius ρ which has the origin as center. The contact points of the circle are given by

$$x(x+\ldots)+y(\lambda y+\ldots)=0, \quad x^2+y^2=\rho^2$$

and it is clear that when ρ is sufficiently small the simultaneous equations have no solution. The origin is therefore a node. As the differential equation has only the remaining possible directions $\theta=\frac{\pi}{2}$, $\theta=\frac{3\pi}{2}$, the characteristics in the neighbourhood of the origin must have these directions at the origin.

Case $\lambda=1$. In this case the equation for possible directions becomes an identity; and all directions at the origin are possible. Also there is no exceptional direction. The origin is therefore a node; and in each direction at the origin there is only one characteristic.

Case $\lambda>1$. Writing the equation in the form

$$\frac{d\xi}{d\eta}=\frac{\frac{1}{\lambda}\xi+\ldots}{\eta+\ldots} \qquad 0<\frac{1}{\lambda}<1,$$

and considering η as the independent variable, it reduces to the case $0<\lambda<1$.

3° $$\frac{d\eta}{d\xi}=\frac{\eta+\lambda\xi+\ldots}{\xi+\ldots}, \ \lambda\neq 0.$$

In this case $a=\infty$ is a double root of the equation (22). By criterion 1 § 7, we see that all characteristics in $[-ly\leqq x\leqq ly$, $0<y\leqq Y]$, where l and Y are sufficiently small positive values, touch the positive y-axis, and those in $[ly\leqq x\leqq -ly, -Y\leqq y<0]$ touch the negative y-axis at the origin. Further, the ellipse $x=\rho\cos\theta$, $y=b\rho\sin\theta$ has no contact point when $b>|2\lambda|$, for sufficiently small ρ. The origin is therefore a node. As there are only two possible directions $\theta=\frac{\pi}{2}, \theta=\frac{3\pi}{2}$, all the characteristics in the neighbourhood of the origin must have these directions at the origin.

The finding concerning characteristics in the neighbourhood of a singular point of the first order may be summed up into the following:

Theorem 10. In the equation

$$\frac{dy}{dx} = \frac{ax+by+\ldots}{Ax+By+\ldots},\quad D \equiv \begin{vmatrix} a & b \\ A & B \end{vmatrix} \neq 0$$

i) *When* $E \equiv (A-b)^2 + 4aB < 0$, *the origin is generally a focus. It may be a center only when the additional condition* $A+b=0$ *is satisfied;*

ii) *When* $E > 0$ *the origin is a col* (*Fig.* 27) *or a node according as* D *is positive or negative; and specially a*) *when* $E > 0$, $D < 0$, *denoting by* t_2 *that one of the possible directions, for which the absolute value of* $Bt+A$ *is greater than that at the other root* t_1, *there are indefinitely many characteristics that have the slope* t_2 *on both sides of the origin, and there is a single characteristic on each side of the origin, that has the slope* t_1 (*Fig.* 28); *b*) *when* $a=0, B=0, A=b$ *on each direction at the origin there is a single characteristic;*

iii) *When* $E=0$ *the origin is a node and all characteristics have the same slope* $\frac{A-b}{2B}$ *at the origin* (*Fig.* 29).

Equation (16), given in § 8

$$x\frac{dy}{dx} = \frac{ax+by+\ldots}{1+A_1x+B_1y+\ldots},\quad b \neq 0$$

is a special case of the above. The possible directions at the origin are $\frac{a}{1-b}$ and ∞; and the origin is a *col* or a node according as b is negative or positive.

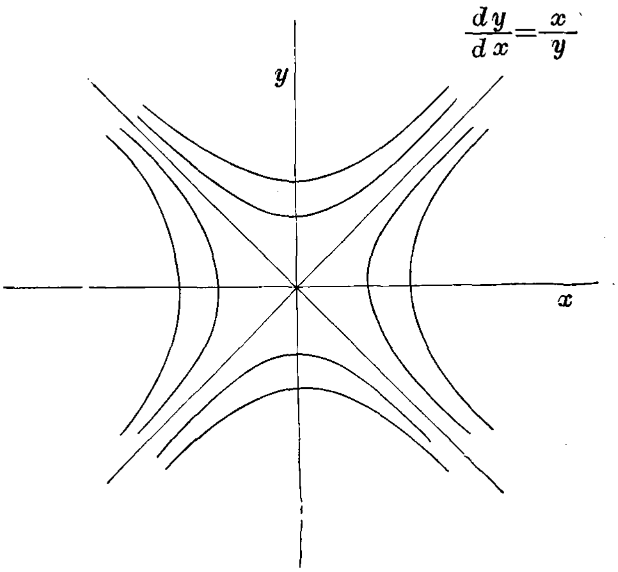


Fig. 27.

