On the Electromagnetic Field due to a Moving Electric Doublet.

By

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The electromagnetic field due to an electric doublet moving uniformly in a direction perpendicular to its axis or in a direction of its axis has already been examined by Prof. T. Mizuno[1] as an application of the Principle of Relativity. Six years ago the writer[2] derived the generalized Lorentz-Einstein transformation equations and noticed that they would have an important bearing on problems concerning motions whose directions are not confined in one of the coordinate axes. The field due to an electric doublet moving in any direction with a constant velocity is easily obtained by making use of the generalized transformation equations.

As is well known, the electric force $\mathbf{E}$ and the magnetic force $\mathbf{H}$, due to a stationary electric doublet, are given by

$$\mathbf{E} = -\frac{\Phi}{r^3}\mathbf{s}_1 + \frac{3\Phi}{r^5}\mathbf{r}(\mathbf{s}_1\mathbf{r}), \quad \mathbf{H} = 0, \tag{1}$$

where Φ is the electric moment of the doublet, $\mathbf{s}_1$ the unit vecter along its axis, and $\mathbf{r}$ the radius vector of any point drawn from its centre.

If the doublet is in motion with a constant velocity $\mathbf{v}$ with respect to a stationary system, and if we consider a moving system, at whose origin the doublet is situated, the electric and magnetic forces measured in the moving system are given by

$$\mathbf{E}' = -\frac{\Phi'}{r'^3}\mathbf{s}_1' + \frac{3\Phi'}{r'^5}\mathbf{r}'(\mathbf{s}_1'\mathbf{r}'), \quad \mathbf{H}' = 0. \tag{2}$$

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1 Mem. Col. Sci. Eng. Kyoto Imp. Univ., **2**, 157 (1909).

2 Ibid. **3**, 103 (1911).

If the two systems coincide at the time $t=t'=0$, we have

$$\mathbf{r}' = \mathbf{r}+(\gamma-1)\,\mathbf{v}_1(\mathbf{v}_1\mathbf{r})-\gamma\mathbf{v}t \tag{3}$$

where

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

and $\mathbf{v}_1$ is the unit vector along $\mathbf{v}$, c being the velocity of light in vacuum.

Multiplying $\mathbf{v}$ scalarly and vectorially to (3) and putting

$$\mathbf{r}-\mathbf{v}t = \mathbf{R}, \tag{4}$$

we get

$$\begin{cases} (\mathbf{v}\mathbf{r}') = \gamma(\mathbf{v}\mathbf{R}), & (5) \\ [\mathbf{v}\mathbf{r}'] = [\mathbf{v}\mathbf{r}] = [\mathbf{v}\mathbf{R}]. & (6) \end{cases}$$

Consequently (3) may be written

$$\mathbf{r}' = \mathbf{R}+(\gamma-1)\mathbf{v}_1(\mathbf{v}_1\mathbf{R}), \tag{7}$$

or

$$\mathbf{r}' = \gamma\mathbf{R}+(\gamma-1)\,[\mathbf{v}_1[\mathbf{v}_1\mathbf{R}]]. \tag{8}$$

Squaring the last equation we have

$$\mathbf{r}'^2 = \gamma^2\mathbf{R}^2\{1-\beta^2[\mathbf{v}_1\mathbf{R}_1]^2\}, \tag{9}$$

where $\mathbf{R}_1$ is the unit vector along $\mathbf{R}$.

Similarly putting

$$\mathbf{s}-\mathbf{v}t = \mathbf{S}, \tag{10}$$

we obtain

$$\mathbf{s}' = \mathbf{S}+(\gamma-1)\mathbf{v}_1(\mathbf{v}_1\mathbf{S}), \tag{11}$$

and consequently

$$\mathbf{r}'(\mathbf{s}\mathbf{r}') = \{\mathbf{R}+(\gamma-1)\mathbf{v}_1(\mathbf{v}_1\mathbf{R})\}\left\{(\mathbf{S}\mathbf{R})+\left(\frac{\gamma}{c}\right)^2(\mathbf{v}\mathbf{R})(\mathbf{v}\mathbf{S})\right\}. \tag{12}$$

By the aid of equations (9), (11) and (12), the equation (2) may be written

$$\begin{aligned} \mathbf{E}' = \frac{e}{\gamma^5R^5(1-\beta^2[\mathbf{v}_1\mathbf{R}_1]^2)^{5/2}} \Big[& 3\left\{(\mathbf{S}\mathbf{R})+\left(\frac{\gamma}{c}\right)^2(\mathbf{v}\mathbf{R})(\mathbf{v}\mathbf{S})\right\}\mathbf{R} \\ & -\gamma^2R^2(1-\beta^2[\mathbf{v}_1\mathbf{R}_1]^2)\mathbf{S} \\ & +3(\gamma-1)(\mathbf{v}_1\mathbf{R})\left\{(\mathbf{S}\mathbf{R})+\left(\frac{\gamma}{c}\right)^2(\mathbf{v}\mathbf{R})(\mathbf{v}\mathbf{S})\right\}\mathbf{v}_1 \\ & -(\gamma-1)\gamma^2R^2(\mathbf{v}_1\mathbf{S})\{1-\beta^2[\mathbf{v}_1\mathbf{R}_1]^2\}\mathbf{v}_1\Big], \end{aligned}$$

$$\mathbf{H}' = 0,$$

because it is well known that the electric charge e is an invariant quantity.

