General Metrics in Hermitian Space.

By

Teikichi Nishiuchi.

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§ 1. Moment of two multi points.

Let the variables $x_1 x_2 \dots x_n$ denote the homogeneous point-coordinates in a complex space of n-dimensions. We shall use the symbol (x) to denote the ratio $x_1 : x_2 : \dots : x_n$. Suppose r points (x'), (x'')...$(x^{(r)})$ $(r < n)$ be given, then the totality of the points given by the coordinates

$$\lambda'(x') + \lambda''(x'') + \dots + \lambda^{(r)}(x^{(r)}),$$

where $\lambda_{,}, \lambda'', \dots \lambda^{(r)}$ are complex parameters, forms a manifoldness $\infty^{2(r-1)}$ This is called r-point and denoted by the symbol M_r.

A point (x) belongs to M_r if the equations

$$x_i + \lambda' x_i' + \lambda'' x_i'' + \dots + \lambda^{(r)} x_i^{(r)} = 0 \; (i = 1, 2, 3 \dots n).$$

are satisfied. The conditions are equivalent to the vanishing of all the determinants of the matrix

$$\begin{vmatrix} x_0 & x_2 & x_3 & \dots & x_n \\ x_1' & x_2' & x_3' & \dots & x_n' \\ \dots & \dots & \dots & \dots & \dots \\ x_1^r & x_2^r & x_3^r & \dots & x_n^r \end{vmatrix}$$

M_r is determined by r points none of which belongs to the same r-k-points.

In place of $(x') \dots (x^{(r)})$, we may take r other points

$$\lambda_{i1}(x') + \lambda_{i2}(x'') + \dots + \lambda_{ir}(x^{(r)}) \quad (i = 1, 2, \dots r),$$

provided that the determinant $|\lambda_{ij}|$ is not equal to zero.

M_{n-1} is a plane in this space.

M_2 is a line in this space.

The coordinates (x) of every point of M_r satisfy the $n-r$ equations for M_r.

$$(\xi^{(i)}x) = 0^{1} \quad (i=1, 2, \dots n-r),$$

which are called $n-r$ equations for M_r.

We may replace these equations by others of the form

$$(\lambda_1\xi_i)\, x_1 + (\lambda_2\xi_i)\, x_2 + \dots + (\lambda_n\xi_i)\, x_n = 0 \quad (i=1, 2, \dots n-r)$$

$$|\lambda_{ij}| \neq 0,$$

where

$$(\lambda_i\xi_j) \equiv \lambda_{i1}\xi_i' + \lambda_{i2}\xi_j'' + \dots + \lambda_{i.\,n-r}\, \xi_j^{(n-r)}.$$

M_r can be considered as envelope of ∞^{n-r} planes and is called $n-r-$plane.

Multi-points and multi-planes constitute two systems of fundamental forms which correspond dually.

We can associate two by two r-points and n-r-points. Then we have the reciprocal relation between the coordinates

$$(xy) = 0.$$

Conversely, if the equation represents n-r-plane which envelopes an r-point, then the planes $(\xi'), (\xi'') \dots (\xi^{n-r})$ determines an n-r-plane associated with the r-point. Specially, every point of one corresponds uniquely to a plane of the other.

If an r-point M_r be given by means of r-points $(x'), (x''), \dots (x^r]$ or by the equations $(\xi'x)=0, (\xi''x)=0, \dots (\xi^{n-r}x)=0$, then the determinants of two matrices

$$|x'x'' \dots x^r|, \ |\xi'\xi'' \dots \xi^{n-r}|^{1}$$

are proportional.

Now put

$$\overset{r}{\mathfrak{X}}_{b\cdots c} \equiv \begin{vmatrix} x_b' \dots\dots\dots \\ \vdots \quad \dots \\ \vdots \qquad x_c^r \end{vmatrix},$$

$$\overset{n-r}{\Xi}_{d\dots e} \equiv \begin{vmatrix} \xi'_d \dots\dots\dots \\ \vdots \quad \dots \\ \vdots \qquad \xi_e^{n-r} \end{vmatrix},$$

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[1] $(\xi x) \equiv \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$.

[1] By the symbal $|x' x'' \dots x^{(r)}|$, we mean the matrix

$$\begin{vmatrix} x_1' \, x_2' \dots x^{n\prime} \\ \dots\dots\dots\dots\dots\dots \\ x_1^{(r)} x_2^{(r)} \dots x_n^{(r)} \end{vmatrix}.$$

then we can easily see that there are relations between the coordinates:

$$\sum_{d\ldots g} (-1)^{\omega} \overset{r}{\mathfrak{X}}_{b\ldots cd\ldots e} \overset{r}{\mathfrak{X}}_{b\ldots cf\ldots g} = 0.$$

We have similar results for $\overset{n-r}{\Xi}_{d\ldots e\ldots}$. And we have also

$$\overset{r}{\mathfrak{X}}_{b\ldots c} : \overset{n-r}{\Xi}_{d\ldots e} = \text{Constant},$$

where $b\ldots c$, $d\ldots e$ is a paramutation of $12\ldots n$.