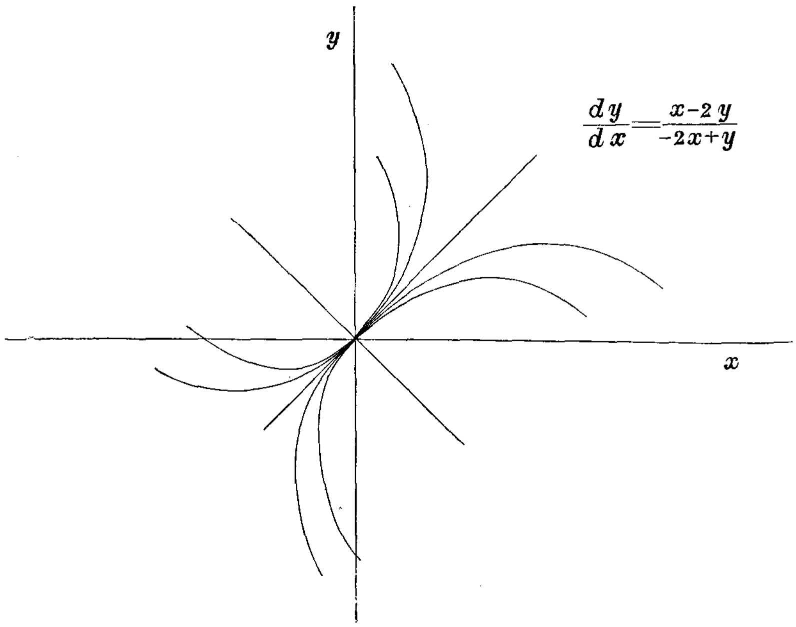


Fig. 28.

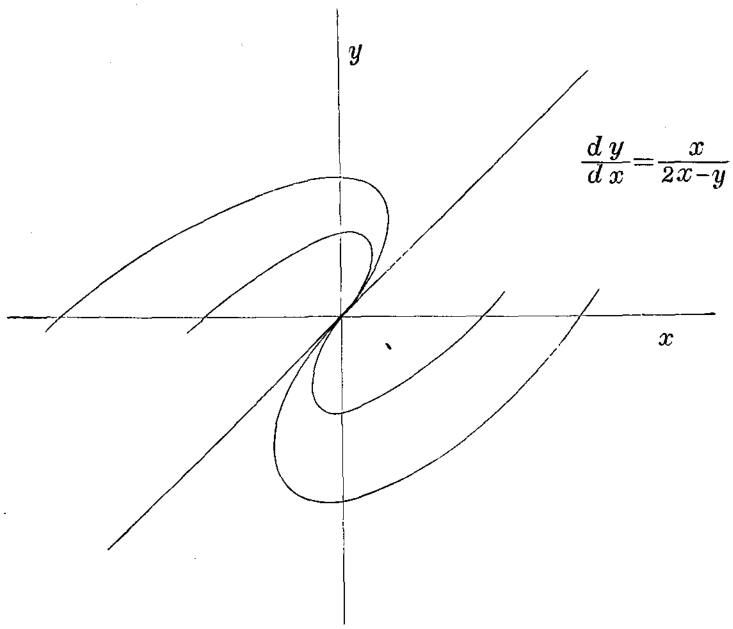


Fig. 29.

14. Examples.

The general principle for tracing characteristics defined by the equation $\frac{dy}{dx}=\frac{P(x,y)}{Q(x,y)}$ is due to the investigation of

1° $f(x,y)=\frac{P(x,y)}{Q(x,y)}$ i.e. a) curve $P(x,y)=0$, b) curve $Q(x,y)=0$ c) sign of $f(x,y)$;

2° $F(x,y)$ i,e. a) curve $F(x,y)=0$, b) sign of $F(x,y)$, c) $\chi\{(x,y)\}_{F=0}=\left(\frac{dy}{dx}\right)_{F=0}-(f(x,y))_{F=0}$;

3° Nature of singular points;

4° Nature of characteristics at infinite distance;

5° Limiting cycles.

Examples of tracing characteristics from their equations.