If we use θ, ψ, φ to express angles between the radius vector $\mathbf{R}$ and the velocity $\mathbf{v}$, the axis of the doublet $\mathbf{S}$ and the velocity $\mathbf{v}$, and the radius vector $\mathbf{R}$ and the axis of the doublet $\mathbf{S}$, respectively, and moreover if we denote by Φ the electric moment of the doublet measured in the stationary system, the above expressions take the form

$$\begin{aligned} \mathbf{E}' = \frac{\Phi}{\gamma^3 R^3(1-\beta^2\sin^2\theta)^{5/2}} & [3\{(1-\beta^2)\cos\varphi+\beta^2\cos\theta\cos\psi\}\mathbf{R}_1 \\ & -(1-\beta^2\sin^2\theta)\,\mathbf{S}_1 \\ & +(\gamma-1)\{3(1-\beta^2)\cos\theta\cos\varphi+2\beta^2\cos^2\theta\cos\psi \\ & -(1-\beta^2)\cos\psi\}\ \mathbf{v}_1], \\ \mathbf{H}' = 0. & \end{aligned}$$

Substituting these values in the transformation equations of the electric and magnetic forces, namely

$$\begin{cases} \mathbf{E} = \gamma\left\{\mathbf{E}' - \dfrac{1}{c}[\mathbf{v}\mathbf{H}']\right\} - (\gamma-1)\,\mathbf{v}_1(\mathbf{v}_1\mathbf{E}'), \\ \mathbf{H} = \gamma\left\{\mathbf{H}' + \dfrac{1}{c}[\mathbf{v}\mathbf{E}']\right\} - (\gamma-1)\,\mathbf{v}_1(\mathbf{v}_1\mathbf{H}'), \end{cases}$$

we get

$$\left.\begin{aligned} \mathbf{E} = \frac{\Phi(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{5/2}} & [3\{(1-\beta^2)\cos\varphi+\beta^2\cos\theta\cos\psi\}\mathbf{R}_1 \\ & \qquad -(1-\beta^2\sin^2\theta)\,\mathbf{S}_1], \\ \mathbf{H} = \frac{\Phi\beta(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{5/2}} & [3\{(1-\beta^2)\cos\varphi+\beta^2\cos\theta\cos\psi\}\sin\theta\ \mathbf{R}_1^* \\ & \qquad -(1-\beta^2\sin^2\theta)\sin\psi\ \mathbf{S}_1^*], \end{aligned}\right\} \quad (13)$$

where $\mathbf{R}_1^*$ and $\mathbf{S}_1^*$ are the unit vectors along the directions $[\mathbf{v}_1\mathbf{R}_1]$ and $[\mathbf{v}_1\mathbf{S}_1]$ respectively. These are the electric and magnetic forces due to an electric doublet moving uniformly in any direction. It is interesting to note that the electric force $\mathbf{E}$ may be considered as made up of two vectors, one in the direction of the axis of the doublet and the other in the direction radial from the centre of the doublet. The electric force is wholly independent of the vector $\mathbf{v}_1$.

Now let us apply the above results to the following special cases :

(i) The doublet is moving along its axis.

In this case, since $\psi=0$ and $\theta=\varphi$, equations (13) reduce to

$$\begin{cases} \mathbf{E} = \dfrac{\Phi(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{5/2}}\left[3\cos\theta\,\mathbf{R}_1-(1-\beta^2\sin^2\theta)\,\mathbf{S}_1\right], \\ \mathbf{H} = \dfrac{\Phi\beta(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{5/2}}\left[3\cos\theta\sin\theta\,\mathbf{R}_1{}^*\right]. \end{cases}$$

If we take the axis of the doublet as the z-axis we get the following scalar equations

$$\begin{cases} E_x = \dfrac{3\Phi(1-\beta^2)XZ}{R^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_y = \dfrac{3\Phi(1-\beta^2)YZ}{R^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_z = \dfrac{3\Phi(1-\beta^2)Z^2}{R^5(1-\beta^2\sin^2\theta)^{5/2}} - \dfrac{\Phi(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{3/2}}, \end{cases} \qquad \begin{cases} H_x = -\dfrac{3\Phi\beta(1-\beta^2)YZ}{R^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ H_y = \dfrac{3\Phi\beta(1-\beta^2)XZ}{R^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ H_z = 0, \end{cases}$$

where

$$R^2 = X^2+Y^2+Z^2.$$

(ii) The doublet is moving perpendicularly to its axis.

In this case, since $\psi=\pi/2$, we get

$$\begin{cases} \mathbf{E} = \dfrac{\Phi(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{5/2}}\left[3(1-\beta^2)\cos\varphi\,\mathbf{R}_1-(1-\beta^2\sin^2\theta)\,\mathbf{S}_1\right], \\ \mathbf{H} = \dfrac{\Phi\beta(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{5/2}}\left[3(1-\beta^2)\cos\varphi\sin\theta\,\mathbf{R}_1{}^*-(1-\beta^2\sin^2\theta)\,\mathbf{S}_1{}^*\right]. \end{cases}$$

Hence, if the axis of the doublet be taken as the z-axis as before and the direction of motion as the x-axis, we obtain

$$\begin{cases} E_x = \dfrac{3\Phi(1-\beta^2)^2XZ}{R^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_y = \dfrac{3\Phi(1-\beta^2)^2YZ}{R^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_z = \dfrac{3\Phi(1-\beta^2)^2Z^2}{R^5(1-\beta^2\sin^2\theta)^{5/2}} - \dfrac{\Phi(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{3/2}}, \end{cases}$$

$$\begin{cases} H_x = 0, \\ H_y = -\dfrac{3\Phi\beta(1-\beta^2)Z^2}{R^5(1-\beta^2\sin^2\theta)^{5/2}} + \dfrac{\Phi\beta(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{3/2}}, \\ H_z = \dfrac{3\Phi\beta(1-\beta^2)YZ}{R^5(1-\beta^2\sin^2\theta)^{5/2}}. \end{cases}$$