The quantities $\overset{r}{\mathfrak{X}}_{b\ldots c}$ which are different in absolute values have $\frac{|\underline{n}}{|\underline{r}\,|\underline{n-r}}$ $-(n-r)r-1$ relations, also the same for $\overset{n-r}{\Xi}_{d\ldots e}$. We will use $\overset{r}{\mathfrak{X}}_{b\ldots c}$ as the homogeneous coordinate of an r-point M_r and $\overset{n-r}{\Xi}_{d\ldots e}$ as that of the n-r-plane.

Now let us consider a bilinear form of point-coordinates

$$f(x\bar{y}) \equiv \sum a_{ij}\, x_i\, \bar{y}_j,$$

where

$$i, j = 1, 2 \ldots n,$$

$$a_{ij} = \bar{a}_{ji},$$

$$|a_{ij}| \neq 0.$$

The type of bilinear form is called Hermitian form[1]. Since

$$f(x\bar{y}) = \sum a_{ij}\, x_i\, \bar{y}_j,\quad f(y\bar{x}) = \sum a_{ij}\, y_i\, \bar{x}_j.$$

we have the relation

$$f(y\bar{x}) = \overline{f(x\bar{y})}.$$

We will have a polar system called absolute Hermitian polar system represented by the equation

$$f(x\bar{y}) = 0 \quad \text{or} \quad f(y\bar{x}) = 0.$$

In complex space, Segre considered four types of projective transformations. They are represented respectively by the transformation-equations:

$$x_i' = (A_i\, x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\text{i}),$$

$$x_i' = (A_i\, \bar{x}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\text{ii}),$$

$$x_i' = (a_i\, \xi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\text{iii}),$$

$$x_i' = (a_i\, \bar{\xi}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\text{iv}),$$

where

[1] See Hermite. Remarque sur un théorème de M. Cauchy. Comptes Rendus Vol. 41.

$$i,j = 1, 2, 3 \dots n,$$

$$|A_{ij}| \neq 0, \quad |a_{ij}| \neq 0.$$

The transformations (ii) and (iv) are called anticollineation[1] and correction[2] respectively by him. The projective transformations in the complex space of $n-1$ dimension by which the absolute Hermitian polar system remains the same, form a group of n^2-1 essential real parameters. If n be odd or even, then and only then the associated Hermitian form is not composed from $\frac{n}{2}$ positive and $\frac{n}{2}$ negative products of the form $x_i\bar{x}_i$, this transformation is constituted by four distinct continuous series of transformations in the exceptional case.

The subgroup G_{n^2-1} of this group is transitive, primitive and simple. If an Hermitian form be given, then every four of the transformations form a closed continium.

Hereafter we shall confine our discussion to such Hermitian forms that can be brought to one of the forms

$$\varkappa^2 x_1\bar{x}_1 + x_2\bar{x}_2 + \dots + x_n\bar{x}_n, \quad (\varkappa^2 = \pm 1).$$

by proper linear transformations. The groups of collineations G_{n^2-1} and that of correlotions $\mathfrak{H}_{n^2-1}$ by which the absolute Hermitian polar system remains invariable are called Hermitian motion and symmetric transformation respectively.

Now put

$$\alpha_{ij} = \frac{1}{|a_{ij}|}\frac{\partial |a_i|}{\partial a_{ij}},$$

then the bilinear equation

$$\varphi(\xi\bar{\eta}) \equiv \sum \alpha_{ij}\xi_i\bar{\eta}_i = 0$$

represents the absolute Hermitian polar system. Here we see that

$$|a_{ij}|\cdot|\alpha_{ij}| = 1.$$

and

$$a_{ij} = \frac{1}{|\alpha_{ij}|}\frac{\partial |\alpha_{ij}|}{\partial \alpha_{ij}}.$$

Further we construct two systems of the Hermitian forms

[1, 2] See Segre. An nuovo Campo di ricerche geometriche. Atti della R. Accademia delle Scienze di Torino. Ser. 2a. Vol. 38.

