On the Solutions of Partial Differential Equations of First Order at the Singular Points.

By

Toshizô Matsumoto.

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Introduction.

Briot and Bouquet were the first to study in a general way the solution of the differential equation of the form

$$x\frac{dy}{dx}=ax+by+\dots,$$

at its singular point $x=0, y=0$. Afterwards Picard and Poincaré found, at the same time, the form of solutions which are not holomorph. On this occasion, Poincaré treated in his *Thèse*, the solution of the partial differential equation of the form

$$\xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\dots+\xi_n(x)\frac{\partial f}{\partial x_n}=0,$$

where $\xi_1(x), \xi_2(x), \dots\xi_n(x)$ are holomorphic functions of n variables x_1, $x_2, \dots x_n$ about the origin $(0, 0, \dots 0)$, and each function begins with terms of first order of $x_1, x_2, \dots x_n$. Under certain conditions he found $n-1$ solutions of the equation. Afterwards, by several mathematicians, the exceptional cases were studied, for the case of two variables, rather as an ordinary differential equation. The literature on the study is given by Painlevé in *Encyclopédie d. Sc. Math.*, (Tome II, vol. 3); but researches upon the partial differential equation are comparatively rare; and the problem is extremely difficult. I will, at first, repeat the discussion of the foregoing equation and next discuss the solutions of complete systems.

1. There is given a partial differential equation of first order between n independent variables $x_1, x_2, \dots x_n$ such that

$$Xf \equiv \xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\ldots+\xi_n(x)\frac{\partial f}{\partial x_n}=0,$$

where $\xi_1(x), \xi_2(x), \ldots \xi_n(x)$ are functions of these n variables. By the symbol X, we always mean the operation of the form

$$\xi_1(x)\frac{\partial}{\partial x_1}+\xi_2(x)\frac{\partial}{\partial x_2}+\ldots+\xi_n(x)\frac{\partial}{\partial x_n},$$

exerted upon a function $f(x)$ of the variables $x_1, x_2, \ldots x_n$.

To find the solution of this equation, consider two equations of the form

$$\xi_1(x)\frac{\partial f_1}{\partial x_1}+\xi_2(x)\frac{\partial f_1}{\partial x_2}+\ldots+\xi_n(x)\frac{\partial f_1}{\partial x_n}=F_1(f_1),$$

and

$$\xi_1(x)\frac{\partial f_2}{\partial x_1}+\xi_2(x)\frac{\partial f_2}{\partial x_2}+\ldots+\xi_n(x)\frac{\partial f_2}{\partial x_n}=F_2(f_2),$$

where $F_1(f_1)$ and $F_2(f_2)$ are functions of f_1 resp. f_2 alone. If we find the solutions f_1 and f_2, then, putting

$$\int\frac{df_1}{F_1(f_1)}\equiv\theta_1(f_1), \quad \int\frac{df_2}{F_2(f_2)}\equiv\theta_2(f_2),$$

we have

$$X\theta_1(f_1)=1, \quad X\theta_2(f_2)=1.$$

Hence the difference $\theta_1(f_1)-\theta_2(f_2)$ is a solution of our equation $Xf=0$.

By this method, Poincaré[1] proved the existence of $n-1$ solutions of the partial differential equation

$$Xf \equiv \xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\ldots+\xi_n(x)\frac{\partial f}{\partial x_n}=0, \qquad (1)$$

where

$$\begin{aligned}\xi_1(x)&=\lambda_1x_1+\ldots,\\ \xi_2(x)&=\lambda_2x_2+\ldots,\\ &\ldots\ldots\ldots\ldots\\ \xi_n(x)&=\lambda_nx_n+\ldots.\end{aligned}$$

The functions $\xi_1(x), \xi_2(x), \ldots \xi_n(x)$ are holomorphic about $(0, 0, \ldots 0)$ and the dotted parts are terms of higher order than the first. To solve

1 Poincaré, Thèse (Paris, 1879) and Acta Mathematica **13** (1890), *Sur le problème des trois corps*... .
Picard, Traité D'analyse, III.

this equation, take for $F_1(f_1)$, the form $\lambda_1 f$. For the existence of a holomorphic solution of the equation

$$Xf = \xi_1(x)\frac{\partial f}{\partial x_1} + \xi_2(x)\frac{\partial f}{\partial x_2} + \ldots + \xi_n(x)\frac{\partial f}{\partial x_n} = \lambda_1 f, \qquad (2)$$

Poincaré gave the following conditions between the coefficients $\lambda_1, \lambda_2, \ldots \lambda_n$, namely :

1°, *the relations*

$$\lambda_1 p_1 + \lambda_2 p_2 + \ldots + \lambda_i(p_i - 1) + \ldots + \lambda_n p_n = 0, \quad i = 1, 2, \ldots n,$$

are not satisfied by any positive integral values of $p_1, p_2, \ldots p_n$, provided,

$$p_1 + p_2 + \ldots + p_n \geqq 2\,;$$

2°, *if we denote $\lambda_1, \lambda_2, \ldots \lambda_n$ by the points on a plane, then we can trace a convex polygon in which these n points lie but which does not contain the origin ; or we may say that this is a straight line through the origin, on one side of which all the n points lie.*

Let us, in the following, call these two conditions, *Poincaré's conditions*, for simplicity.

For the equation (2), the conditions for $i=2, 3, \ldots n$ are unnecessary. Now differentiating the equation (2) p_1 times with respect to x_1, p_2 times with respect to $x_2, \ldots p_n$ times with respect to x_n and putting $x_1 = x_2 = \ldots = x_n = 0$, we have the value of

$$(\lambda_1(p_1 - 1) + \lambda_2 p_2 + \ldots + \lambda_n p_n)\left(\frac{\partial^{p_1+p_2+\ldots+p_n} f}{\partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_n^{p_n}}\right)_0,$$

expressed by the values of partial derivatives lower than $p_1 + p_2 + \ldots + p_n$ at $x_1 = x_2 = \ldots = x_n = 0$. Under Poincaré's conditions, the coefficient of the derivative is not zero, moreover the quotient

$$\left|\frac{\lambda_1(p_1 - 1) + \lambda_2 p_2 + \ldots + \lambda_n p_n}{p_1 - 1 + p_2 + \ldots + p_n}\right|$$

is greater than a certain number $\varepsilon > 0$, for any $p_1, p_2, \ldots p_n$. Using Mr. Picard's notations, put

$$f = Ax_1 + v,$$

where A is an arbitrary constant. Then the equation (2) becomes

$$\lambda_1 x_1 \frac{\partial v}{\partial x_1} + \lambda_2 x_2 \frac{\partial v}{\partial x_2} + \ldots + \lambda_n x_n \frac{\partial v}{\partial x_n} - x_1 v =$$

$$= \varphi_1 \frac{\partial v}{\partial x_1} + \varphi_2 \frac{\partial v}{\partial x_2} + \dots + \varphi_n \frac{\partial v}{\partial x_n} + \psi, \tag{3}$$

where the functions $\varphi_1, \varphi_2, \dots \varphi_n$ and ψ commence at least by terms of second order. Let M be the maximum modulas of $\varphi_1, \varphi_2, \dots \varphi_n$ and ψ in the convergent circles with radius a, about the origin. Then the equation

$$\varepsilon\left(x_1 \frac{\partial V}{\partial x_1} + x_2 \frac{\partial V}{\partial x_2} + \dots + x_n \frac{\partial V}{\partial x_n} - V\right)$$

$$= \left\{\frac{M}{1 - \frac{x_1 + x_2 + \dots + x_n}{a}} - M - M \frac{x_1 + x_2 + \dots + x_n}{a}\right\}$$

$$\times \left(\frac{\partial V}{\partial x_1} + \frac{\partial V}{\partial x_2} + \dots + \frac{\partial V}{\partial x_n} + 1\right)$$

give a horomorphic solution $V(x)$ commencing at least by terms of second order. This function $V(x)$ serves as *fonction majorante* of the solution $v(x)$ of the equation (3). Hence the equation (2) has a holomorphic solution of the form

$$f = Ax_1 + v(x).$$

For the details, we refer to Mr. Picard's book.

2. When we differentiate both sides of the equation (3), § 1, p_1 times with respect to x_1, p_2 times with respect to $x_2, \dots p_n$ times with respect to x_n and put $x_1 = x_2 = \dots = x_n = 0$, then from the left-hand side, we obtain

$$(\lambda_1(p_1 - 1) + \lambda_2 p_2 + \dots + \lambda_n p_n)\left(\frac{\partial^{p_1+p_2+\dots+p_n} v}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}}\right)_0.$$

The right-hand side is a *linear homogeneous* function of

$$\left(\frac{\partial^{p_1+p_2+\dots+p_n} \psi}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}}\right)_0$$

and

$$\left(\frac{\partial^{q_1+q_2+\dots+q_n} v}{\partial x_1^{q_1} \partial x_2^{q_2} \dots \partial x_n^{q_n}}\right)_0,$$

where

$$0 \leqq q_1 \leqq p_1, \ 0 \leqq q_2 \leqq p_2, \dots 0 \leqq q_n \leqq p_n,$$

$$2 \leqq q_1 + q_2 + \dots + q_n < p_1 + p_2 + \dots + p_n,$$

since the functions $\varphi_1, \varphi_2, \ldots \varphi_n$ and ψ contain no terms lower than second order and $v(x)$ commences also with terms of second order at least. Therefore

$\left(\dfrac{\partial^{p_1+p_2+\ldots+p_n} v}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\right)_0$ is a *linear homogeneous* function of $\left(\dfrac{\partial^{p_1+p_2\ldots+p_n}\psi}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\right)_0$ and $\left(\dfrac{\partial^{q_1+q_2\ldots+\ldots q_n} v}{\partial x_1^{q_1}\partial x_2^{q_2}\ldots\partial x_n^{q_n}}\right)_0$, where $q_1, q_2, \ldots q_n$ obey the above conditions.

Hence for $p_1+p_2+\ldots+p_n = 2$,

$$\left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_0 \quad i, j = 1, 2, \ldots n,$$

are linear homogeneous functions of

$$\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right)_0, \quad i, j = 1, 2, \ldots n.$$

For $p_1+p_2+\ldots+p_n = 3$, all derivatives of third order at $x_1 = x_2 = \ldots = x_n = 0$, are linear homogeneous functions of

$$\left(\frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k}\right)_0 \quad \text{and} \quad \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_0, \qquad i, j, k = 1, 2, \ldots n,$$

and hence they become linear homogeneous functions of

$$\left(\frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k}\right)_0, \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right)_0, \qquad i, j, k = 1, 2, \ldots n.$$

Proceeding in this way, all the derivatives of $v(x)$ at $x_1 = x_2 = \ldots = x_n = 0$ are linear homogeneous functions of some number of derivatives of ψ at $x_1 = x_2 = \ldots = x_n = 0$. But a derivative of a power-series at a point, divided by a certain positive integer, becomes a coefficient of the power-series. Hence we can enunciate the above result as follows. *All the coefficients of the expansion of $v(x)$ at $x_1 = x_2 = \ldots = x_n = 0$ are certain linear homogeneous function of some number of coefficients of the power series $\psi(x)$.*

If we compare equations (2) and (3), § 1, we see that

$$\psi(x) = -A(\xi_1(x) - \lambda_1 x_1).$$

That is, all the coefficients of $\psi(x)$ are multiplied by A. Whence we know that all the coefficients of $v(x)$ are linear homogeneous functions of A, or put

$$v(x) = A\,w(x),$$

then $w(x)$ is independent of A. Therefore the solution of the equation (2), § 1 becomes

$$\begin{aligned} f &= Ax_1 + v(x) \\ &= A(x_1 + (w(x)) \\ &\equiv Af_1(x), \end{aligned}$$

where
$$f_1(x) \equiv x_1 + w(x),$$

and clearly $f_1(x)$ *is a holomorphic solution af the preceding equation* and *its coefficients have no arbitrarity.*

This solution $f_1(x)$ is unique, for if not, the equation (3), § 1, would have two solutions, say $v_1(x)$ and $v_2(x)$, therefore $v_1(x) - v_2(x)$ should satisfy the equation

$$\lambda_1 x_1 \frac{\partial v}{\partial x_2} + \lambda_2 x_2 \frac{\partial v}{\partial x_2} + \dots + \lambda_n x_n \frac{\partial v}{\partial x_n} - \lambda_1 v$$

$$= \varphi_1 \frac{\partial v}{\partial x_1} + \varphi_2 \frac{\partial v}{\partial x_2} + \dots + \varphi_n \frac{\partial v}{\partial x_n}.$$