Ex. 1. $$\frac{dy}{dx}=y^2-x^2 \quad \text{(Fig. 32).}$$

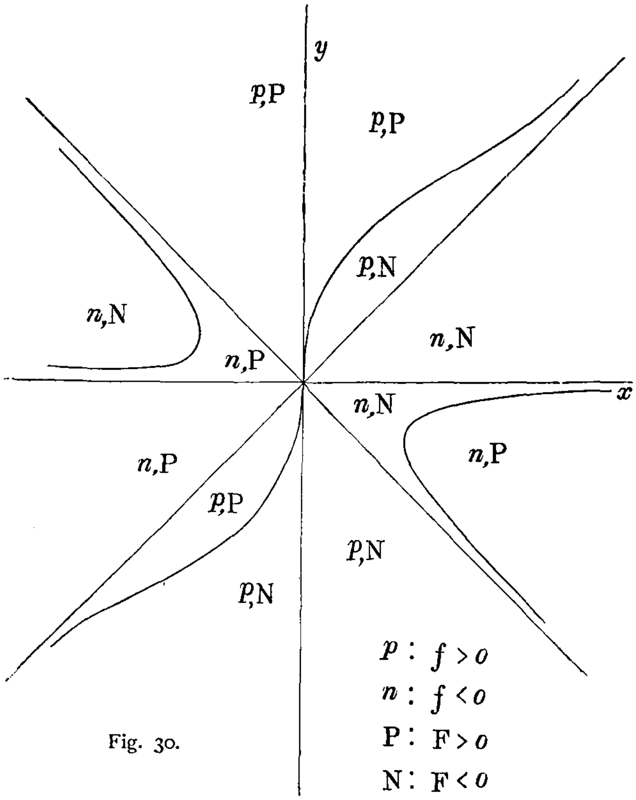


Fig. 30.

We have $f(x,y)=y^2-x^2,\ F(x,y)=2\{y(y^2-x^2)-x\}$

and the configuration of slope and concavity is shewn in Fig. 30.

On the curve $F(x,y)=0$,

$$[\chi(x,y)]_{F=0}=\left[\frac{y^2-x^2}{y^2(3y^2-x^2)}\right]_{F=0}$$

and we find

$$\chi>0 \quad \text{when} \quad 1<\frac{y}{x}<\infty \quad \text{and} \quad -\frac{1}{\sqrt{3}}<\frac{y}{x}<0,$$

$$\chi<0 \quad \text{when} \quad -1<\frac{y}{x}<-\frac{1}{\sqrt{3}},$$

$$\chi=\infty \quad \text{when} \quad \frac{y}{x}=-\frac{1}{\sqrt{3}}$$

and $[\chi(x,y)]_{F=0}=0$ has no root, so there is no contact point on the curve $F=0$.

The given equation has no singular point.

At infinite distance the possible directions are (1) $a_1=\infty$, (2)

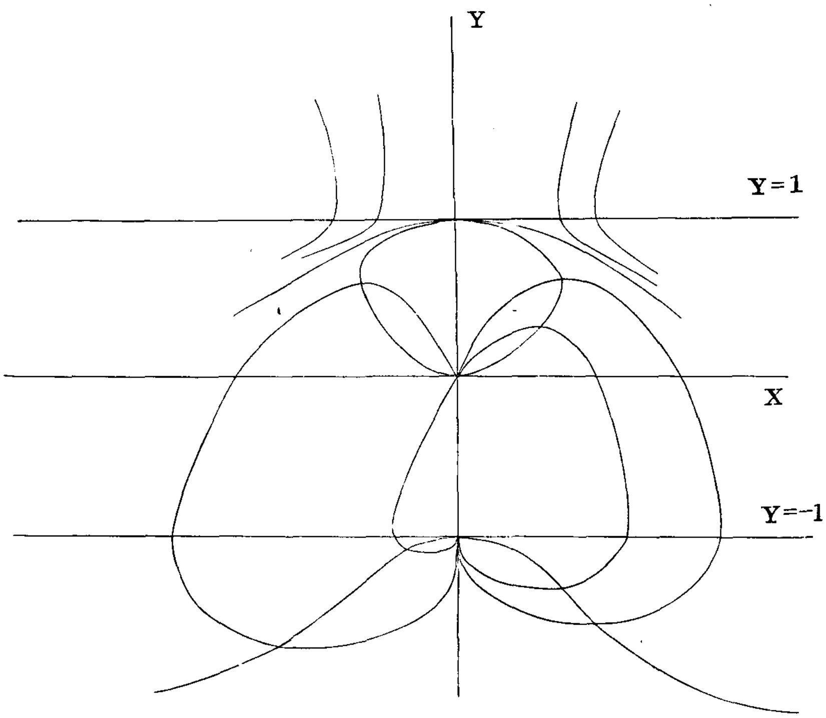


Fig. 31.