$$f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}}) \equiv \sum a_{ij\ldots pq\ldots}\, \overset{r}{\mathfrak{X}}_{ij\ldots}\, \overset{r}{\bar{\mathfrak{X}}}_{pq\ldots}\,,$$

$$\varphi(\overset{r}{\Xi}\overset{r}{\bar{\Xi}}) \equiv \sum a_{ij\ldots pq\ldots}\, \overset{r}{\Xi}_{ij\ldots}\, \overset{r}{\bar{\Xi}}_{pq\ldots}\,,$$

$$r = 1, 2, \cdots n,$$

where $ij\ldots pq\ldots$ denotes a permutation of $12\ldots n$. The discriminants of the forms $f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})$ and $\varphi(\overset{r}{\Xi}\overset{r}{\bar{\Xi}})$ are:

$$|a_{ij\ldots pq\ldots}| = |a_{ij}|^{\binom{n-1}{n-r}},$$

$$|a_{ij\ldots pq\ldots}| = |a_{ij}|^{\binom{n-1}{n-r}}$$

respectively. We see that

$$\overset{r}{\mathfrak{X}}_{ij\ldots} : \overset{n-r}{\bar{\Xi}}_{kl\ldots} = \sqrt{|a_{ij}|} = \frac{1}{\sqrt{|a_{ij}|}}\,,$$

$$\overset{r}{\bar{\mathfrak{X}}}_{ij\ldots} : \overset{n-r}{\bar{\Xi}}_{lk\ldots} = \sqrt{|a_{ij}|} = \frac{1}{\sqrt{|a_{ij}|}}\,,$$

where $ij\ldots kl\ldots$ denotes a permutation of $1, 2, \ldots n$. Thus we have:

$$f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}}) = \varphi(\overset{n-r}{\Xi}\overset{n-r}{\bar{\Xi}}).$$

In general, if we form the bilinear functions

$$f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{Y}}}) = \sum a_{ij\ldots pq\ldots}\, \overset{r}{\mathfrak{X}}_{ij\ldots}\, \overset{r}{\bar{\mathfrak{Y}}}_{pq\ldots}\,,$$

$$\varphi(\overset{n-r}{\Xi}\overset{n-r}{\bar{Y}}) = \sum a_{ij\ldots pq\ldots}\, \overset{n-r}{\Xi}_{ij\ldots}\, \overset{n-r}{\bar{Y}}_{pq\ldots}\,,$$

then we have:

$$f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{Y}}}) = \varphi(\overset{n-r}{\Xi}\overset{n-r}{\bar{Y}}).$$

Further, we have the relations:

$$\frac{\partial f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})}{\partial \overset{r}{\mathfrak{X}}_{ij\ldots}} : \frac{\partial \varphi(\overset{n-r}{\Xi}\overset{n-r}{\bar{\Xi}})}{\partial \overset{n-r}{\Xi}_{kl\ldots}} = |a_{ij}| = \frac{1}{|a_{ij}|}\,,$$

$$\frac{\partial f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})}{\partial \overset{r}{\bar{\mathfrak{X}}}_{ij\ldots}} : \frac{\partial \varphi(\overset{n-r}{\Xi}\overset{n-r}{\bar{\Xi}})}{\partial \overset{n-r}{\bar{\Xi}}_{kl\ldots}} = |a_{ij}| = \frac{1}{|a_{ij}|}\,.$$

Two reciprocal substitutions

$$x_i = \frac{\partial \varphi(\xi\bar{\xi})}{\partial \xi_i}\,, \quad \xi_i = \frac{\partial f(x, \bar{x})}{\partial x_i}\,.$$

which can transform $f(x\bar{x})$ and $\varphi(\xi\bar{\xi})$ from one to the other establish one to one correspondence between the points and planes in the given space. We shall call the plane corresponding to a point (x') and a plane (ξ') conjugate if there be the relations,

$$(\xi' x') = 0 \quad \text{or} \quad f(x'\bar{x}') = 0.$$

and

$$(x'\xi') = 0 \quad \text{or} \quad \varphi(\xi'\bar{\xi}') = 0.$$

We can easily see that the planes which are absolute Hermitian Conjugate to every point of an r-point form an $r-1$-plane or an $n-r+1$-point. We shall call such two multi-points absolute Hermitian Conjugate to each other. Two absolute Hermitian multi-points have no common points.

Let the planes (ξ) and (η) be absolute Hermitian Conjugates to two given points (x) and (y) respectively, then we have:

$$f(x\bar{y}) = \varphi(\xi\bar{\eta}) = (x\eta) = (\xi y)$$
$$= -\frac{1}{|a_{ij}|}\cdot\begin{vmatrix} x & \\ \bar{y} & a \end{vmatrix}.$$
$$= -\frac{1}{|a_{ij}|}\cdot\begin{vmatrix} \xi & \\ \bar{\eta} & a \end{vmatrix},$$

where

$$\begin{vmatrix} x & \\ \bar{y} & a \end{vmatrix} = \begin{vmatrix} 0 & x_1 & x_2 & \dots & x_n \\ \bar{y}_1 & a_{11} & a_{12} & \dots & a_{1n} \\ \bar{y}_2 & a_{21} & a_{12} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \bar{y}_n & a_{n1} & \dots & \dots & a_{nn} \end{vmatrix}.$$