But in this equation, the function ψ is absent, therefore by the result just obtained, it must follow that

$$v_1(x) - v_2(x) \equiv 0.$$

This is against our assumption ; hence *the solution* $f_1(x)$ *is unique.* Hence any holomorphic solution such as

$$f = Ax_1 + \dots$$

can be written in the form

$$f = Af_1(x).$$

3. Returning to the former, under Poincaré's conditions, we find the holomorphic solutions

$$f_1 = x_1 + \dots$$
$$f_2 = x_2 + \dots$$

of the equations

$$Xf_1 = \lambda_1 f_1, \quad Xf_2 = \lambda_2 f_2,$$

respectively. Hence, $\log f_1^{\frac{1}{\lambda_1}} - \log f_2^{\frac{1}{\lambda_2}}$, and therefore $f_1^{-\frac{1}{\lambda_1}} f_2^{\frac{1}{\lambda_2}}$ is a solution of the equation (1), § 1. Proceeding in this way, the proposed equation has $n-1$ solutions of the form

$$f_1^{-\frac{1}{\lambda_1}} f_2^{\frac{1}{\lambda_2}},\quad f_1^{-\frac{1}{\lambda_1}} f_3^{\frac{1}{\lambda_3}},\ \ldots\ldots\ f_1^{-\frac{1}{\lambda_1}} f_n^{\frac{1}{\lambda_n}}.$$

If in the equation, some number of $\lambda_1, \lambda_2, \ldots \lambda_n$ be equal, say $\lambda_1 = \lambda_2$, put

$$f = Ax_1 + Bx_2 + v(x),$$

then

$$\psi = -A(\xi_1(x) - \lambda_1 x_1) - B(\xi_2(x) - \lambda_1 x_2).$$

Then all the coefficients of $v(x)$ are linear homogeneous functions of A and B. Putting at first $A = 1, B = 0$ then $A = 0, B = 1$, we obtain two solutions

$$f_1 = x_1 + \ldots$$

$$f_2 = x_2 + \ldots,$$

and any other holomorphic solution of the form

$$f = Ax_1 + Bx_2 + \ldots$$

can be written in this way

$$f = Af_1(x) + Bf_2(x).$$

The case where more than two λ's are equal may be treated in the same way.

4. The partial differential equation

$$Xf = \xi_1(x)\frac{\partial f}{\partial x_1} + \xi_2(x)\frac{\partial f}{\partial x_2} + \ldots + \xi_n(x)\frac{\partial f}{\partial x_n} = 0 \qquad (1)$$

is equivalent to the system of ordinary differential equations

$$\frac{dx_1}{\xi_1(x)} = \frac{dx_2}{\xi_2(x)} = \ldots = \frac{dx_n}{\xi_n(x)}, \qquad (2)$$

where, as before, writing only the first terms,

$$\xi_1(x) = \lambda_1 x_1 + \ldots,$$

$$\xi_2(x) = \lambda_2 x_2 + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\xi_n(x) = \lambda_n x_n + \ldots.$$

When $\lambda_1, \lambda_2, \ldots \lambda_n$ don't satisfy Poincaré's conditions, but *when* $\lambda_1, \lambda_2, \ldots \lambda_\nu$, $(\nu < n)$ *satisfy the conditions and moreover the conditions that the equality*

$$\lambda_1 p_1+\lambda_2 p_2+\ldots+\lambda_\nu p_\nu = \lambda_i, \quad i = \nu+1, \ldots n,$$

for
$$p_1+p_2+\ldots+p_\nu \geqq 2,$$

cannot be satisfied, then the solutions $x_1, x_2, \ldots x_n$ of the equation (2), equated to $\frac{dt}{t}$, can be expressed as power-series of t^{λ_1}, t^{λ_2}, $\ldots t^{\lambda_\nu}$ and converge, provided the moduli of these quantities are sufficiently small. To prove this Mr. Picard transformed the equations (2) by

$$y_1 = t^{\lambda_1},\ y_2 = t^{\lambda_2},\ \ldots\ y_\nu = t^{\lambda_\nu}.$$

Considering $x_1, x_2, \ldots x_n$ as functions of $y_1, y_2, \ldots y_\nu$, the transformed equations are

$$\begin{aligned}
&\lambda_1 y_1 \frac{\partial x_1}{\partial y_1}+\lambda_2 y_2 \frac{\partial x_1}{\partial y_2}+\ldots+\lambda_\nu y_\nu \frac{\partial x_1}{\partial y_\nu} = \xi_1(x) = \lambda_1 x_1+\ldots,\\
&\lambda_1 y_1 \frac{\partial x_2}{\partial y_1}+\lambda_2 y_2 \frac{\partial x_2}{\partial y_2}+\ldots+\lambda_\nu y_\nu \frac{\partial x_2}{\partial y_\nu} = \xi_2(x) = \lambda_2 x_2+\ldots, \quad (3)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\lambda_1 y_1 \frac{\partial x_n}{\partial y_1}+\lambda_2 y_2 \frac{\partial x_n}{\partial y_2}+\ldots+\lambda_\nu y_\nu \frac{\partial x_n}{\partial y_\nu} = \xi_n(x) = \lambda_n x_n+\ldots.
\end{aligned}$$

Since the coefficients $\lambda_1, \lambda_2, \ldots \lambda_\nu$ satisfy Poincaré's conditions and the other just mentioned, this system of partial differential equations has n holomorphic solutions $x_1(y), x_2(y), \ldots x_\nu(y)$. For the detail, we refer to Mr. Picard Book.

5. The preceding system of partial differential equations can be obtained from another point of view, i.e.

Ta determine the substitution

$$\begin{aligned}
x_1 &= \theta_1(y_1, y_2, \ldots y_\nu),\\
x_2 &= \theta_2(y_1, y_2, \ldots y_\nu),\\
&\ldots\ldots\ldots\ldots\\
x_n &= \theta_n(y_1, y_2, \ldots y_\nu)
\end{aligned}$$

by which the differential expression

$$Xf = \xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\ldots+\xi_n(x)\frac{\partial f}{\partial x_n}$$

is to be reduced to the differential expression

$$Yf \equiv \eta_1(y)\frac{\partial f}{\partial y_1}+\eta_2(y)\frac{\partial f}{\partial y_2}+\ldots+\eta_\nu(y)\frac{\partial f}{\partial y_\nu}, \qquad (2)$$

where $\eta_1, \eta_2, \ldots \eta_\nu$ *are any functions of* $y_1, y_2 \ldots y_\nu$.
Now by this substitution, we have

$$\begin{aligned} Yf &= \sum_{j=1}^{\nu} \eta_j(y)\frac{\partial f}{\partial y_j} \\ &= \sum_{j=1}^{\nu} \eta_j(y)\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_j} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{\nu}\eta_j(y)\frac{\partial x_i}{\partial y_j}\frac{\partial f}{\partial x_i}. \end{aligned}$$

This is identical with the differential expression Xf. Hence, equating the coefficient of $\frac{\partial f}{\partial x_i}$, $i=1, 2, \ldots n$, we have

$$\begin{aligned} &\eta_1(y)\frac{\partial x_1}{\partial y_1}+\eta_2(y)\frac{\partial x_1}{\partial y_2}+\ldots+\eta_\nu(y)\frac{\partial x_1}{\partial y_\nu} = \xi_1(x), \\ &\eta_1(y)\frac{\partial x_2}{\partial y_1}+\eta_2(y)\frac{\partial x_2}{\partial y_2}+\ldots+\eta_\nu(y)\frac{\partial x_2}{\partial y_\nu} = \xi_2(x), \qquad (3) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\eta_1(y)\frac{\partial x_n}{\partial y_1}+\eta_2(y)\frac{\partial x_n}{\partial y_2}+\ldots+\eta_\nu(y)\frac{\partial x_n}{\partial y_\nu} = \xi_n(x). \end{aligned}$$

The n solutions $x_1=\theta_1(y)$, $x_2=\theta_2(y)$, $\ldots$ $x_n=\theta_n(y)$ are the required substitions.

If we take for the arbitrary functions

$$\eta_1 = \lambda_1 y_1, \ \eta_2 = \lambda_2 y_2, \ldots \eta_n = \lambda_n y_n,$$

this system of equations (3) coincides with the system given by Mr. Picard. If we put for $\eta_1, \eta_2, \ldots \eta_\nu$ some holomorphic functions commencing by terms $\lambda_1 y_1$, $\lambda_2 y_2$, $\ldots$ $\lambda_\nu y_\nu$ respectively, under the conditions written in the last section between $\lambda_1, \lambda_2 \ldots \lambda_\nu$, the system of partial differential equations (3) has always n holomorphic solutions $\theta_1(y)$, $\theta_2(y)$, $\ldots$ $\theta_n(y)$.

6. Before the proof of the existence-theorem of the system of equations (3) of the last section, some remarks should be given about the system of equations (3), § 4, whose existence-theorem is given by Mr. Picard.

Suppose, at first, all λ's are different from one another, then the solutions of the system of equations (3), § 4, take the form.

$$\begin{aligned} x_1 &= \theta_1(y) = a_1 y_1 + \theta_1'(y), \\ x_2 &= \theta_2(y) = a_2 y_2 + \theta_2'(y), \\ &\cdots\cdots\cdots \\ x_\nu &= \theta_\nu(y) = a_\nu y_\nu + \theta_\nu'(y), \\ x_{\nu+1} &= \theta_{\nu+1}(y), \\ &\cdots\cdots\cdots \\ x_n &= \theta_n(y), \end{aligned} \tag{1}$$

where $\theta_1', \theta_2' \ldots\ldots \theta_\nu'$, and $\theta_{\nu+1}, \ldots \theta_n$ are power series of $y_1, y_2 \ldots y_\nu$ which don't contain terms of first order and $a_1, a_2, \ldots a_\nu$ are arbitrary constants not zero. Hence we have

$$\left[\frac{\partial(x_1, x_2, \ldots x_\nu)}{\partial(y_1, y_2, \ldots y_\nu)}\right]_0 = a_1 . a_2 . \ldots a_\nu \neq 0.$$

Therefore, from the first ν equations of (1), we can find $y_1, y_2, \ldots y_\nu$ as functions of $x_1, x_2, \ldots x_\nu$:

$$\begin{aligned} y_1 &= \chi_1(x_1 . x_2 , \ldots x_\nu), \\ y_2 &= \chi_2(x_1, x_2, \ldots x_\nu), \\ &\cdots\cdots\cdots \\ y_\nu &= \chi_\nu(x_1, x_2, \ldots x_\nu). \end{aligned} \tag{2}$$

Substituting these values of $y_1, y_2, \ldots y_\nu$ in the remaining $n-\nu$ equations (1), we obtain $n-\nu$ cohditions between $x_1, x_2, \ldots x_n$:

$$\begin{aligned} \Psi_1(x_1, x_2, \ldots x_n) &= 0, \\ \Psi_2(x_1, x_2, \ldots x_n) &= 0, \\ &\cdots\cdots\cdots \\ \Psi_{n-\nu}(x_1, x_2 \ldots x_n) &= 0. \end{aligned} \tag{3}$$

When some number of $\lambda_1, \lambda_2, \ldots \lambda_n$ are equal, by the remark of last paragraph of § 3, similar systems of equations as (2) and (3) will de obtained. Now by the substitution (1), there holds the identity $Yf \equiv Xf$, in the full expression

$$\lambda_1 y_1 \frac{\partial f}{\partial y_1} + \lambda_2 y_2 \frac{\partial f}{\partial y_2} + \ldots + \lambda_\nu y_\nu \frac{\partial f}{\partial y_\nu}$$

$$\equiv \xi_1(x) \frac{\partial f}{\partial x_1} + \xi_2(x) \frac{\partial f}{\partial x_2} + \ldots + \xi_n(x) \frac{\partial f}{\partial x_n}.$$