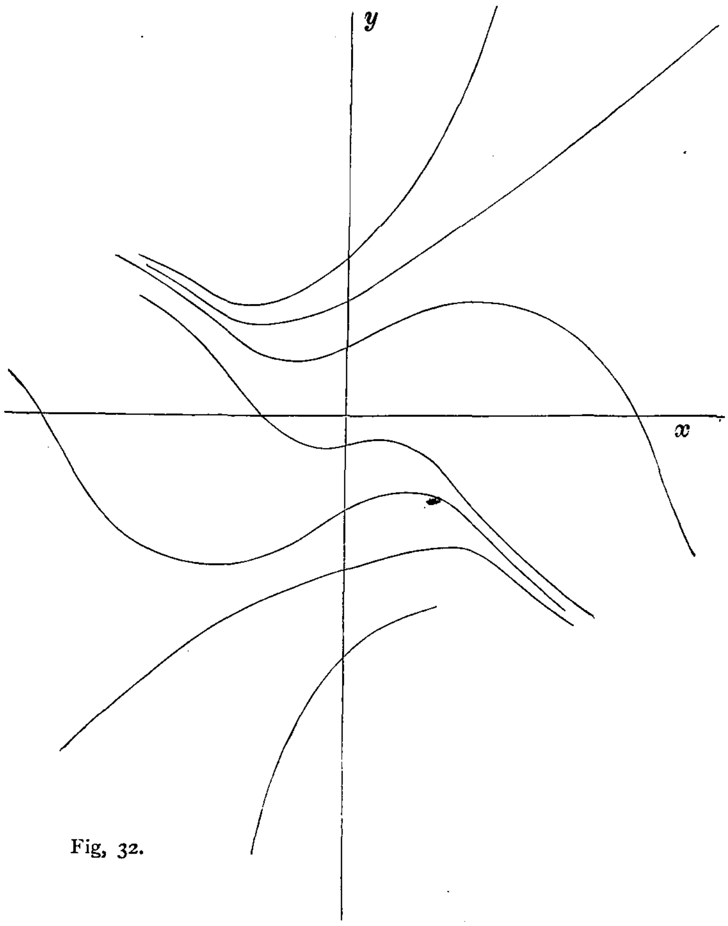
$\alpha_2 = 1$ and (3) $\alpha_3 = -1$. By the substitution $x = \frac{Y}{X}$, $y = \frac{1}{X}$ the differential equation becomes

$$\frac{dY}{dX} = \frac{Y - Y^3 - X^2}{X(1 - Y^2)} \qquad \text{(Fig. 31)}$$

and the three directions go to the points: (1) $X=0$, $Y=0$; (2) $X=0$, $Y=1$ and (3) $X=0$, $Y=-1$.

(1) The point $X=0$, $Y=0$ is a node and in each direction there is a single characteristic.

(2) Put $Y = Y_1 + 1$, then $\frac{dY_1}{dX} = \frac{2Y_1 + 3Y_1^2 + Y_1^3 + X^2}{X(Y_1^2 + 2Y_1)} \equiv f_1(X, Y_1)$ and the possible directions at $X=0$, $Y_1=0$ are $\theta=0$, $\theta=\pi$, $\theta=\frac{\pi}{2}$ and $\theta=\frac{3\pi}{2}$. The integral $X=0$ is the only characteristic in the direction



Fig, 32.

$\theta = \frac{\pi}{2}$, $\theta = \frac{3\pi}{2}$. There is a single characteristic on each direction $\theta = 0$, $\theta = \pi$.

(3) $X = 0$, $Y = -1$ is a node and all the characteristics near the point have the direction $-\frac{\pi}{2}$ at $X = 0$, $Y = -1$.

Thus the differential equation at infinite distance has (1) characteristics which have arbitrary straight lines parallel to the y-axis as asymptotes, (2) single characteristic on each of first and third quadrants, which has the line $x = y$ as asymptote, (3) indefinitely many characteristics which have the direction $-\frac{\pi}{4}$ in second and fourth quadrants.

Ex. 2. $$x\frac{dy}{dx} = x - 2y + y^2 \quad \text{(Fig. 35)}.$$

Here $$f(x, y) = \frac{x - 2y + y^2}{x}$$

$$F(x, y) = \frac{-2x + 6y + 2xy - 7y^2 + 2y^3}{x^2},$$

$$\{\chi(x, y)\}_{F=0} = \frac{y^2 - 2y}{(6 - 14y + 13y^2 - 4y^3)(3 - 2y)}$$ and the curve