This formula remains true when $(x) \equiv (y)$. Let $(\eta'), (\eta'') \dots$ be absolute Hermitian Conjugates to the points $(x'), (x''), \dots$ respectively, then the product of the two matrices becomes as follows:

$$|x' x'' \dots|\cdot|\bar{\eta}' \bar{\eta}'' \dots|$$
$$= \sum \begin{vmatrix} x_i' & x_j' & \dots \\ x_i'' & x_j'' & \dots \\ \dots & \dots & \dots \end{vmatrix} \cdot \begin{vmatrix} \bar{\eta}_i' & \bar{\eta}_j' & \dots \\ \bar{\eta}_i'' & \bar{\eta}_j'' & \dots \\ \dots & \dots & \dots \end{vmatrix}$$
$$= \sum \begin{vmatrix} x_i' & x_j' & \dots \\ x_i'' & x_j'' & \dots \\ \dots & \dots & \dots \end{vmatrix} \cdot \left(\left| \bar{y}' \bar{y}'' \dots \right| \cdot \left| a_i a_j \dots \right| \right)$$

$$=\sum \begin{vmatrix} x_i' & x_j' & \dots \\ x_i'' & x_j'' & \dots \\ \dots & \dots & \dots \end{vmatrix} \cdot \begin{vmatrix} \bar{y}_r' & \bar{y}_q' & \dots \\ \bar{y}_p'' & \bar{y}_i'' & \\ \dots & & \dots \end{vmatrix} \cdot \begin{vmatrix} a_{ir} & a_{iq} & \dots \\ a_{jr} & a_{jq} & \dots \\ \dots & & \dots \end{vmatrix}$$

$$=\sum a_{ij\dots pq\dots} \overset{r}{\mathfrak{X}}_{ij\dots} \overline{\overset{r}{\mathfrak{Y}}}_{pq\dots}$$

$$= f(\overset{r}{\mathfrak{X}} \overline{\overset{r}{\mathfrak{Y}}}_{pt\dots})$$

$$= \frac{(-1)^r}{|a_i|} \begin{vmatrix} 0 & \dots\dots & 0 & x_1' & x_2' \dots\dots & x_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots\dots & 0 & x_1^{(r)} & x_2^{(r)} \dots & x_n^{(r)} \\ \bar{y}_1' & \bar{y}_1'' \dots & \bar{y}_1^{(r)} & a_{11} & a_{12} & \dots\dots \\ \vdots & & \vdots & a_{21} & a_{22} & \\ \bar{y}_n' & \dots\dots & \bar{y}_n^{(r)} & \dots & \dots & \dots \end{vmatrix}$$

$$= \frac{(-1)^r}{|a_{ij}|} \begin{vmatrix} x' \; x'' \dots x^{(r)} & \\ & a \\ \bar{y}' \; \bar{y}'' \dots \bar{y}^{(r)} & \end{vmatrix}.$$

And

$$\begin{vmatrix} f(x'\bar{y}') & f(x'\bar{y}'') & \dots \\ f(x''\bar{y}') & f(x''\bar{y}'') & \dots \\ \dots & & \end{vmatrix}$$

$$= \frac{(-1)^r}{|a_{ij}|} \cdot \begin{vmatrix} x' \; x'' \dots & \\ & a \\ \bar{y}' \; \bar{y}'' \dots & \end{vmatrix} = f(\overset{r}{\mathfrak{X}} \overline{\overset{r}{\mathfrak{Y}}})$$

$$= \frac{(-1)^r}{|a_{ij}|} \cdot \begin{vmatrix} \xi' \; \xi'' \dots & \\ & a \\ \bar{\eta}' \; \bar{\eta}'' \dots & \end{vmatrix} = \varphi(\overset{n-r}{\Xi} \overline{\overset{n-r}{Y}}).$$

The determinant

$$\begin{vmatrix} f(x'\bar{y}') \dots & \\ \dots & f(x''\bar{y}'') \dots \end{vmatrix}$$

changes its value by the factor $|\lambda_{ij}| \cdot |\bar{\lambda}_{ij}|$ after the transformation

$$x_i' = (\mu_i x).$$

Now

$$\overset{r}{\mathfrak{X}}_{ij\dots} = \begin{vmatrix} x_i' & x_j' & \dots \\ x_i'' & x_j'' & \\ \dots & & \end{vmatrix} = \begin{vmatrix} \left(\frac{\partial\varphi}{\partial\xi_i}\right)' & \left(\frac{\partial\varphi}{\partial\xi_j}\right)' & \dots \\ \left(\frac{\partial\varphi}{\partial\xi_i}\right)'' & \left(\frac{\partial\varphi}{\partial\xi_j}\right)'' & \\ \dots & & \end{vmatrix}$$

$$= | a_i\, a_j \ldots | \cdot | \bar{\xi}' \bar{\xi}'' \ldots |$$

$$= \sum a_{ij\ldots pq\ldots}\, \bar{\Xi}_{pq\ldots}.$$

Therefore

$$\overset{r}{\mathfrak{X}}_{ij\ldots} = \frac{\partial\varphi(\overset{n-r}{\Xi}\,\overset{n-r}{\bar{\Xi}})}{\partial \overset{n-r}{\Xi}_{pq\ldots}}.$$