Therefore any solution $F(y_1, y_2, \ldots y_\nu)$ of the equation

$$Yf = \lambda_1 y_1 \frac{\partial f}{\partial y_1} + \lambda_2 y_2 \frac{\partial f}{\partial y_2} + \ldots + \lambda_\nu y_\nu \frac{\partial f}{\partial y_\nu} = 0,$$

by aid of the transformation (2) and the conditions (3), must satisfy the equation

$$Xf = \xi_1(x) \frac{\partial f}{\partial x_1} + \xi_2(x) \frac{\partial f}{\partial x_2} + \ldots + \xi_n(x) \frac{\partial f}{\partial x_n} = 0. \tag{4}$$

The former equation, as we have seen, has $\nu - 1$ solutions

$$y_1^{-\frac{1}{\lambda_1}} y_2^{\frac{1}{\lambda_2}}, \quad y_1^{-\frac{1}{\lambda_1}} y_3^{\frac{1}{\lambda_3}}, \quad \ldots y^{-\frac{1}{\lambda_1}} y_\nu^{\frac{1}{\lambda_\nu}}.$$

Substituting the values (2), we arrive at the result :

When the constants $\lambda_1, \lambda_2, \ldots \lambda_\nu$ satisfy Poincaré's conditions and no relation $\lambda_1 p_1 + \lambda_2 p_2 + + \ldots + \lambda_\nu p_\nu = \lambda_i$, $i = \nu + 1, \ldots n$, hold, provided $p_1 + p_2 + \ldots + p_\nu \geqq 2$; then the equation (4) has $\nu - 1$ solutions of the form

$$\chi_1^{-\frac{1}{\lambda_1}} \chi_2^{\frac{1}{\lambda_2}}, \; \chi_1^{-\frac{1}{\lambda_1}} \chi_3^{\frac{1}{\lambda_3}}, \; \ldots \chi_1^{-\frac{1}{\lambda_1}} \chi_\nu^{\frac{1}{\lambda_\nu}},$$

and the variables $x_1, x_2 \ldots x_n$ satisfy the conditions (3).

7. We shall apply this result to the problem of § 1.

When $\lambda_1, \lambda_2, \ldots \lambda_n$ satisfy Poincaré's conditions, let us consider the following system of n equations

$$\lambda_1 y_1 \frac{\partial x_1}{\partial y_1} + \lambda_2 y_2 \frac{\partial x_1}{\partial y_2} + \ldots + \lambda_n y_n \frac{\partial x_1}{\partial y_n} = \xi_1(x),$$

$$\lambda_1 y_1 \frac{\partial x_2}{\partial y_1} + \lambda_2 y_2 \frac{\partial x_2}{\partial y_2} + \ldots + \lambda_n y_n \frac{\partial x_2}{\partial y_n} = \xi_2(x) \tag{1}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_1 y_1 \frac{\partial x_n}{\partial y_1} + \lambda_2 y_2 \frac{\partial x_n}{\partial y_2} + \ldots + \lambda_n y_n \frac{\partial x_n}{\partial y_n} = \xi_n(x).$$

This system is quite analogous with the system (3), § 4. Moreover if we write n for ν, Mr. Picard's proof of the existence-theorem may be applied, term by term, and give n holomorphic solutions

$$\begin{aligned} x_1 &= \theta_1(y_1, y_2, \dots y_n), \\ x_2 &= \theta_2(y_1, y_2, \dots y_n), \\ &\dots\dots\dots \\ x_n &= \theta_n(y_1, y_2, \dots y_n). \end{aligned} \tag{2}$$

The inverse functions are

$$\begin{aligned} y_1 &= \chi_1(x_1, x_2, \dots x_n), \\ y_2 &= \chi_2(x_1, x_2, \dots x_n), \\ &\dots\dots\dots \\ y_n &= \chi_n(x_1, x_2, \dots x_n), \end{aligned} \tag{3}$$

and, by the theorem of the last section, the $n-1$ functions $y_1^{-\frac{1}{\lambda_1}} y_2^{\frac{1}{\lambda_2}}$, $y_1^{-\frac{1}{\lambda_1}} y_3^{\frac{1}{\lambda_3}}, \dots y_1^{-\frac{1}{\lambda_1}} y_n^{\frac{1}{\lambda_n}}$ are solutions of the equation

$$Xf = \xi_1(x)\frac{\partial f}{\partial x_1} + \xi_2(x)\frac{\partial f}{\partial x_2} + \dots + \xi_n(x)\frac{\partial f}{\partial x_n} = 0,$$

where between $x_1, x_2, \dots x_n$, no further relation exists.

Since Xf and Yf are equivalent, we see at once that each

$$y_i = \chi_i(x_1, x_2, \dots x_n), \quad i = 1, 2, \dots n,$$

is a holomorphic solution of the equation

$$Xf = \xi_1(x)\frac{\partial f}{\partial x_1} + \xi_2(x)\frac{\partial f}{\partial x_2} + \dots + \xi_n(x)\frac{\partial f}{\partial x_n} = \lambda_i f, \quad i = 1, 2, \dots n.$$

i.e., *the differential expression Xf, transformed by the n holomorphic solutions* (3) *of $Xf = \lambda_i f$, $i = 1, 2, \dots n$, will take the form*

$$Yf = \lambda_1 y_1 \frac{\partial f}{\partial y_1} + \lambda_2 y_2 \frac{\partial f}{\partial y_2} + \dots + \lambda_n y_n \frac{\partial f}{\partial y_n}.$$

We shall often meet this transformation in the future.

8. Now let us return to the proof of the existence-theorem of the system of equations

$$\eta_1(y)\frac{\partial x_1}{\partial y_1} + \eta_2(y)\frac{\partial x_1}{\partial y_2} + \dots + \eta_\nu(y)\frac{\partial x_1}{\partial y_\nu} = \xi_1(x),$$

$$\eta_1(y)\frac{\partial x_2}{\partial y_1}+\eta_2(y)\frac{\partial x_2}{\partial y_2}+\ldots+\eta_\nu(y)\frac{\partial x_2}{\partial y_\nu}=\xi_2(x), \tag{1}$$

....

$$\eta_1(y)\frac{\partial x_n}{\partial y_1}+\eta_2(y)\frac{\partial x_n}{\partial y_2}+\ldots+\eta_\nu(y)\frac{\partial x_n}{\partial y_\nu}=\xi_n(x),$$

where the $\eta_1, \eta_2, \ldots \eta_\nu$ are holomorphic and commence with terms $\lambda_1 y_1$, $\lambda_2 y_2, \ldots \lambda_\nu y_\nu$ respectively. This system is not escentially different from the system (3), § 4 which Picard has used. For, as we have just seen, our system (1) will be transformed by the ν solutions z_i, $i=1, 2, \ldots \nu$ of the equations

$$\eta_1(y)\frac{\partial z_i}{\partial y_1}+\eta_2(y)\frac{\partial z_i}{\partial y_2}+\ldots+\eta_\nu(y)\frac{\partial z_i}{\partial y_\nu}=\lambda_i z_i,\ i=1, 2, \ldots n, \tag{2}$$

into the form

$$\lambda_1 z_1\frac{\partial x_1}{\partial z_1}+\lambda_2 z_2\frac{\partial x_1}{\partial z_2}+\ldots+\lambda_\nu z_\nu\frac{\partial x_1}{\partial z_\nu}=\xi_1(x),$$

$$\lambda_1 z_1\frac{\partial x_2}{\partial z_1}+\lambda_2 z_2\frac{\partial x_2}{\partial z_2}+\ldots+\lambda_\nu z_\nu\frac{\partial x_2}{\partial z_\nu}=\xi_2(x),$$

....

$$\lambda_1 z_1\frac{\partial x_n}{\partial x_1}+\lambda_2 z_2\frac{\partial x_n}{\partial x_2}+\ldots+\lambda_\nu z_\nu\frac{\partial x_n}{\partial z_\nu}=\xi_n(x).$$

The existence of holomorphic solutions of this system as well as equations (2) are already known. Hence the systen (1) has n holomorphic solutions. The arbitrary choice of $\eta_1, \eta_2 \ldots \eta_\nu$ in the problem of the reduction of § 5 serves us nothing

At the end of the section, we remark that the system (1) is an extended case of the problem of § 1, and hence the proof of the existence-theorem should go parallel with each other.

Poincaré's conditions, are not necessary conditions for the existence of solutions; but there are infininitely may equations in which Poincaré's conditions are not satisfied. This case will be considered in the future, and enter into the problem where the first terms of the functions $\xi_1(x)$, $\xi_2(x)$, ... $\xi_n(x)$ consist from linear homogeneous functions of the variables.

9. Being given the equation

$$Xf=\xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\ldots+\xi_n(x)\frac{\partial f}{\partial x_n}=0,$$

where
$$\xi_1(x) = \lambda_{11}x_1 + \lambda_{12}x_2 + \dots + \lambda_{1n}x_n + \dots ,$$
$$\xi_2(x) = \lambda_{21}x_1 + \lambda_{22}x_2 + \dots + \lambda_{2n}x_n + \dots ,$$
$$\dots\dots\dots$$
$$\xi_n(x) = \lambda_{n1}x_1 + \lambda_{n2}x_2 + \dots + \lambda_{nn}x_n + \dots ,$$

in which only the terms of first order are written, we try to transform these terms of first order in the simplest forms by the transformation of variables
$$x_1 = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n,$$
$$x_2 = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n,$$
$$\dots\dots\dots$$
$$x_n = a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n.$$

Let L and A mean the matrices formed of $\lambda_{11}, \lambda_{12}, \dots \lambda_{1n}, \dots \lambda_{nn}$, and $a_{11}, a_{12}, \dots a_{1n}, \dots a_{nn}$ respectively, then the matrix formed of the coefficients of first order of the variables $y_1, y_2, \dots y_n$, in the transformed equation, is
$$A^{-1} LA.$$

When the n roots $\lambda_1, \lambda_2, \dots \lambda_n$ of the characteristic equation of L
$$\begin{vmatrix} \lambda_{11}-\lambda & \lambda_{12} & \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22}-\lambda & \dots & \lambda_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nn}-\lambda \end{vmatrix} = 0$$
are different from each other, then by suitable choice of A, the matrix can be transformed in a multiplication $(\lambda_1, \lambda_2 \dots \lambda_n)$. This case is treated in the preceding sections,

The case where the characteristic equation has multiple roots was not treated by Poincaré. Whether this case has been treated by other mathematicians, I don't know. In this, by the theory of substitutions, we can so find a matrix A that the transformed matrix of L, i.e., $A^{-1}LA$ will take the form, the number of the multiple roots being ν,
$$\begin{pmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & L_\nu \end{pmatrix},$$

where $L_i, i=1, 2, \dots \nu$ is itself a matrix whose diagonal elements are all $\lambda_i, i=1, 2, \dots \nu$, a multiple root of the preceding characteristic equation, and the elements over the diagonal are all zero. We remark that when the matrix L belongs to a group of substitution of finite order, it can be transformed into a multiplication.

10. Suppose that the equation has been already transformed and is given in the form

$$Xf \equiv \xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\dots+\xi_{n_1}(x)\frac{\partial f}{\partial x_{n_1}}$$
$$+\dots+\xi_n(x)\frac{\partial f}{\partial x_n}=0, \qquad (1)$$

where, writing only the terms of first order

$$\begin{aligned}
\xi_1(x) &= \lambda_1 x_1+\dots, \\
\xi_2(x) &= \lambda_{21}x_1+\lambda_1 x_2+\dots, \\
&\dots\dots\dots \\
\xi_{n_1}(x) &= \lambda_{n_1 1}x_1+\lambda_{n_1 2}x_2+\dots+\lambda_1 x_{n_1}+\dots, \\
\xi_{n_1+1}(x) &= \lambda_2 x_{n_1+1}+\dots, \\
&\dots\dots\dots
\end{aligned} \qquad (2)$$

The matrix L made of the coefficients of terms of first order of the variables $x_1, x_2, \dots x_n$ has a normal determinant $|L|$, the elements over the diagonal are all zero. The vanishing of certain elements under the diagonal elements is not at present necessary.

In the following, we investigate whether the successive calculation of the differential equation

$$Xf=\lambda_1 f \qquad (3)$$

is possible or not and then search for a holomorphic solution.