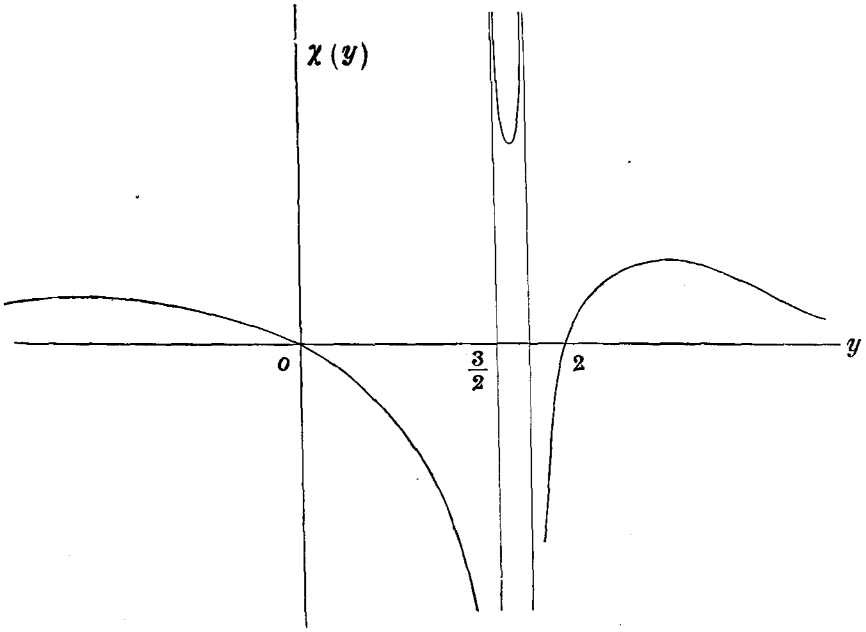


Fig. 33.

$\{\chi(x,y)\}_{F=0} = \chi(y)$ is shewn in Fig. 33.

The singular points are $x=0, y=0$ and $x=0, y=2$.

The origin is a *col* through which, besides the characteristic $x=0$, a characteristic with the slope $\frac{1}{3}$ passes.

The point $x=0, y=2$ is a node. At that point all characteristics except $x=0$ have the slope -1.

The possible directions at infinite distance are $\alpha=\infty$ and $\alpha=0$.

$\alpha=\infty$. Put $x=\frac{Y}{X}$, $y=\frac{1}{X}$ then

$$\frac{dY}{dX}=\frac{Y(1-3X+XY)}{X(1-2X+XY)}.$$

Hence $X=0$, $Y=0$ is a node through which in each direction a single characteristic passes.

$\alpha=0$. Put $x=\frac{1}{X_1}$, $y=\frac{Y_1}{X_1}$, then

$$\frac{dY_1}{dX_1}=\frac{-X_1+3X_1Y_1-Y_1^2}{X_1^2} \quad \text{(Fig. 34).}$$

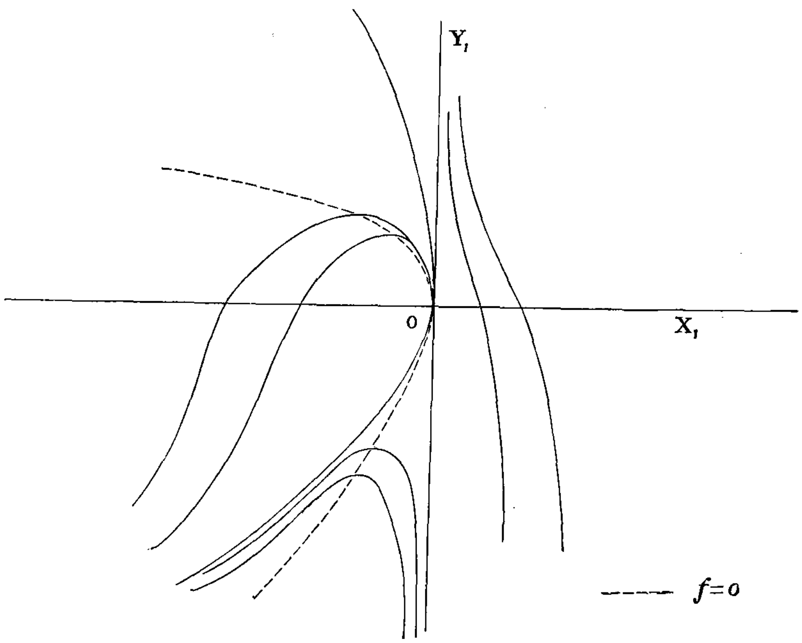


Fig. 34.

At $X_1=0$, $Y_1=0$ the possible direction is ∞. Using the configuration of slope and the criterions given in § 7, we find: $X_1=0$ is a characteristic; in the first and fourth quadrants there is no characteristic through the origin; in the third quadrant there is a single characteristic C_1 which has the slope $+\infty$ at the origin; the characteristics below C_1 are parabolic and those above C_1 are rays having the slope $-\infty$ at the origin.