In the same manner, we can find

$$\overset{n-r}{\Xi}_{ij\ldots} = \frac{\partial f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})}{\partial \overset{r}{\mathfrak{X}}_{ij\ldots}}.$$

We shall define the length between two points in Hermitian space by the expression

$$D = \frac{\varkappa}{2i} \log \frac{\sqrt{f(x\bar{y})f(y\bar{x})} + \sqrt{f(x\bar{y})f(y\bar{x}) - f(x\bar{x})f(y\bar{y})}}{\sqrt{f(x\bar{y})f(y\bar{x})} - \sqrt{f(y\bar{x})f(y\bar{x}) - f(x\bar{x})f(y\bar{y})}}$$

$$(\varkappa^2 = \pm 1).$$

where $\varkappa^2$ has the same sign as that of the determinant $| a_{ij} |$, Then we see that

$$\text{Cos}^2 \frac{D}{\varkappa} = \frac{f(x\bar{y})\, f(y\bar{x})}{f(x\bar{x})\, f(y\bar{y})}.$$

$$\text{Sin}^2 \frac{D}{\varkappa} = \frac{\begin{vmatrix} f(x\bar{x}) & f(y\bar{x}) \\ f(x\bar{y}) & f(y\bar{x}) \end{vmatrix}}{f(x\bar{x})\, f(y\bar{y})}.$$

When the Hermitian form of reference is

$$[x\bar{x}] \equiv \varkappa^2 x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_n\bar{x}_n,$$

$$(\varkappa^2 = \pm 1),$$

these expressions are reduced to those given by Fubini[1] and Study. In fact,

$$D = \frac{\varkappa}{2i} \log \frac{\sqrt{[x\bar{y}][y\bar{x}]} + \sqrt{[x\bar{y}][y\bar{x}] - [x\bar{x}][y\bar{y}]}}{\sqrt{[x\bar{y}][y\bar{x}]} - \sqrt{[x\bar{y}][y\bar{x}] - x\bar{x}][y\bar{y}]}}$$

$$\text{Cos}^2 \frac{D}{\varkappa} = \frac{[x\bar{y}][y\bar{x}]}{[x\bar{x}][y\bar{y}]},$$

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1 See Fubini. (Sulle metriche definite da una forma Hermitiana. Atti del R. Istituto di Veneto di Scienze, lettre ed Arti Serie, 8 (63 2).

$$\operatorname{Sin}^2 \frac{D}{\varkappa} = \frac{\begin{vmatrix} [x\bar{x}][y\bar{x}] \\ [x\bar{y}][y\bar{y}] \end{vmatrix}}{[x\bar{x}][y\bar{y}]}.$$

Any point lying in the plane $f(x\bar{y})=0$ is called an orthogonal point to a point (y). This definition shall be extended to two planes and two multi points.

When a point is given, there is r-1-point orthogonal to a given point. A point as orthogonal to R if it is orthogonal to r independent points of R in our sense.

In R, there is $\infty^{s(r-1)-\binom{s}{2}}$ groups of s-point which are orthogonal in our sense two by two among them.

The totality of the orthogonal points to R in our sense forms $(n-r)$-conjugate point R_0.

When r-point R and r'-point R' be given $(r \geqq r')$, R' do not generally contain orthogonal points to R in our sense but it is possible that R' contains l-point L of orthogonal points in our sense to R and consequently R contains l'-point L' (where $r-r'+l=l'$) of orthogonal points to R' in our sense. In that case we shall say that R and R' have multiple Hermitian orthogonality. R' and R_0 cut along L, R and R_1 along L'.

R and R' can be further r' times orthogonal in our sense. Then R and R', are said to be perfectly orthogonal in our sense and R' is included in R_0, R in R_1.

All the lines which cut R_0 are orthogonal to R in our sense and all the lines which cut R and R_0 are perpendicular to R (and to R_0) in one sense. There are lines which cut $RR'\,R_0R_1$ simultaneously at distinct points: they are perpendicular to R and R' (also to R_0 and R_1) in our sense at distinct points.

There are many cases:

1) if R and R' have no Hermitian orthogonality.

2) if they have no common points when $r+r' \leqq n$,

3) if they have no common k-points when $r+r'=n+k$; the number of the Hermitian perpendiculars is r' or $r-k\,(r \geqq r')$. They are perfectly orthogonal two by two and orthogonal to common multi point to R, R', R_0, R_1. They cut R, R', R_0, R_1 in groups of points which are mutually orthogonal in our sense.