If we differentiate the equation (3) p_1 times with respect to x_1, p_2 times with respect to $x_2, \dots$ p_n times with respect to x_n and then put $x_1=x_2=\dots=x_n=0$, we will obtain an equation between several differential coefficient at $x_1=x_2=\dots=x_n=0$. Put $p_1+p_2+\dots+p_n\equiv m$, then the highest derivatives are of order m, in which the derivative $\left(\frac{\partial^m f}{\partial x_1^{p_1}\partial x_2^{p_2}\dots\partial x_n^{p_n}}\right)_0$ appears; but in general this is not the only derivative of order m. Therefore, to obtain the values of derivatives of mth order at $x_1=x_2=\dots=x_n=0$, we must solve a system of simultaneous equations of first order. The following shows that *the determi-*

nant made with the coefficients of derivatives of m^{th} order in this system of simultaneous equations is a normal determinant.

To prove this, let us introduce some abbriviations and definitions.

a. Let us write, for simplicity

$$(x_i^{p_i}\, x_j^{p_j} \ldots x_l^{p_l}) \equiv \left(\frac{\partial^{p_i+p_j+\ldots+p_l} f}{\partial x_i^{p_i}\, \partial x_j^{p_j} \ldots \partial x_l^{p_l}} \right)_{x_1=x_2=\ldots=x_n=0},$$

where the *order of the suffixes* are so fixed that

$$1 \leq i < j < \ldots < l \leq n,$$

and call this new expression the *differential quotient* or *quotient* of $(p_i+p_j+\ldots+p_l)$th order, often omiting the order.

b. With two quotients of the same order

$$\text{(I).}\quad (x_a^{p_a} x_b^{p_b} \ldots x_h^{p_h} \ldots), \quad 1 \leq a < b < \ldots < h < \ldots \leq n,$$

$$\text{(II).}\quad (x_\alpha^{q_\alpha} x_\beta^{q_\beta} \ldots x_\eta^{q_\eta} \ldots), \quad 1 \leq \alpha < \beta < \ldots < \eta < \ldots \leq n,$$

where $$p_a + p_b + \ldots + p_h = q_\alpha + q_\beta + \ldots + q_\eta,$$

if $x_h^{p_h}$ and $x_\eta^{q_\eta}$ be such in the first pair, lying in the same place counted from the left, that

$$h \neq \eta, \quad \text{or} \quad p_h \neq q_\eta \quad \text{and} \quad h = \eta.$$

When $h < \eta$, let us say that the quotient (I) is *of higher stage* than the quotient (II), or simply, (I) is *higher* than (II); or the latter is lower than the former, when $h=\eta$ yet $p_h > q_\eta$, we also say that the quotient (I) is *higher* than the quotient (II), and the like.

c. Now take the quotient $(x_i^{p_i} \ldots x_h^{p_h} x_k^{p_k})$ of $(p_i + \ldots + p_h + p_k)$th order. If $h+1=k$, then with respect to $(x_i^{p_i} \ldots x_h^{p_h} x_{h+1}^{p_{h+1}})$, we say that

$$(x_i^{p_i} \ldots x_h^{p_h+1} x_n^{p_{h+1}-1}) \quad \text{is } \textit{next higher than} \quad (x_i{}^{i} \ldots x_h^{p_h}\, x_{h+1}^{p_{h+1}}).$$

If $h+1<k$, then with respect to $(x_i^{p_i} \ldots x_h^{p_h} x_k^{p_k})$, we say that

$$(x_i^{p_i} \ldots x_h^{p_h}\, x_{k-1}\, x_n^{p_k-1}) \quad \text{is } \textit{next higher than} \quad (x_i^{p_i} \ldots x_h^{p_h}\, x_k^{p_k}).$$

Thus we could give a definite stage to each differential qnotients of mth order at $x_1=x_2=\ldots=x_n=0$; and, according to this definition, (x_n^m) is of the lowest and (x_1^m) is of the highest stage.

Now differentiate the equation

$$Xf = \xi_1(x)\frac{\partial f}{\partial x_1} + \xi_2(x)\frac{\partial f}{\partial x_2} + \dots + \xi_n(x)\frac{\partial f}{\partial x_n} = \lambda_1 f, \qquad (3)$$

p_i times with respect to $x_i, \dots p_h$ times with respect to $x_h, \dots$, where $p_i + \dots + p_h + \dots = m$, and put $x_1 = x_2 = \dots = x_n = 0$, then we obtain a linear equation containing the quotient $(x_i^{p_i} \dots x_h^{p_h} \dots)$. The remaining quotients of mth order in the equation must be of lower stage than $(x_i^{p_i} \dots x_h^{p_h} \dots)$. For, in general, with respect to the suffix h, the coefficients $\xi_1(x)$, $\xi_2(x), \dots \xi_{h-1}(x)$ don't contain the variable x_h in their terms of first order. Therefore any quotient, the exponent of whose variabie x_h is less than p_h and whose other variable before x_h is greater than that of $(x_i^{p_i} \dots x_h^{p_h} \dots)$ cannot appear, i.e., any quotient higher than $(x_i^{p_i} \dots x_h^{p_h} \dots)$ cannot appear in that equation.

The number of the simultaneous linear equations is equal to the number of the derivatives of order m. It is given by the coefficients of mth order in the expansion of $(1 + x + x^2 + \dots + x^m)^n$. Now arrange all the linear equations in such an order that, when any two consecutive equations are taken, the highest quotient in the one is *next higher* (or next lower) than the highest quotient in the other, the equations containing the quotient (x_1^m) and (x_n^m) standing at both ends. The equation which contains (x_n^m) as the highest cannot contain any quotient of mth order. Thus *the determinant made with the coefficients of all the quotients of mth order in the simultaneous linear equations, arranged in the order stated above, is a normal determinant; and hence, in each solution of a linear equation containing one unknown, all the quotients of mth order may be calculated successively, from a lower to a higher stage.*

Let us again take up the differential equation (3);

$$\xi_1(x)\frac{\partial f}{\partial x_1} + \xi_2(x)\frac{\partial f}{\partial x_2} + \dots + \xi_n(x)\frac{\partial f}{\partial x_n} \equiv \lambda_1 f. \qquad (3)$$

Differentiate this equation by $x_1, x_2, \dots x_n$ respectively and put $x_1 = x_2 = \dots x_n = 0$, then we know that there is no contradiction provided

$$\left(\frac{\partial f}{\partial x_2}\right)_0 = \left(\frac{\partial f}{\partial x_3}\right)_0 = \dots = \left(\frac{\partial f}{\partial x_n}\right)_0 = 0, \qquad (4)$$

and $$\left(\frac{\partial f}{\partial x_1}\right)_0 = \text{arbitrary.}$$

Let us fix so that

$$\left(\frac{\partial f}{\partial x_1}\right)_0 = +1. \tag{4}$$

Now write the equation (3) as follows:

$$\lambda_1 x_1 \frac{\partial f}{\partial x_1} + \lambda_1 x_2 \frac{\partial f}{\partial x_2} + \dots + \lambda_1 x_{n_1} \frac{\partial f}{\partial x_{n_1}} + \dots + \lambda_\nu x_n \frac{\partial f}{\partial x_n} - \lambda_1 f$$
$$= \varphi_1 \frac{\partial f}{\partial x_1} + \varphi_2 \frac{\partial f}{\partial x_2} + \dots + \varphi_n \frac{\partial f}{\partial x_n}, \tag{5}$$

where the function $\varphi_i, (i=1, 2, \dots n)$ is obtained by replacing the function $\xi_i(x), (i=1, 2, \dots n)$ except the term containing the first power of $x_i (i=1, 2, \dots n)$ from the left-hand side to the right-hand side. Differentiate the equation (5) p_1 times with respect to x_1, p_2 times with respect to $x_2, \dots p_n$ times with respect to x_n and put $x_1=x_2=\dots=x_n=0$, then on the left-hand side, there remains the term

$$\{((p_1-1)+p_2+\dots+p_{n_1})\lambda_1+\dots+(\dots+p_n)\lambda_\nu\}(x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}).$$

The right-hand side is a homogeneous linear expression between quotients of orders lower than $p_1+p_2+\dots+p_n$ and of stages lower than $(x_1^{p_1}x_2^{p_2}\dots x_n^{p_n})$. Now assume that $\lambda_1, \lambda_2, \dots \lambda_\nu$ satisfy Poincare's conditions. Then we can find a definite positive number ε such that

$$\varepsilon \leq \left| \frac{(p_1-1+p_2+\dots+p_{n_1})\lambda_1+\dots+(\dots+p_n)\lambda_\nu}{p_1-1+p_2+\dots+p_{n_1}+\dots+p_n} \right|, \tag{6}$$

for all positive integral numbers $p_1, p_2, \dots p_n$ which satisfy the relation

$$p_1+p_2+\dots+p_n \geqq 2.$$

Therefore, the coefficient of $(x_1^{p_1}x_2^{p_2}\dots x_n^{p_n})$ does not vanish; and hence, for the reason stated above, *all differential quotients of any order at* $x_1=x_2=\dots=x_n=0$ *can be calculated successively, the initial value being given by* (4).

On the other hand, it is always possible to determine *n positive* numbers $\varepsilon_1, \varepsilon_2, \dots \varepsilon_n$ which are *different from one another* and satisfy the following inequalities

$$\frac{\varepsilon_1}{2} < \varepsilon_i < \frac{\varepsilon+\varepsilon_1}{2} < \varepsilon, \quad i=2, 3, \dots n. \tag{7}$$

For, take a positive number ε_1 less than ε, then it satisfies the inequalities

$$\frac{\varepsilon_1}{2} < \frac{\varepsilon+\varepsilon_1}{2} < \varepsilon,$$

and we can insert between $\frac{\varepsilon_1}{2}$ and $\frac{\varepsilon+\varepsilon_1}{2}$, $n-1$ positive numbers ε_2, $\varepsilon_3 \ldots \varepsilon_n$, which are different from one another and from ε_1.
These inequalities show that

a. $$2\varepsilon_i > \varepsilon_1, \; i=2,3, \ldots n, \tag{8}$$

and also

$$2\varepsilon_j > \varepsilon_1, \; j=2, 3, \ldots n.$$

Adding these two inequalities and dividing by 2, we have

$$\varepsilon_i+\varepsilon_j > \varepsilon_1, \; i,j = 2, 3, \ldots n. \tag{9}$$

By (8) and (9) it follows that

$$-\varepsilon_1+p_2\varepsilon_2+p_3\varepsilon_3+\ldots+p_n\varepsilon_n > 0,$$

for any positive integral values $p_2, p_3, \ldots p_n$ (zero inclusive) satisfying the relation

$$p_2+p_3+\ldots+p_n \geqq 2\,;$$

whence it follows tnat, since $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ are all positive,

$$(p_1-1)\varepsilon_1+p_2\varepsilon_2+\ldots+p_n\varepsilon_n > 0$$

for any positive integral values $p_1, p_2, \ldots p_n$ (zero inclusive) such that

$$p_1+p_2+\ldots+p_n \geqq 2.$$

Thus *the n constants* $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ *satisfy Poincaré's conditions.*

b. The inequalities (7) show that

$$\varepsilon^i < \varepsilon, \; i = 1, 2, \ldots n\,; \tag{10}$$

moreover that

$$2\varepsilon_i < \varepsilon+\varepsilon_1, \quad \text{or} \quad -\varepsilon_1+2\varepsilon_i < \varepsilon, \; i = 2, 3, \ldots n, \tag{11}$$

and also

$$-\varepsilon_1+2\varepsilon_j < \varepsilon, \qquad j = 2, 3, \ldots n.$$

Adding these two inequalities and dividing by 2, we obtain

$$-\varepsilon_1+\varepsilon_i+\varepsilon_j < \varepsilon, \; i,j = 2, 3, \ldots n. \tag{12}$$

By (10), (11) and (12), we have

$$(p_1-1)\varepsilon_1+p_2\varepsilon_2+\ldots+p_n\varepsilon_n<(p_1-1+p_2+\cdots+p_n)\varepsilon,$$

provided

$$p_1+p_2+\ldots+p_n\geqq 2;$$

whence it follows that

$$\frac{(p_1-1)\varepsilon_1+p_2\varepsilon_2+\ldots+p_n\varepsilon_n}{p_1-1+p_2+\ldots+p_n}<\varepsilon.$$

Comparing this inequality with the inequality (6), we see that

$$(p_1-1)\varepsilon_1+p_2\varepsilon_2+\ldots+p_n\varepsilon_n<|(p_1-1+p_2+\ldots+p_{n_1})\lambda_1+\ldots+(\ldots+p_n)\lambda_\nu|. \tag{13}$$

Now consider the equation

$$\begin{aligned}&\varepsilon_1x_1\frac{\partial f}{\partial x_1}+\varepsilon_2x_2\frac{\partial f}{\partial x_2}+\ldots+\varepsilon_nx_n\frac{\partial f}{\partial x_n}-\varepsilon_1 f\\ &=\varphi_1'\frac{\partial f}{\partial x_1}+\varphi_2'\frac{\partial f}{\partial x_2}+\ldots+\varphi_n'\frac{\partial f}{\partial x_n},\end{aligned} \tag{14}$$

where the modulas of each coefficient of $\varphi_i(i=1,2,\ldots n)$ is the corresponding coefficient of $\varphi_i'\,(i=1,2,\ldots n)$.
Now take for initial values, the values given by (4), then from (14) all values of differential quotients of any order at $x_1=x_2=\ldots=x_n=0$ can be calculated successively, and moreover they are all positive. Since the laws of calculation of the values of differential quotients at $x_1=x_2=\ldots=x_n=0$ from the equations (5) and (14), comparing the constants in (5) and (14), and noticing the inequality (13), we conclude that *all the absolute values of differential quotients of any order at $x_1=x_2=\ldots=x_n=0$, calculated from the equation* (5) *are less than the values of corresponding differential quotients at $x_1=x_2=\ldots=x_n=0$ calculated from the equation* (14).