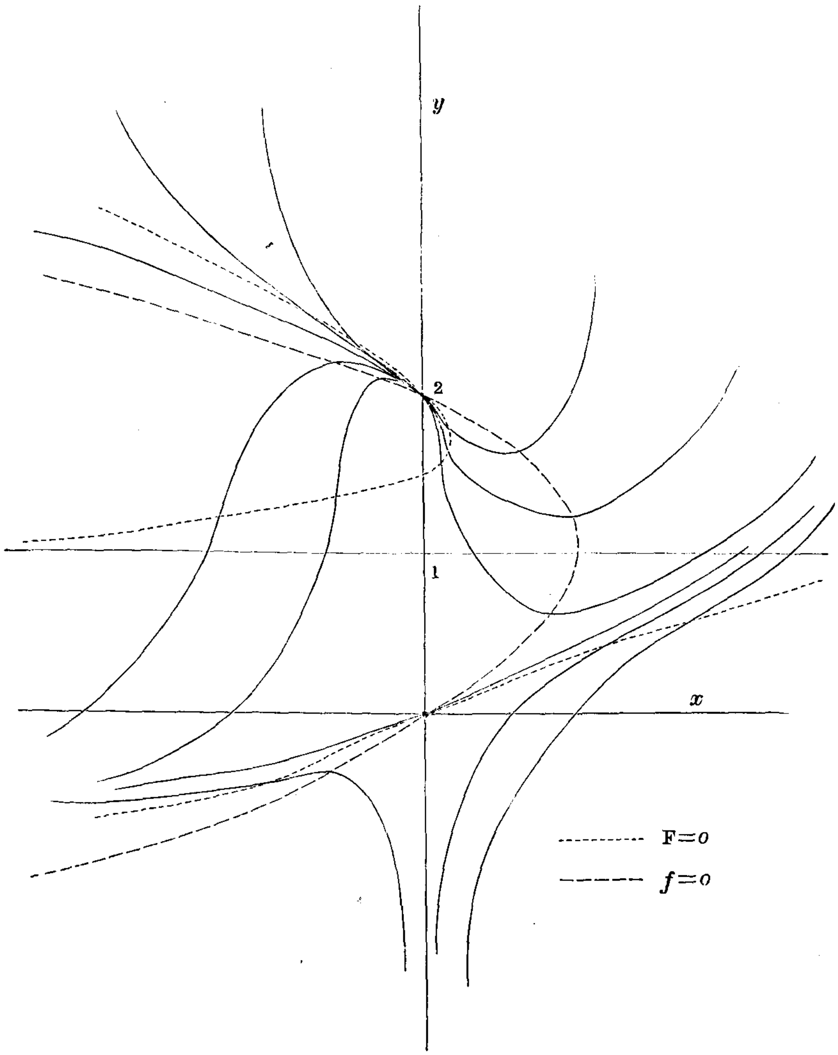


Fig. 35.

Hence of the original equation we may say: 1° there are two characteristics, one of which lies above the x-axis and the other below it, that have any line $x=c$ as asymptote; 2° in the second quadrant there is a single characteristic which has the slope 0 at infinity. This is the characteristic corresponding to C_1; 3° in the third quadrant there is an infinite number of characteristics which have the slope 0 at infinite distance.

Ex. 3. $$\frac{dy}{dx} = \frac{4x^3y}{y^2+x^4} \qquad \text{(Fig. 36).}$$

We have $f(x,y) = \dfrac{4x^3y}{y^2+x^4}$, $F(x,y) = \dfrac{4x^2y\,(3y^4-2x^4y^2+3x^8)}{(y^2+x^4)^3}$, $\dfrac{\partial f}{\partial y} = \dfrac{4x^3(x^4-y^2)}{(y^2+x^4)^2}$ and we know that $f(x,y)$, $F(x,y)$ are positive or negative according as $\dfrac{y}{x}$ is positive or negative.

The equation has only one singular point $x=0, y=0$ and evidently $y=0$ is a characteristic through it.

To find if there will be characteristics besides this, we apply criterion 4 § 7. Assume

$$\eta = a\xi^n,\ n > 1$$

then $$\chi(\xi,\eta) = \frac{d\eta}{d\xi} - f(\xi,\eta) = na\xi^{n-1} - \frac{4a\xi^{n+3}}{a^2\xi^{2n}+\xi^4}$$