When R and R' have a common k-point $(r+r' < n+k)$, l times orthogonality in our sense and two cases occur at the same time, the number of these lines reduce respectively to $r'-k$, $r'-l$, $r'-k-l$. Let us denote the number of the perpendiculars by ρ.

Through a given point, we can draw one Hermetian perpendicular to a given multi-point. The intersection point of the Hermitian perpendicular and the multi-point shall be called Hermitian projection from the given point to the given multi-point. Hermitian projection of an s-point on an r-point shall be defined as the locus of the Hermitian projections of every point in the s-point on the r-point. The same line projects the given point on the given multi-point and its Hermitian conjugate.

Let a point P and a r-point R be given, then we see that there is $(r-1)$-point which is orthogonal to P in our sense and in the same R there will be a point P' which is orthogonal to the r-1-point in our sense. P' shall be called the projection of P in the r-point. If we give an s-point and an r-point, then there will be an r-s-point in R which is orthogonal to this. S and s-point S' orthogonal to this r-s-point in our sense. S' will be the projection of S in R.

Now we will find the coordinates of Hermitian projection of the given point (y) in a given r-point $R(x')(x'')\dots(x^{(r)})$. In two points $(x')(x'')$, there is a point $\lambda'(x')+\lambda''(x'')$ which is orthogonal to the point (y) in our sense and this can be found by the equation

$$f(\lambda' x'+\lambda'' x'', \bar{y}) = 0,$$

$$f(x'\bar{y})\,\lambda' + f(x''\bar{y})\,\lambda'' = 0.$$

Thus, we have

$$\lambda'(x)+\lambda''(x'') = \frac{(x')}{f(x''\bar{y})} - \frac{(x'')}{f(x'\bar{y})}\,.$$

Similarly in the lines $(x')(x'')$, ..., $(x')(x^r)$, there are the points

$$\frac{(x')}{f(x'''\bar{y})} - \frac{(x''')}{f(x'\bar{y})}\,,$$

$$\frac{(x')}{f(x^{\mathrm{IV}}\bar{y})} - \frac{(x^{\mathrm{IV}})}{f(x'\bar{y})}\,,$$

....

which are orthogonal to the point (y) in our sense. These points are orthogonal to (y) in our sense and exist in R. If we indicate with

$$\lambda'(x')+\lambda''(x'')+\dots+\lambda^{(r)}(x^{(r)})$$

the orthogonal point to the $r-1$-point which is the projection of (y)

on R, then the coordinates of this point will satisfy the condition of Hermitian orthogonality

$$f\left(\lambda' x' + \lambda'' x'' + \dots, \frac{\bar{x}'}{f(y\bar{x}'')} - \frac{\bar{x}''}{f(y\bar{x}')}\right) = 0, \dots$$

Hence we have

$$\frac{f(x'\bar{x}')\lambda' + f(x''\bar{x}')\lambda'' + \dots}{f(y\bar{x}')}$$

$$= \frac{f(x'\bar{x}'')\lambda' + f(x''\bar{x}'')\lambda'' + \dots}{f(y\bar{x}'')}$$

$$= \dots\dots\dots$$

$$= \sigma \text{ (say)}.$$

Therefore

$$f(x'\bar{x}')\lambda' + f(x''\bar{x}')\lambda'' + \dots + f(x^r\bar{x}')\lambda^{(r)} = f(y\bar{x})\sigma,$$

$$f(x'\bar{x}'')\lambda' + f(x''\bar{x}'')\lambda'' + \dots + f(x^r\bar{x}'')\lambda^{(r)} = f(y\bar{x}'')\sigma.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The solution of the simultaneous equations gives us

$$-\sigma : \lambda : \lambda'' \vdots = \left\| \begin{matrix} f(y\bar{x}') & f(x'\bar{x}') & f(x''\bar{x}') & \dots \\ f(y\bar{x}'') & f(x'\bar{x}'') & f(x''\bar{x}'') & \dots \\ \dots & \dots & & \end{matrix} \right\| \dots (\text{V}).$$

Then we see that the point

$$\left\{ \left| \begin{matrix} 0 & x' & x'' & \dots \\ f(y\bar{x}') & f(x'\bar{x}') & f(x'\bar{x}'') & \dots \\ f(y\bar{x}'') & f(x'\bar{x}'') & f(x''\bar{x}'') & \dots \\ \dots & & & \end{matrix} \right| \right\}$$

is the Hermitian projection of the point (y) in R.