Since the determinant made with the coefficients of terms of first order in $x_1, x_2, \ldots x_n$ in (14) is a normal determinant whose diagonal elements are $\varepsilon_1, \varepsilon_2, \ldots\varepsilon_n$, the roots of the characteristic equation corresponding to the determinant are $\varepsilon_1, \varepsilon_2, \ldots\varepsilon_n$ and they are all different one from another. Therefore by § 9, the equotion (14) may be transformd into the form;

$$(\varepsilon_1y_1+\ldots)\frac{\partial f}{\partial y_1}+(\varepsilon_2y_2+\ldots)\frac{\partial f}{\partial y_2}+\ldots+(\varepsilon_ny_n+\ldots)\frac{\partial f}{\partial y_n}=\varepsilon_1 f, \tag{15}$$

where the dotted parts mean terms of higher order than first. Now

$\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ satisfy Poincaré's conditions, and this equation has a holomorphic solution as stated in § 1. Therefore *the equation* (14) *has a holomorphic solution by the initial conditions* (4), *since in transforming the equation* (14) *into* (15) *the vasiable* x_1 *does not change, i.e.,* $x_1 \equiv y_1$, Therefore by the preceding paragraph, *our proposed equation* (5), *i.e.,* (3) *has a holomorphic solution such as*

$$f_1 = x_1 + \ldots ,$$

writing only the term of first order.

11. Next we search for a holomorphic solution of the differential equation

$$Xf = \lambda_1 f + \lambda_{21} f_1, \tag{1}$$

or, writing as equation (5), § 10, and using the functions $\varphi_1, \varphi_2, \ldots \varphi_n$,

$$\begin{aligned} \lambda_1 x_1 \frac{\partial f}{\partial x_1} + \lambda_1 x_2 \frac{\partial f}{\partial x_2} + \ldots + \lambda_\nu x_n \frac{\partial f}{\partial x_n} - \lambda_1 f \\ = \varphi_1 \frac{\partial f}{\partial x_1} + \varphi_2 \frac{\partial f}{\partial x_2} + \ldots + \varphi_n \frac{\partial f}{\partial x_n} + \lambda_{21} f_1. \end{aligned} \tag{2}$$

In calculating the values of derivatives at $x_1 = x_2 = \ldots = x_n = 0$, we may take the following initial conditions

$$\left(\frac{\partial f}{\partial x_2}\right)_0 - 1 = \left(\frac{\partial f}{\partial x_i}\right)_0 = 0, \quad i = 1, 3, 4, \ldots n. \tag{3}$$

Thereby consider the equation as in (14), § 10, using the functions $\varphi_1' \varphi_3' \ldots \varphi_n'$,

$$\begin{aligned} \varepsilon_2 x_1 \frac{\partial f}{\partial x_1} + \varepsilon_1 x_2 \frac{\partial f}{\partial x_2} + \ldots + \varepsilon_n x_n \frac{\partial f}{\partial x_n} - \varepsilon_1 f \\ = \varphi_1' \frac{\partial f}{\partial x_1} + \varphi_2' \frac{\partial f}{\partial x_2} + \ldots + \varphi_n' \frac{\partial f}{\partial x_n} + |\lambda_{21}| f_1', \end{aligned} \tag{4}$$

where $|\lambda_{21}|$ means the absolute value of λ_{21} and the coefficients of f_1' are the absolute values of the corresponding coefficients in f_1. Now differentiate the equation (4) by x_1 and put $x_1 = x_2 = \ldots = x_n = 0$, then between the quotients of first order we have the equation

$$\varepsilon_2(x_1) - \varepsilon_1(x_1) = |\lambda_{21}|(x_2) + |\lambda_{31}|(x_3) + \ldots + |\lambda_{21}|.$$

Assume $(x_i) = 0$, $i = 3, 4, \ldots n$, then we have

$$(x_2) = \frac{\varepsilon_2 - \varepsilon_1}{|\lambda_{21}|}(x_1) - 1.$$

We may take, without any contradiction against the inequalities (7), § 10, so that $\varepsilon_2 > \varepsilon_1$. Hence as the initial condition we must take

$$\left(\frac{\partial f}{\partial x_1}\right)_0 = \frac{2|\lambda_{21}|}{\varepsilon_2 - \varepsilon_1} > 0,$$

$$\left(\frac{\partial f}{\partial x_2}\right)_0 - 1 = \left(\frac{\partial f}{\partial x_i}\right)_0 = 0, \quad i = 3, 4, \dots n. \tag{5}$$

under these initial conditions, we see easily that the solution calculated from the equation (4) has all positive coefficients and serves as a *fonction majorante* of that calculated from (2).

To prove the existence of a holomorphic solution under these initial conditions, put

$$f = \frac{2|\lambda_{21}|}{\varepsilon_2 - \varepsilon_1} x_1 + x_2 + v,$$

then since φ_1' does not contain any term of first order, while φ_2' has $|\lambda_{21}|x_1$, we have the following equation

$$\varepsilon_2 x_1 \frac{\partial v}{\partial x_1} + \varepsilon_1 x_2 \frac{\partial v}{\partial x_2} + \dots + \varepsilon_n x_n \frac{\partial v}{\partial x_n} - \varepsilon_1 v$$

$$= \theta_1 \frac{\partial v}{\partial x_1} + \theta_2 \frac{\partial v}{\partial x_2} + \dots + \theta_n \frac{\partial v}{\partial x_n} + \theta,$$

where

$$\theta_2 = \varphi_2' - |\lambda_{21}|x_1, \quad \theta = |\lambda_{21}|(f_1' - x_1) + \frac{2|\lambda_{21}|}{\varepsilon_2 - \varepsilon_1}\varphi_1' + \theta_2,$$

$$\theta_i = \varphi_i', \; i = 1, 3, 4, \dots n,$$

and θ does not contain terms of first order. Now by the transformation of variables, as considered in the foregoing section, this equation may be transformed into the form

$$(\varepsilon_2 y_1 + \dots) \frac{\partial v}{\partial y_1} + (\varepsilon_1 y_2 + \dots) \frac{\partial v}{\partial y_2} + \dots + (\varepsilon_n y_n + \dots) \frac{\partial v}{\partial y_n} = \varepsilon_1 v + \chi,$$

wehre χ *does not contain terms of first order.* Compare this equation with the equation (3), § 1. In our case ψ will increase by χ. But for proof of the existence of the integral, it is enough that the function ψ does not contain terms of first order. Hence our equation, also has a holomorphic solution $v(y)$ with no terms of first order. Consequently the equation (4) has a holomorphic solution under the initial conditions (5). Hence *our proposed equation* (2), *i.e.*, (1) *has a holomorphic solution under the initial conditions* (3), *a solution such as*

$$f_2 = x_2 + \dots .$$

Specially when $\lambda_{21}=0$, the equation (2) will be identical with (5), § 10. If we replace x_2 for x_1, we have also a holomorphic solution f_2.

The preceding considerations are quite general, and we arrive at the result: *There exist n_1 holomorphic solutions $f_1, f_2, \dots f_{n_1}$ of the differential equations*

$$\begin{aligned} Xf_1 &= \lambda_1 f_1, \\ Xf_2 &= \lambda_{21} f_1 + \lambda_1 f_2, \\ &\dots\dots\dots \\ Xf_{n_1} &= \lambda_{n_1 1} f_1 + \lambda_{n_1 2} f_2 + \dots + \lambda_1 f_{n_1}, \end{aligned} \tag{6}$$

where, writing only the terms of first oder

$$\begin{aligned} f_1 &= x_1 + \dots \\ f_2 &= x_2 + \dots \\ &\dots\dots\dots \\ f_{n_1} &= x_{n_1} + \dots \end{aligned} \tag{7}$$

Since there are ν multiple roots, proceeding as in the foregoing, we can prove the existence of $\nu-1$ groups of holomorphic solutions as (7), each group corresponding to the multiple root $\lambda_2, \lambda_3, \dots$ or, λ_ν.

Let us take these n holomorphic solutions as new independent variables such that

$$\begin{aligned} y_1 &= f_1, \\ y_2 &= f_2, \\ &\dots\dots\dots \\ y_{n_1} &= f_{n_1}, \quad \text{and the like}; \end{aligned}$$

and transform the differential expression Xf of the equation (1), § 10, then we obtain

$$Yf \equiv \eta_1(y)\frac{\partial f}{\partial y_1} + \eta_2(y)\frac{\partial f}{\partial y_2} + \dots + \eta_{n_1}(y)\frac{\partial f}{\partial y_{n_1}} + \dots + \eta_n(y)\frac{\partial f}{\partial y_n}, \tag{8}$$

where

$$\begin{aligned} \eta_1(y) &= \lambda_1 y_1, \\ \eta_2(y) &= \lambda_{21} y_1 + \lambda_1 y_2, \end{aligned}$$

....

$$\eta_{n_1}(y) = \lambda_{n_1 1} y_1 + \lambda_{n_1 2} y_2 + \dots + \lambda_1 y_{n_1},$$

and the like. This transformation is a more general case of that in § 7.

Thus the problem of the differential equation

$$Xf = 0 \qquad (9)$$

is reduced to the problem of the differential cquation

$$Yf = 0. \qquad (10)$$

12. According to the *Encyclopédie*[1] (loc. cit.), Mr. Bendixson solved, after Poincaré, the system of differential equations

$$\frac{dx_1}{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n} = \frac{dx_2}{a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n} = \dots$$
$$= \frac{dx_n}{a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n}, \qquad (1)$$

for the case where the characteristic equation

$$\Delta(\lambda) \equiv \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} - \lambda & \dots a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots a_{nn} - \lambda \end{vmatrix} = 0,$$

has multiple roots and satisfy Poincaré's conditions. When λ be a j-ple root of the characteristic equation, he showed that the equation has $j-1$ integrals of the form

$$\frac{T_1}{t^\lambda}, \quad \frac{T_2 + T_1 \log t}{t^\lambda}, \quad \dots \quad \frac{T_j + T_{j-1} \log t + \dots + \frac{1}{(j-1)!} T_1 (\log t)^{j-1}}{t^\lambda}$$

t being eliminated and $T_1, \dots T_j$ being holomorph. But, according to this book, nothing is said, explicitly, as to whether $a_{11}, \dots a_{nn}$ are constants or unit functions; and reference to his original papers is for me impossible, since his literature is not at my hand. Mr. Dulac treated the case of two independent variables and said in his memoir[2] that his method and that of Mr. Bendixson are applicable for the general

1 Tome II, vol. 3. Fasc. 1, pp. 49–51.

2 Jour. de L'école poly., 9, pp. 50–59 (1904).

case of n variables in so far as the characteristic equation has a root of any order.