and we have, if $\xi > 0$,

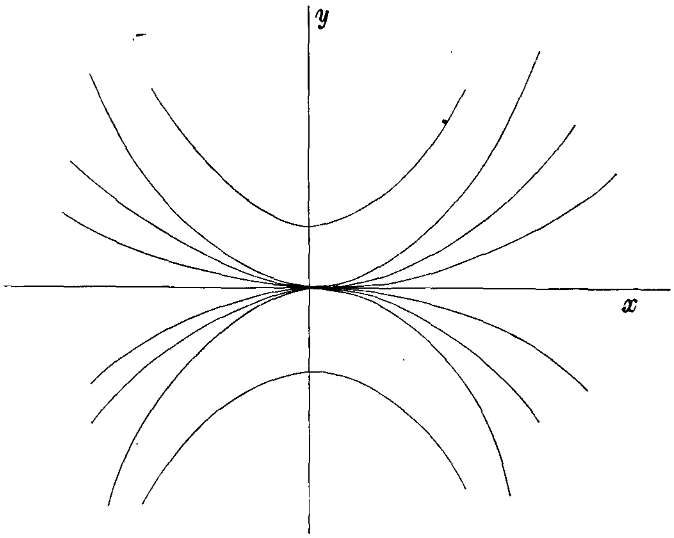


Fig. 36.

i) $\chi > 0$ when $1 < n < 2,\ \alpha > 0$ or $n = 2,\ \alpha > 1$;

ii) $\chi = 0$ when $n = 2,\ \alpha = 1$;

iii) $\chi < 0$ when $n = 2,\ 0 < \alpha < 1$ or $2 < n < 4,\ \alpha > 0$;

iv) $\chi > 0$ when $n > 4,\ \alpha > 0$.

By ii) $y=x^2$ is a characteristic and, since $\dfrac{\partial f}{\partial y} < 0$, above this curve there is no characteristic through the origin. In $(0 \leqq y \leqq x^2,\ x > 0)$ all characteristics pass through the origin and touch the positive x-axis. Also we know that the characteristics are symmetrical with respect to the x or the y-axis.

The possible directions at infinite distance are $\alpha=0$ and $\alpha=\infty$; and all the characteristics except $y=0$ have the slope $+\infty$ at infinity.

Ex. 4. $$\frac{dy}{dx} = \frac{-x-y+y\,(x^2+y^2)}{y}. \quad \text{(Fig. 37).}$$

The configuration of slope is shewn in the figure. The equation has only one singular point $x=0,\ y=0$; and this point is a focus.

Consider the circle
$$\xi^2+\eta^2 = \rho^2$$
then
$$\chi(\xi,\eta) = \frac{d\eta}{d\xi} - f(\xi,\eta) = 1-\rho^2,$$

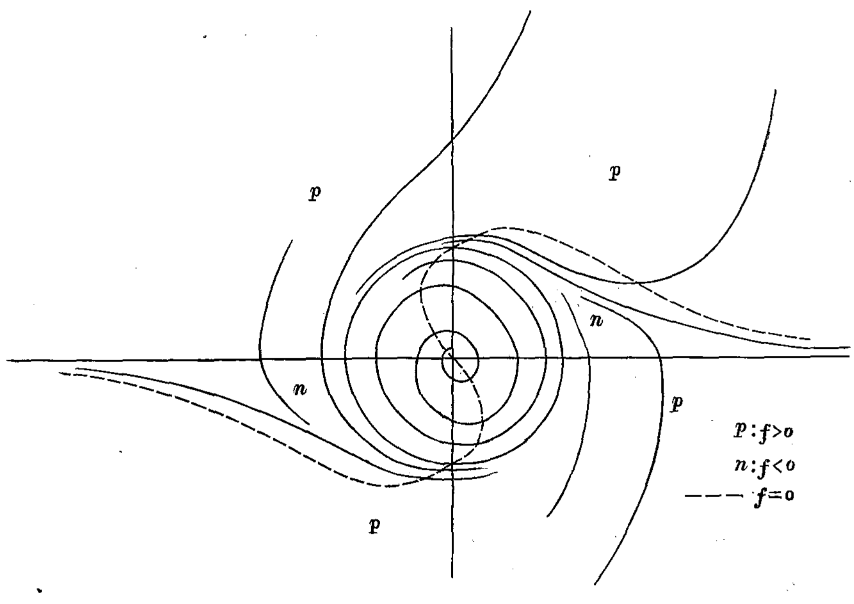


Fig. 37.

therefore
$$\chi(\xi,\eta) > 0 \quad \text{when} \quad 0 < \rho < 1,$$
$$= 0 \quad \text{when} \quad \rho = 1,$$
$$< 0 \quad \text{when} \quad 1 < \rho.$$

Thus the circles $\xi^2+\eta^2=\rho^2$ have no contact points except the points $\xi=\pm\rho, \eta=0$ when $\rho\neq 1$. At the contact points the circle and the characteristics cut each other and we see that the circle $\xi^2+\eta^2=1$ is the only limiting cycle.

In each of the first and third quadrants there is a characteristic which has the x-axis as asymptote. All the other characteristics have the slope ∞ at infinite distance.