We can easily find that the point which is orthogonal in our sense to the Hermitian projection of (y) and lying on the projecting line has the coordinates

$$\left\{ \left| \begin{matrix} y & x' & x'' & \dots \\ f(y\bar{x}') & f(x'\bar{x}') & f(x''\bar{x}') & \dots \\ f(y\bar{x}'') & f(x'\bar{x}'') & f(x''\bar{x}'') & \dots \\ \dots & & & \end{matrix} \right| \right\}.$$

Now we will find the point of R from which the Hermitian distance to the given point (y) will be minimum. A point in $R(x')(x'')$ $\dots (x^{(r)})$ has the coordinates of the form

$$(u) \equiv \lambda'(x')+\lambda''(x'')+\ldots+\lambda^{(r)}(x^{(r)}).$$

But

$$\text{hcos}^2(uy) = \frac{f(y\bar{u})f(u\bar{y})}{f(y\bar{y})f(u\bar{u})}.$$

The differentiate h $\cos^2(uy)$ by $\lambda', \lambda'', \ldots$ respectively, then we have

$$f(y\bar{x})f(u\bar{u})-f(u\bar{x}')f(y\bar{u}) = 0,$$
$$f(y\bar{x}'')f(u\bar{u})-f(u\bar{u}'')f(y\bar{u}) = 0.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Put

$$\frac{f(u\bar{u})}{f(y\bar{u})} \equiv \varepsilon,$$

then

$$f(x'\bar{x}')\lambda'+f(x''\bar{x}')\lambda''+\ldots+f(x^r\bar{x}')\lambda^{(r)} = f(y\bar{x}')\varepsilon,$$
$$f(x'\bar{x}'')\lambda'+f(x''\bar{x}'')\lambda''+\ldots+f(x^r\bar{x}'')\lambda^{(r)} = f(y\bar{x}'')\varepsilon,$$
$$\ldots\ldots\ldots$$
$$\ldots\ldots\ldots$$

The point (u) corresponding to $\lambda^{(i)}$ $(i=1 \ldots r)$ found from the above equations gives us the minimum distance. The $\varepsilon, \lambda^{(i)}$ $(i= 1, 2 \ldots r)$ are the same as those that are obtained in (V).

Therefore

$$\text{h}\cos^2(yR) \equiv \text{h}\cos^2(yu) = \frac{f(u\bar{y})f(y\bar{u})}{f(u\bar{u})f(y\bar{y})}.$$

But

$$f(u\bar{y}) = \begin{vmatrix} 0 & f(x'\bar{y}') & f(x''\bar{y}) & \ldots \\ f(y\bar{x}') & f(x'\bar{x}') & f(x''\bar{x}') & \ldots \\ f(y\bar{x}'') & f(x'\bar{x}'') & f(x''\bar{x}'') & \ldots \\ \ldots & \ldots & & \end{vmatrix},$$

$$f(y\bar{u}) = \begin{vmatrix} 0 & f(y\bar{x}') & f(y\bar{x}'') & \ldots \\ f(x'\bar{y}) & f(x'\bar{x}') & f(x'\bar{x}'') & \ldots \\ f(x''\bar{y}) & f(x''\bar{x}') & f(x''\bar{x}'') & \ldots \\ \ldots & \ldots & & \end{vmatrix},$$

$$f(u\bar{x}') = \begin{vmatrix} 0 & f(x'\bar{x}') & f(x'\bar{x}') & \ldots \\ f(y\bar{x}') & f(x'\bar{x}') & f(x''\bar{x}') & \ldots \\ f(y\bar{x}'') & f(x'\bar{x}'') & f(x''\bar{x}'') & \ldots \\ \ldots & \ldots & & \end{vmatrix},$$

$$= -f(y\bar{x}')\begin{vmatrix} f(x'\bar{x}') & f(x''\bar{x}') & \dots \\ f(x'\bar{x}'') & f(x''\bar{x}'') & \dots \\ \dots \\ \dots \end{vmatrix},$$

and

$$f(u\bar{u}) = \begin{vmatrix} 0 & f(u\bar{x}') & f(u\bar{x}'') & \dots \\ f(x'\bar{y}) & f(x'\bar{x}') & f(x'\bar{x}'') & \dots \\ f(x''\bar{y}) & f(x''\bar{x}') & f(x''\bar{x}'') & \dots \\ \dots \end{vmatrix}.$$

Therefore

$$\mathrm{hcos}^2(yR) = -\frac{1}{f(y\bar{y})\begin{vmatrix} f(x'\bar{x}') & \dots \\ ,, & \cdot \end{vmatrix}}\begin{vmatrix} 0 & f(x'\bar{y}) & f(x''\bar{y}) & \dots \\ f(y\bar{x}') & f(x'\bar{x}') & f(x''\bar{x}') & \dots \\ f(y\bar{x}'') & f(x'\bar{x}'') & f(x''\bar{x}'') & \dots \\ \dots \end{vmatrix}.$$