Now to find the solutions of the epuation (10), § 11 is easy. We consider first the following simple equation

$$Zf = \zeta_1(z)\frac{\partial f}{\partial z_1} + \zeta_2(z)\frac{\partial f}{\partial z_2} + \dots + \zeta_h(z)\frac{\partial f}{\partial z_h} = 0, \qquad (1)$$

where

$$\begin{aligned} \zeta_1(z) &= az_1, \\ \zeta_2(z) &= a_{21}z_1 + az_2, \\ &\dots\dots\dots \\ \zeta_h(z) &= a_{h1}z_1 + a_{h2}z_2 + \dots + az_h. \end{aligned}$$

Using h relations of the forms

$$\begin{aligned} Z(z_1) &= \zeta_1(z), \\ Z(z_2) &= \zeta_2(z), \\ &\dots\dots\dots \\ Z(z_h) &= \zeta_h(z), \end{aligned}$$

we can easily prove that the equation (1) has $h-1$ solutions of the forms

$$f_1 = \frac{z_2}{z_1} + \alpha_1 \log z_1,$$

$$f_2 = \frac{z_3}{z_1} + \beta_1 \frac{z_2}{z_1}\log z_1 + \beta_2 \log z_1 + \beta_3(\log z_1)^2,$$

$$\dots\dots\dots$$

$$f_{h-1} = \frac{z_h}{z_1} + \gamma_1 \frac{z_{h-1}}{z_1}\log z_1 + \dots + \gamma_{h-2}\frac{z_2}{z_1}(\log z_1)^{h-2} + \gamma_{h-1}\log z_1 + \dots + \gamma_{2h-3}(\log z_1)^{h-1}.$$

The equation (10), § 11 is a mere combination of ν equations like as (1). Hence after this lemma, it is clear that the equation (10), § 11 has

$$(n_1-1)+(n_2-1)+\dots+(n_\nu-1) = n-\nu$$

solutions of the form stated above. Moreover since $\lambda_1, \lambda_2, \dots\lambda_\nu$ are different from each other, this equation has $\nu-1$ integrals:

$$y_1^{-\frac{1}{\lambda_1}}\, y_{n_1+1}^{\frac{1}{\lambda_2}}, \quad y_1^{-\frac{1}{\lambda_1}}\, y_{n_1+n_2+1}^{\frac{1}{\lambda_2}}, \dots y_1^{-\frac{1}{\lambda_1}}\, y_{n-n_\nu+1}^{\frac{1}{\lambda_\nu}}.$$

Thus, adding them up, *our equation has $n-1$ independent solutions, and* *hence our proposed equation* (1), § 10 *has* $n-1$ *independent solutions,* $n-\nu$ *of which contain, in general, logarithmic function; and the problem of Poincaré is completed.*

In the following we shall consider system of partial differential equations such as treated by Poincaré, and try the extension of Poincaré's theorem about a complete system.

13. In the first place, let us consider two partial differential equations under certain conditions. Now, let Xf and Yf be, as above

$$Xf \equiv \xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\dots+\xi_n(x)\frac{\partial f}{\partial x_n},$$

$$Yf \equiv \eta_1(x)\frac{\partial f}{\partial x_1}+\eta_2(x)\frac{\partial f}{\partial x_2}+\dots+\eta_n(x)\frac{\partial f}{\partial x_n}.$$

The functions $\xi_i(x)$, $\eta_i(x), i=1, 2, \dots n$ commence with terms of first order. The matrices formed with the coefficients of terms of first order are A and B. Then we obtain by the composition of Xf with Yf, an expression of the same type,

$$(XY)f=\sum_{i=1}^{n}(X\eta_i-Y\xi_i)\frac{\partial f}{\partial x_i},$$

and let C be the corresponding matrix with respect to this, then we see easily that $C=B\cdot A$. Therefore, when $(XY)f\equiv 0$, or using the words of Lie, *when the inifinitesimal transformations Xf and Yf are permutable, then the substitutions A and B must be also permutable.* We shall consider this case alone.

When A is possible of transformation into a multiplication, then B also must be so. Hence we assume, from the beginning, that

$$\xi_i(x) = \lambda_i x_i+\dots,$$
$$\eta_i(x) = \mu_i x_i+\dots, \quad i=1, 2, \dots n,$$

where the dotted parts stand for terms of higher order. If $\lambda_i, (i=1, 2, \dots n)$ satisfy Poincaré's conditions, then, by the transformation given by

$$Xy_i = \lambda_i y_i, \quad i = 1, 2, \dots n,$$

Xf will be transformed into

$$X'f \equiv \lambda_1 y_1\frac{\partial f}{\partial y_1}+\lambda_2 y_2\frac{\partial f}{\partial y_2}+\dots+\lambda_n y_n\frac{\partial f}{\partial y_n}.$$

Yf will be transformed into

$$Y'f \equiv \sum_{i=1}^{n} Yy_i \frac{\partial f}{\partial y_i}.$$

But since $(XY)f$ is invariant for the transformation of variables, it must hold that

$$(X'Y')f = \sum_{i=1}^{n} \left\{ X'(Yy_i) - Y'(\lambda_i\, y_i) \right\} \frac{\partial f}{\partial y_i} \equiv 0.$$

For that we must have

$$X'(Yy_i) \equiv Y'(\lambda_i y_i),\ i = 1, 2, \dots n,$$

or
$$X'(Yy_i) \equiv \lambda_i Yy_i,$$

i.e., Yy_i must be the solution of the evuation

$$X'f = \lambda_i f.$$

As we have said before, the solution of this equation is of the form

$$Ay_i.$$

But, by calculation we know that

$$Yy_i = \mu_i y_i + \dots ,$$

and therefore

$$Yy_i = \mu_i y_i,\ i = 1, 2, \dots n.$$

Thus *when two differential expressions Xf and Yf are such that $(XY)f \equiv 0$, and the coefficients $\lambda_1, \lambda_2, \dots \lambda_n$ of the former satisfy Poincaré's conditions, then by the transformation given by*

$$Xy_i \equiv \lambda_i y_i,\ i = 1, 2, \dots n,$$

both expressions may be transformed into

$$X'f = \sum_{i=1}^{n} \lambda_i y_i \frac{\partial f}{\partial y_i},\ \text{resp. } Y'f = \sum_{i=1}^{n} \mu_i y_i \frac{\partial f}{\partial y_i}.$$

The inverve is also true. In this theorem, whether $\mu_1, \mu_2, \dots \mu_n$ satisfy Poincaré's conditions or not is of no concern; hence the result follows;

The necessary and sufficient condition of existence of $n-1$ algebroidal solutions of the equation $Yf=0$, is that there gives an equation of the same form $Xf=0$, such that $(XY)f \equiv 0$, $\lambda_1, \lambda_2, \dots \lambda_n$ satisfying Poincaré's conditions. This is Lie's theorem at singulality.

14. Next consider a system of r equations which are permutable with one another,

$$X_1 f \equiv \xi_{11}(x)\frac{\partial f}{\partial x_1} + \xi_{12}(x)\frac{\partial f}{\partial x_2} + \dots + \xi_{1n}(x)\frac{\partial f}{\partial x_n} = 0,$$

$$X_2 f \equiv \xi_{21}(x)\frac{\partial f}{\partial x_1} + \xi_{22}(x)\frac{\partial f}{\partial x_2} + \dots + \xi_{2n}(x)\frac{\partial f}{\partial x_n} = 0,$$

....

$$X_r f \equiv \xi_{r1}(x)\frac{\partial f}{\partial x_1} + \xi_{r2}(x)\frac{\partial f}{\partial x_2} + \dots + \xi_{rn}(x)\frac{\partial f}{\partial x_n} = 0,$$

where, writing only the first term.

$$\xi_{ij}(x) = \lambda_{ij}x_j + \dots, \quad i = 1, 2, \dots r; \quad j = 1, 2, \dots n; \quad r < n,$$

and suppose that $\lambda_{11}, \lambda_{12}, \dots \lambda_{1n}$ obey Poincaré's conditions. Then, by the theorem stated above, this system may be transformed into

$$X_1' f = \sum \lambda_{1i} y_2 \frac{\partial f}{\partial y_i},$$

$$X_2' f = \sum \lambda_{2i} y_i \frac{\partial f}{\partial y_i},$$

....

$$X_r' f = \sum \lambda_{r1} y_i \frac{\partial f}{\partial y_i}.$$

When $X_1 f, X_2 f, \dots X_r f$ and hence $X_1' f, X_2' f, \dots X_r' f$ are independent, the rank of the matrix

$$\begin{Vmatrix} \lambda_{11} & \lambda_{.2} & \dots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda_{r1} & \lambda_{r2} & \dots & \lambda_{rn} \end{Vmatrix}$$

must be r. Suppose the determinant made by first r columns does not vanish, then if the function

$$f = y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}$$

satisfy the system of equations, there follows

$$\lambda_{11}m_1 + \lambda_{12}m_2 + \dots + \lambda_{1r}m_r + \dots + \lambda_{1n}m_n = 0,$$

$$\lambda_{21}m_1+\lambda_{22}m_2+\ldots+\lambda_{2r}m_r+\ldots+\lambda_{2n}m_n=0,$$

$$\ldots\ldots\ldots\ldots\ \ldots\ldots$$

$$\lambda_{r1}m_1+\lambda_{r2}m_2+\ldots+\lambda_{rr}m_r+\ldots+\lambda_{rn}m_n=0,$$

whence we have

$$m_1=k_{11}m_{r+1}+k_{12}m_{r+2}+\ldots+k_{1n-r}m_n,$$
$$m_2=k_{21}m_{r+1}+k_{22}m_{r+2}+\ldots+k_{1n-r}m_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$m_r=k_{r1}m_{r+1}+k_{r2}m_{r+2}+\ldots+k_{rn-r}m_n,$$

where k_{ij} are constants. Hence our complete system has $n-r$ independent solutions

$$f_1=y_1^{k_{11}}y_2^{k_{21}}\ldots y_r^{k_{r1}}y_{r+1},$$
$$f_2=y_1^{k_{12}}y_2^{k_{22}}\ldots y_r^{k_{r2}}y_{r+2},$$
$$\ldots\ldots\ldots$$
$$f_{n-r}=y_1^{k_{1\,n-r}}y_2^{k_{2\,n-r}}\ldots y_r^{k_{r\,n-r}}y_n.$$

Thus *a complete system made with* $X_1f=X_2f=\ldots=X_rf=0$, *where* $(X_i,X_j)f\equiv0$, $(i,j=1,2,\ldots r)$ *and* n *coefficients of terms of first order in* X_1f *satisfy Poincaré's conditions, has* $n-r$ *independent solutions.*

This is an extension of Poincaré's research. When we go to the general complete system, we must consider some device. If the system has holomorphic solutions, Jacobi's method is sufficient. In the following I give a lemma, and next prove the general theorem.

15. Let us consider a system of m partial differential equations of the form:

$$\begin{aligned}Xf_1&=F_1(x_1,x_2,\ldots x_n;f_1,f_2,\ldots f_m),\\ Xf_2&=F_2(x_1,x_2,\ldots x_n;f_1,f_2,\ldots f_m),\\ &\ldots\ldots\ldots\\ Xf_m&=F_m(x_1,x_2,\ldots x_n;f_1,f_2,\ldots f_m),\end{aligned}\tag{1}$$

where

$$Xf\equiv(\lambda_1x_1+\ldots)\frac{\partial f}{\partial x_1}+(\lambda_2x_2+\ldots)\frac{\partial f}{\partial x_2}+\ldots+(\lambda_nx_n+\ldots)\frac{\partial f}{\partial x_n},$$

$\lambda_1,\lambda_2,\ldots\lambda_n$ satisfying Poincaré's conditions, and $F_i,(i=1,2,\ldots m)$ are holomorphic functions of $x_1,x_2,\ldots x_n$ in certain convergent circles about the origin and also of $f_1,f_2\ldots f_m$ in certain convergent circles whose radii, without loosing the generality, are unity about $f_1=f_2=\dots=f_m=0$. Adding to them, F_i, $(i=1, 2, \dots m)$ have the following forms:

$$F_i(x;f)=\theta_0^{(i)}(x)+\sum_{s=1}^{m}\theta_s^{(i)}(x)f_s+\sum_{s,t=1}^{m}\theta_{s,t}^{(i)}(x)f_s\cdot f_t+\dots, \qquad i=1,2,\dots m, \tag{2}$$

where, since $F_1, F_2, \dots F_m$ are holomorphic functions, all θ's are holomorphic functions, and we assume that they commence at least with terms of first order.