Ex. 5

$$x\{2(x^2+y^2)-1\}dx+$$
$$[\{2(x^2+y^2)+1\}y+\frac{3}{2}(2x^2-1)\{(x^2+y^2)^2-x^2+y^2\}]dy=0 \quad \text{(Fig. 38).}$$

The equation has four finite singular points $x=0, y=0$; $x=0, y=1$; $x=\frac{1}{\sqrt{2}}, y=0$ and $x=-\frac{1}{\sqrt{2}}, y=0$.

1° $x=0$, $y=0$ is a *col* and the slopes of the characteristics through that point are -1 and $+1$.

2° $x=0$, $y=1$ is also a *col* and the slopes of the characteristics through that point are $-\frac{1}{\sqrt{2}}$, $+\frac{1}{\sqrt{2}}$.

3° Put $x=\frac{1}{\sqrt{2}}+X$, then

$$\frac{dy}{dX}=\frac{-2X+\dots}{2y-\frac{3\sqrt{2}}{8}X+\dots}$$

and we know that the point $x=\frac{1}{\sqrt{2}}$, $y=0$ is a focus.

The equation remains unchanged by putting $-x$ for x, hence the line-elements are symmetrical with respect to the y-axis.

The equation has the characteristic $(x^2+y^2)^2-x^2+y^2=0$; and each loop of this lemniscate is the limiting cycle of the spirals which have the foci at $x=\frac{1}{\sqrt{2}}$, $y=0$ or $x=-\frac{1}{\sqrt{2}}$, $y=0$.

By the configuration of the slope the form of the characteristics may be traced as shewn in Fig. 38.

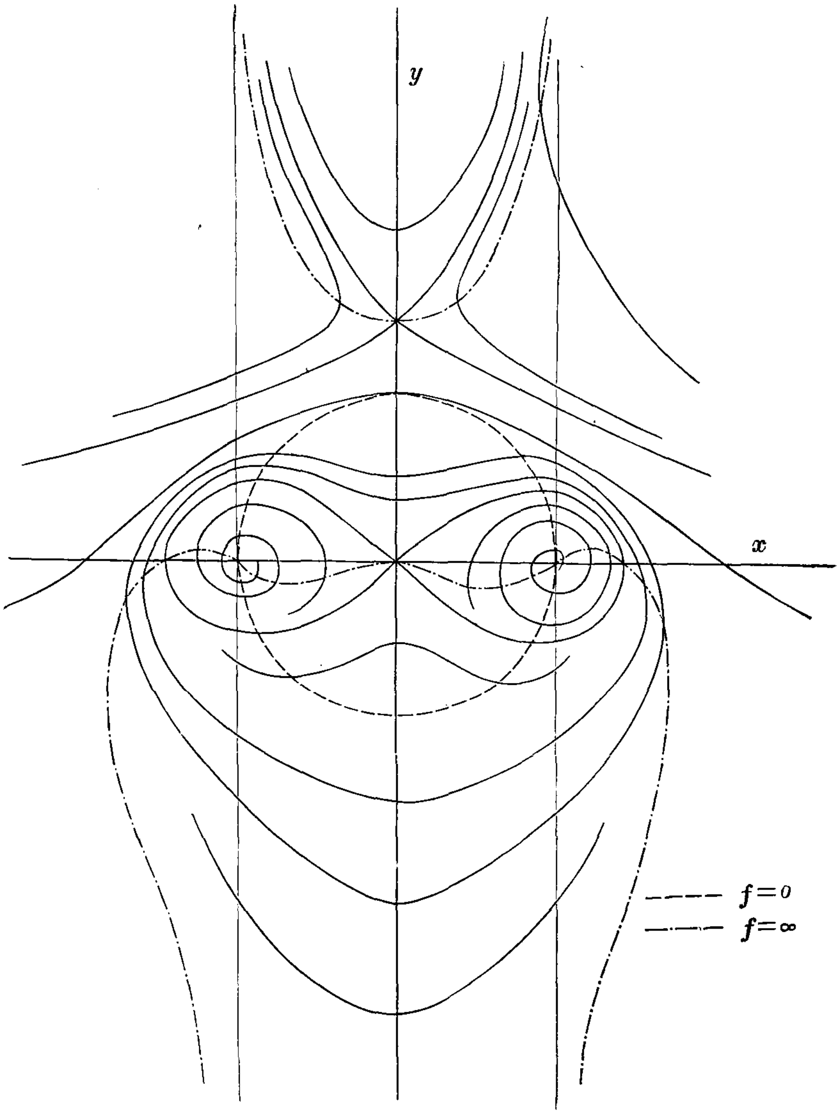


Fig. 38.