To prove the existence of holomorphic solutions of this system, we transform as usual these equations by the n solutions given by

$$Xy_i=\lambda_i y_i,\ i=1,2,\dots n,$$

then the differential expression Xf will become

$$\lambda_1 y_1\frac{\partial f}{\partial y_1}+\lambda_2 y_2\frac{\partial f}{\partial y_2}+\dots+\lambda_n y_n\frac{\partial f}{\partial y_n},$$

and the hypothesis made upon $F_i(i=1, 2, \dots n)$ will also be fulfilled. Therefore, we assume from the beginning that Xf has already been reduced to this form. We take for initial values of $f_1, f_2, \dots f_m$:

$$(f_i)_0\equiv f_0^{(i)}=0,\ i=1,2,\dots m.$$

Next differentiate the equation (1) by x_1 and put $x_1=x_2=\dots=x_n=0$, then we have

$$\lambda_1\left(\frac{\partial f_i}{\partial x_1}\right)_0=\left(\frac{\partial\theta_0^{(i)}(x)}{\partial x_1}\right)_0,\ i=1,2,\dots m,$$

while all the others vanish, since for example,

$$\sum_{s,t,\dots u=1}^{m}\left(\frac{\partial\theta_{s,t,\dots u}^{(i)}(x)f_sf_t\cdots f_u}{\partial x_1}\right)_0$$

$$=\sum_{s,t,\dots u=}^{m}\left\{\left(\frac{\partial\theta_{s,t,\dots u}^{(i)}(x)}{\partial x_1}\right)_0(f_s\cdot f_t\cdots f_u)_0+(\theta_{s,t,\dots u}^{(i)}(x))_0\left(\frac{\partial(f_sf_t\cdots f_u)}{\partial x_1}\right)_0\right\}=0.$$

The same is true for the remaining variables; and therefore we have the equations:

$$\left(\frac{\partial f_i}{\partial x_1}\right)_0=\frac{\left(\frac{\partial\theta_0^{(i)}(x)}{\partial x_1}\right)_0}{\lambda_1},$$

$$\left(\frac{\partial f_i}{\partial x_2}\right)_0 = \frac{\left(\frac{\partial \theta_0^{(i)}(x)}{\partial x_2}\right)_0}{\lambda_2},$$

$$\cdots\cdots\cdots$$

$$\left(\frac{\partial f_i}{\partial x_n}\right)_0 = \frac{\left(\frac{\partial \theta_0^{(i)}(x)}{\partial x_n}\right)_0}{\lambda_n}; \quad i = 1, 2, \dots m.$$

From these conditions, the coefficients of all the terms of first order in $f_1, f_2 \dots f_m$ are determined uniquely. In general, if we differentiate both sides of (1) p_1 times with respect to x_1, p_2 times with respect to $x_2, \dots p_n$ times with respect to x_n and put $x_1 = x_2 = \dots = x_n = 0$, we obtain from the left-hand side

$$(p_1\lambda_1 + p_2\lambda_2 + \dots + p_n\lambda_n)\left(\frac{\partial^{p_1+p_2+\dots+p_n} f_i}{\partial x_1^{p_1}\, \partial x_2^{p_2} \dots \partial x_n^{p_n}}\right)_0, \quad i = 1, 2, \dots m.$$

But in the right-hand side, the differential quotients of order $p_1 + p_2 + \dots + p_n$ are given in the following manner:

$$\sum\left(\frac{\partial^{p_1+p_2+\dots+p_n}\, \theta^{(i)}_{s,t,\dots u}(x) f_s f_t \dots f_n}{\partial x_1^{p_1}\, \partial x_2^{p_2} \dots \partial x_n^{p_n}}\right)_0$$

$$= \sum\left[(\theta^{(i)}_{s,t,\dots u}(x))_0 \left\{\left(\frac{\partial^{p_1+p_2+\dots+p_n} f_s}{\partial x_1^{p_1}\, \partial x_2^{p_2} \dots \partial x_u^{p_u}} f_t \dots f_u\right)_0 + \dots\right\} + \text{lower orders}\right].$$

Since all θ's vanish at $x_1 = x_2 = \dots = x_n = 0$, the left-hand side give no differential quotients of order $p_1 + p_2 + \dots + p_n$. But the coefficients $\lambda_1, \lambda_2, \dots \lambda_n$ obey Poincare's conditions, the coefficient $p_1\lambda_1 + p_2\lambda_2 + \dots + p_n\lambda_n$ don't vanish, and therefore all the values

$$\left(\frac{\partial^{p_1+p_2+\dots+p_n} f_i}{\partial x_1^{p_1}\, \partial x_2^{p_2} \dots \partial x_n^{p_n}}\right)_0, \quad i = 1, 2, \dots m$$

may be calculated gradually.

Next we prove the convergency of the integrals. Let ε be a positive value such that for any positive integers $p_1, p_2, \dots p_n$ which satisfy

$$p_1 + p_2 + \dots + p_n \geq 1,$$

we have

$$\varepsilon \leq \left| \frac{p_1\lambda_1 + p_2\lambda_2 + \dots + p_n\lambda_n}{p_1 + p_2 + \dots + p_n} \right|,$$

and, assume that all $\theta's$ are holomorphic in circles with radius a about $x_1=x_2=\ldots=x_n=0$, and M be the common maximum value of their absolute values, then

$$\text{all}\left|\theta^{(i)}_{s,t,\ldots u}(|x|)\right|\leq\frac{M}{1-\dfrac{|x_1|+|x_2|+\ldots+|x_n|}{a}}-M.$$

Consider the following system of equations:

$$\varepsilon\left(x_1\frac{\partial v_i}{\partial x_1}+x_2\frac{\partial v_i}{\partial x_2}+\ldots+x_n\frac{\partial v_i}{\partial x_n}\right)=\left(\frac{M}{1-\dfrac{x_1+x_2+\ldots+x_n}{a}}-M\right)$$

$$\times\frac{1}{1-(v_1+v_2+\ldots+v_m)},\quad i=1,2,\ldots m. \tag{3}$$

The right-hand side may be written

$$\equiv\theta_0^{(i)}(x)+\sum_{s=1}^{m}\theta_s^{(i)}(x)\,v_s+\sum_{s,t=1}^{m}\theta^{(i)}_{s,t}(x)v_sv_t+\ldots,$$

all the coefficients in $\theta's$ are positive. Moreover

$$\text{all}\left|\theta^{(i)}_{s,t,\ldots u}(|x|)\right|\leq\Theta^{(i)}_{s,t,\ldots u}(|x|);$$

whence the function standing on the right of (3) is a *fonction majorante* of each F_i. On the other hand the coefficient of $\left(\dfrac{\partial^{p_1+p_2+\ldots+p_n}v_i}{\partial x_1^{p_1}\partial x_1^{p_2}\ldots\partial x_n^{p_n}}\right)_0$ is $\varepsilon(p_1+p_2+\ldots+p_n)$. This is, by the assumption upon ε, less than $|p_1\lambda_1+p_2\lambda_2+\ldots+p_n\lambda_n|$. Therefore, taking for v_i, $i=1,2,\ldots m$, an initial value $\dfrac{M}{\varepsilon a}$ which is greater than the absolute values of the initial values of f_i, $i=1,2,\ldots m$, we have always

$$\left|\left(\frac{\partial^{p_1+p_2+\ldots+p_n}f_i}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\right)_0\right|<\left(\frac{\partial^{p_1+p_2+\ldots+p_n}v_i}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\right)_0\quad i=1,2,\ldots m.$$

Hence the integrals of (3) may be taken as *a fonction majorante* of the solutions of (1).

Now we have to prove the existence of integrals of (3). For which put

$$x_1+x_2+\ldots+x_n=u,$$

then by the symmetry, it is sufficient to consider the equation

$$\varepsilon u \frac{dv}{du} = \left(\frac{M}{1 - \frac{u}{a}} - M \right) \frac{1}{1 - mv}.$$

This equation has a holomorphic solution of the form

$$v = \frac{M}{\varepsilon a} u + \dots ,$$

about the origin. Therefore our proposed equation (1) has a system of m solutions $f_1, f_2, \dots f_m$ which are holomorphic about and vanish at $x_1 = x_2 = \dots = x_n = 0$.

If we want to obtain such a system of solutions $f_1, f_2, \dots f_m$ that

$$f_i(0, 0, \dots 0) = f_0^{(i)}, \ i = 1, 2, \dots m,$$

where $f_i^{(0)}, (i = 1, 2, \dots m)$ are arbitrary constants but the points $f_i^{(0)}$, $(i = 1, 2, \dots m)$ lie in the convergent circles of $F_i(x; f)$, $(i = 1, 2, \dots m)$; then we put

$$\varphi_i = f_i - f_0^{(i)}, \ i = 1, 2, \dots m.$$

We have

$$\begin{aligned} Xf_i &= X(\varphi_i + f_0^{(i)}) = X\varphi_i \\ &= F_i(x_1, x_2, \dots x_n; \ \varphi_1 + f_0^{(1)}, \varphi_2 + f_0^{(2)}, \dots \varphi_m + f_0^{(m)}), \ i = 1, 2, \dots m. \end{aligned}$$

Since the points $f_0^{(i)}, (i = 1, 2, \dots m)$ lie in their convergent circles, we may expand the Functions:

$$F_i(x, \varphi + f_0^{(\ )}) = \tau_0^{(i)}(x) + \sum_{s=1}^{m} \tau_s^{(i)}(x) \frac{\varphi_s}{b_s} + \sum_{s,t=1}^{m} \tau_{s,t}^{(i)}(x) \frac{\varphi_s}{b_s} \frac{\varphi_t}{b_t} + \dots ,$$
$$i = 1, 2, \dots m,$$

where $b_1, b_2, \dots b_m$ are the radii of convergent-circles of $F_i(x; \varphi + f_0^{(\ )})$, $(i = 1, 2, \dots m)$, with respect to the variables $\varphi_1, \varphi_2, \dots \varphi_m$. Since all the functions $\theta's$ commence at least with terms of first order of $x_1, x_2, \dots x_n$, all the new functions $\tau's$ also fail to contain any constant terms, and moreover are holomorphic about $x_1 = x_2 = \dots = x_n = 0$. Now put

$$\frac{\varphi_i}{b_i} \equiv \psi_i, \ \ i = 1, 2, \dots m,$$

then the new epuations take the forms of the equations (1). From these we conclude that *our equations have a system of solutions* $f_1, f_2, \dots f_m$, *such that*

$$f_i(0, 0, \dots 0) = f_0^{(i)}, \ i = 1, 2, \dots m,$$

where $f_0^{(i)}, (i=1, 2, \ldots m)$ *are integration constants and that*

$$f_i = f_0^{(i)} + \sum_{s=1}^{n} f_s^{(i)} x_s + \sum_{s,t=1}^{n} f_{s,t}^{(i)} x_s x_t + \ldots, \quad i = 1, 2, \ldots m,$$

where $f_s^{(i)}, f_{s,t}^{(i)} \ldots, (i=1, 2, \ldots m)$ *are constants.*

16. The system of equations (1), § 15, cannot have such solutions

$$f_i \equiv 0, \; i = 1, 2, \ldots m,$$

in so far as all $\theta_0^{(i)}(x), (i=1, 2, \ldots m)$ are identically zero. Conversely when all $\theta_0^{(i)}(x), (i=1, 2, \ldots m)$ are identically zero, all solutions f_i, $(i=1, 2, \ldots m)$ which are zero at $x_1=x_2=\ldots=x_n=0$ must vanish identically. The proof is easy. In the proof of the last theorem, we saw that all values of differential quotients of order m at $x_1=x_2=\ldots=x_n=0$ are given by certain linear homogeneous functions of the coefficients of the functions $\theta_0^{(i)}(x), (i=1, 2, \ldots m)$ and of the values of differential quotients of order lower than m, at $x_1=x_2=\ldots=x_n=0$. This fact shows that all the coefficients of the solutions $f_i, (i=1, 2, \ldots m)$ are linear homogeneous functions of the coefficients of the functions $\theta_0^{(i)}(x), (i=1, 2, \ldots m)$. Therefore, when

$$\theta_0^{(i)}(x) \equiv 0, \; i = 1, 2, \ldots m,$$

then

$$f_i(x) \equiv 0, \; i = 1, 2, \ldots m;$$

i.e., when at least one of the functions $\theta_0^{(i)}(x), (i=1, 2, \ldots m)$ does not vanish, then at least one of the functions $f_i, (i=1, 2. \ldots m)$ does not vanish.

From this remark, it follows that the system of solutions $f_i, (i=1, 2, \ldots m)$ which become $f_0^{(i)}, (i=1, 2, \ldots m)$ respectively at $x_1=x_2=\ldots=x_n=0$ is unique.

17. Now there is given a complete system of r equations

$$X_1 f \equiv \xi_{11}(x)\frac{\partial f}{\partial x_1} + \xi_{12}(x)\frac{\partial f}{\partial x_2} + \ldots + \xi_{1n}(x)\frac{\partial f}{\partial x_n} = 0,$$

$$X_2 f \equiv \xi_{21}(x)\frac{\partial f}{\partial x_1} + \xi_{22}(x)\frac{\partial f}{\partial x_2} + \ldots + \xi_{2n}(x)\frac{\partial f}{\partial x_n} = 0, \qquad (1)$$

....

$$X_r f \equiv \xi_{r1}(x)\frac{\partial f}{\partial x_1} + \xi_{r2}(x)\frac{\partial f}{\partial x_2} + \ldots + \xi_{rn}(x)\frac{\partial f}{\partial x_n} = 0,$$

where all the functions ξ's are holomorphic about $x_1=x_2=\ldots=x_n=0$, and have the forms

$$\xi_{ij}(x) = \lambda_{ij}\,x_j+\ldots,\quad \begin{array}{l} i=1,2,\ldots r, \\ j=1,2,\ldots n, \end{array}$$

the dotted part meaning terms of higher order.

We assume that $\lambda_{11}, \lambda_{12}, \ldots \lambda_{1n}$ satisfy Poincaré's conditions; moreover, the rank of the matrix

$$\left\| \begin{array}{lll} \lambda_{11} & \lambda_{12} \ldots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} \ldots & \lambda_{2n} \\ \ldots & \ldots & \ldots \\ \lambda_{r1} & \lambda_{r2} \ldots & \lambda_{rn} \end{array} \right\| \tag{2}$$

is r. If the rank be less than r, multiplying certain constants into $X_1f, X_2f\ldots X_rf$ and adding, we have the equation

$$\bar{\xi}_1(x)\frac{\partial f}{\partial x_1}+\bar{\xi}_2(x)\frac{\partial f}{\partial x_2}+\ldots+\bar{\xi}_n(x)\frac{\partial f}{\partial x_n}=0,$$

whose coefficients $\bar{\xi}_i, (i=1, 2, \ldots n)$ commence at least with terms of second order. Such an equation has not yet been treated generally. In the following we shall prove that this system has $n-r$ solutions.

Since $X_1f, X_2f, \ldots X_rf$ form a complete system, such relations hold

$$(X_i, X_j)f=\sum_{s=1}^{r} C_{ijs}X_sf,\quad i,j=1,2,\ldots r, \tag{3}$$

where C_{ijs} are holomorphic functions; for Lie's group these are constants. Between the functions C_{ijs}, there hold such relations:

$$C_{ijs}+C_{jis}=0,\ i,j,s=1,2,\ldots r. \tag{4}$$

Take any three of $X_if, (i=1, 2, \ldots r)$, then we have, after Jacobi,

$$(X_k, (X_i, X_j))f+(X_i, (X_j, X_k))f+(X_j, (X_k, X_i))f\equiv 0.$$

Therefore we know, after some easy calculations, that

$$\sum_{s=1}^{r}(C_{ijs}\,C_{kst}+C_{jks}\,C_{ist}+C_{kis}\,C_{jst}) \tag{5}$$

$$+X_kC_{ijt}+X_iC_{jkt}+X_jC_{kit}\equiv 0,\qquad i,j,k,t=1,2,\ldots r.$$

Now consider the following $r-1$ alternants

$$
\begin{aligned}
(X_1, X_2) f &= \sum_{s=1}^{r} C_{12s} X_s f, \\
(X_1, X_3) f &= \sum_{s=1}^{r} C_{13s} X_s f, \\
&\cdots\cdots\cdots \\
(X_1, X_r) f &= \sum_{s=1}^{r} C_{1rs} X_s f.
\end{aligned} \tag{6}
$$

Put

$$
Y_i f = X_i f + \sum_{j=1}^{r} R_{ij}(x) X_j f, \tag{7}
$$

where $R_{ij}(x), (j=1, 2, \dots r)$ are yet unknown functions; then we have by (6)

$$
\begin{aligned}
(X_1, Y_i) f &= (X_1, X_i + \sum_{j=1}^{r} R_{ij}(x) X_j) f \\
&= (X_1, X_i) f + \sum_{j=1}^{r} R_{ij} (X_1, X_j) f + \sum_{j=1}^{r} X_1 R_{ij} X_j f, \\
&= \sum_{s=1}^{r} C_{1is} X_s f + \sum_{j=1}^{r} R_{ij} \sum_{s=1}^{r} C_{1js} X_s f + \sum_{j=1}^{r} X_1 R_{ij} X_j f, \\
&= \sum_{s=1}^{r} \left\{ C_{1is} + \sum_{j=1}^{r} R_{ij} C_{1js} + X_1 R_{is} \right\} X_s f.
\end{aligned}
$$

Therefore, to have the relation

$$
(X_1, Y_i) f \equiv 0,
$$

we must take $R_{ij}, (j=1, 2, \dots r)$ such that

$$
X_1 R_{is} + C_{1is} + \sum_{j=1}^{r} C_{1js} R_{ij} = 0, \quad s = 1, 2, \dots r. \tag{8}
$$

This is a system of r partial differential equations. When the functions C_{1is}, $(s=1, 2, \dots r)$ are not all zero and all the functions $C_{1js}, (j, s= 1, 2, \dots r)$ commence at least with terms of first order, then, by the theorem of § 15, this system of equations has r holomorphic solutions $R_{i1}, R_{i2}, \dots R_{ir}$ which do not contain any constant terms.
By this system of solutions, we have

$$
(X_1, Y_i) f \equiv 0.
$$

When all $C_{1is}, (s=1, 2, \dots r)$ are zero, then put $X_i \equiv Y_i$ and the result is the same. By the property of $R_{ij}, (j=1, 2, \dots r)$, we have

$$Y_i f = (\lambda_{i1}x_1 + \dots)\frac{\partial f}{\partial x_1} + (\lambda_{i2}x_2 + \dots)\frac{\partial f}{\partial x_2} + \dots + (\lambda_{in}x_n + \dots)\frac{\partial f}{\partial x_n}, \quad (9)$$

and moreover

$$X_i f = (1 + R_{ii})^{-1}(Y_i f - {\sum_{j=1}^{r}}' R_{ij}X_j f), \qquad (10)$$

where, in the summation Σ', $X_i f$ is excepted. Thus $X_i f$ can be expressed linearly in terms of $X_1 f, \dots X_{i-1}, Y_i f, X_{i+1} f, \dots X_r f$. Moreover from (10), we have

$$\begin{aligned}(X_1, X_k)f &= \sum_{s=1}^{r} C_{1ks}X_s f \\ &= {\sum_{s=1}^{r}}' C_{1ks}X_s f + C_{1ki}(1+R_{ii})^{-1}(Y_i f - {\sum_{j=1}^{r}}' R_{ij}X_j f) \\ &\equiv C'_{1k1}\,X_1 f + \dots + C'_{1ki}\,Y_i f + \dots + C'_{1kr}\,X_r f.\end{aligned}$$

Thus we see that when the functions C_{1ks}, $(k, s = 1, 2, \dots r)$ don't contain constant terms, then the functions C'_{1ks}, $(s = 1, 2, \dots r)$ also don't contain any constant terms. Now operate this process for $X_2 f, X_3 f, \dots X_r f$ successively and replace our complete system (1) by thc complete system

$$Y_1 f \equiv X_1 f, \quad Y_2 f, \dots Y_r f, \qquad (11)$$

where

$$Y_i f = (\lambda_{i1}x_1 + \dots)\frac{\partial f}{\partial x_1} + (\lambda_{i2}x_2 + \dots)\frac{\partial f}{\partial x_2} + \dots + (\lambda_{in}x_n + \dots)\frac{\partial f}{\partial x_n},$$
$$i = 1, 2, \dots r$$

$$(Y_1\ Y_i)f \equiv 0, \ i = 2, 3, \dots r. \qquad (12)$$

But in general

$$(Y_i, Y_j)f = \sum_{s=1}^{r} C'_{ijs}\, Y_s f.$$

Since

$$C'_{1js} = -C_{j1s} = 0, \qquad (13)$$

and

$$C_{iis} = 0, \quad i, j, s = 1, 2, \dots r,$$

we may prove that all the remaining functions C_{ijs} are also zero. For apply the formula (5), then we have

$$\begin{aligned}&\sum_{s=1}^{r}(C'_{ijs}\,C'_{kst} + C'_{jks}\,C'_{ist} + C'_{kis}\,C'_{jst}) \\ &+ Y_k C'_{ijt} + Y_i\,C'_{jkt} + Y_j\,C'_{kit} \equiv 0, \quad i, j, k, t = 1, 2, \dots r.\end{aligned}$$

Put $k = 1$, then by virtue of (13),

$$Y_1\,C'_{ijt} = 0, \quad i, j, t = 1, 2, \dots r.$$

But since C'_{ijt} is not a constant, we must have

$$all\ C'_{ijt} = 0,$$

or

$$(Y_i, Y_j)f \equiv 0, \quad i, j = 1, 2, \ldots r.$$

Thus our new system is permutable.

We remark that under the condition that the matrix (2) has the rank r, it follows that all C_{ijs} can not contain constant terms. For otherwise, putting

$$C_{ijs} = c_{ijs} + \ldots,$$

where c_{ijs} is the constant, since in

$$(X_i, X_j)f = \sum_{s=1}^{r} (X_i\, \xi_{js} - X_j\, \xi_{is}) \frac{\partial f}{\partial x_s},$$

$X_i\, \xi_{js} - X_j\, \xi_{is}, (s = 1, 2, \ldots r)$ don't contain terms of first order, hence in $\sum_{s=1}^{r} C_{ijs} X_s f$ it must be so, i.e., we must have

$$\sum_{s=1}^{r} c_{ijs}\, \lambda_{st} = 0, \quad \begin{matrix} i, j = 1, 2, \ldots r \\ t = 1, 2, \ldots n. \end{matrix}$$

Hence the rank of the matrix (2) must be less than r; which is contrary to our assumption. Thus $c_{ijs}, (i, j, s = 1, 2, \ldots r)$ must be zero.

Now, by the theorem of § 14, our new complete system has $n-r$ solutions, hence the result:

A complete system of r partial differential equations

$$X_i f = \xi_{i1}(x) \frac{\partial f}{\partial x_1} + \xi_{i2}(x) \frac{\partial f}{\partial x_2} + \ldots + \xi_{in}(x) \frac{\partial f}{\partial x_n} = 0, \quad i = 1, 2, \ldots r$$

where

$$\xi_{ij}(x) = \lambda_{ij}\, x_j + \ldots, \quad i = 1, 2, \ldots r, \quad j = 1, 2, \ldots n,$$

and $\lambda_{11}, \lambda_{12}, \ldots \lambda_{1n}$ *satisfy Poincaré's conditions, and the rank of the matrix*

$$\left\| \begin{matrix} \lambda_{11} & \lambda_{12} \ldots \lambda_{1n} \\ \lambda_{21} & \lambda_{22} \ldots \lambda_{2n} \\ \ldots\ldots & \ldots\ldots \\ \lambda_{r1} & \lambda_{r2} \ldots \lambda_{rn} \end{matrix} \right\|$$

is r, always has n−r independent solutions.

This theorem is an extension of Poincaré's theorem given in his thèse, as well as Lie's method of integration.

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