Hence

$$\mathrm{hsin}^2(yR) = \frac{\begin{vmatrix} f(y\bar{y}) & f(x'\bar{y}) & f(x''\bar{y}) & \dots \\ f(y\bar{x}') & f(x'\bar{x}') & f(x''\bar{x}') & \dots \\ f(y\bar{x}'') & f(x\bar{x}'') & f(x''\bar{x}'') & \dots \\ \dots \end{vmatrix}}{f(y\bar{y})\begin{vmatrix} f(x'\bar{x}') & f(x''\bar{x}') & \dots \\ f(x'\bar{x}'') & f(x''\bar{x}'') & \dots \\ \dots \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} y\, x'\, x'' \dots \\ \bar{y}\, \bar{x}'\, \bar{x}'' \dots \end{vmatrix}_\alpha}{\begin{vmatrix} y \\ \bar{y} \end{vmatrix}_\alpha \cdot \begin{vmatrix} x'\, x'' \dots \\ \bar{x}'\, \bar{x}'' \dots \end{vmatrix}_\alpha}.$$

We can easily prove that the common Hermitian perpendiculars between two multi-points R and R' give the couples of points which give the mininum distances of R and R'. Therefore such distances shall be defined as Hermitian distances between R and R'.

The moment of two multi-points in Hermitian space shall be defined as the product of the sines of ρ Hermitian distances between them and will be denoted by the symbol *hm*. Suppose that the points (z'), (z''), ... (z^k) determine k-point K which is common to the given multi-points M and M', and that (x'), (x'') ... $(x^{(r-k)})$ and (y'), (y'')

$\dots(y^{(r-k)})$ with $(z'), (z'')\dots(z^{(k)})$ determine M and M' respectively, then we shall have the relation:

$$\text{hm}^2(M.M') \equiv \Pi\,\text{hsin}^2\,(\text{distance})$$

$$= \frac{\begin{vmatrix} f(z'\bar{z}') & f(z''\bar{z}' & \dots \\ f(z''\bar{z}') & f(z''\bar{z}'') & \dots \\ \dots \end{vmatrix} \cdot \begin{vmatrix} f(x'\bar{x}') & \dots & & \\ & \dots & f(y'\bar{y}') & \\ & & \dots & \\ & & & f(z'\bar{z}') \\ & & & \dots \end{vmatrix}}{\begin{vmatrix} f(x'\bar{x}') & f(x'\bar{x}'') & \dots \\ f(x''\bar{x}') & f(x''\bar{x}'') & \\ \dots & & f(z'\bar{z}') \\ & & \dots \end{vmatrix} \cdot \begin{vmatrix} f(y'\bar{y}') & f(y'\bar{y}'') & \dots \\ f(y''\bar{y}'') & f(y''\bar{y}'') & \dots \\ \dots & & f(z'\bar{z}') \\ & & \dots \end{vmatrix}}.$$

We will prove that the right hand side of the equality is equal to the left hand side.

Substitute

$$\lambda'(z') + \lambda''(z'') + \dots,\ \mu'(z') + \mu''(z'') + \dots,\ \dots$$

for (z'), (z''), ...
then the determinant

$$\begin{vmatrix} f(z'\bar{z}') & f(z'\bar{z}'') & \dots \\ f(z'\bar{z}'') & f(z''\bar{z}'') & \dots \\ \dots \end{vmatrix}$$

will be multiplied by the factor

$$|\lambda\mu\dots|\cdot|\bar{\lambda}\bar{\mu}\dots|$$

after the transformation. This substitution is equivalent to the substitution of

$$\begin{aligned} &(x),\ \dots \text{ in } 1(x)+0(x)+\dots+0(y)+\dots+0(z)+\dots,\dots \\ &(y),\ \dots \text{ in } 0(x)+\dots\dots\dots\dots+1(y)+\dots+0(z)+\dots,\dots \\ &(z),\ \dots \text{ in } 0(x)+\dots\dots\dots\dots+0(y)+\dots+1(z)+\dots,\dots. \end{aligned}$$

Therefore, by this substitution, the determinant

$$\begin{vmatrix} f(x'\bar{x}') & f(x'\bar{x}'') & \dots & & \\ f(x''\bar{x}') & f(x''\bar{x}'') & & & \\ \dots & & \dots & & \\ \dots & & f(y'\bar{y}') & & \\ \dots & & & \dots & \\ \dots & & & & f(z'\bar{z}') \\ \dots & & & & \dots \end{vmatrix}$$

gets the same factor

$$|\lambda\mu \dots|\cdot|\bar{\lambda}\bar{\mu} \dots|.$$

For the same reason, we can see that the determinants

$$\begin{vmatrix} f(x'\bar{x}') & f(x'\bar{x}'') & \dots\dots\dots & \\ f(x''\bar{x}') & f(x''\bar{x}'') & & \\ & & \ddots & \\ \dots & & & f(z'\bar{z}') \\ \dots & & & \dots \end{vmatrix}$$

and

$$\begin{vmatrix} f(y'\bar{y}') & f(y'\bar{y}'') & \dots\dots\dots & \\ f(y''\bar{y}') & f(y''\bar{y}'') & & \\ & & \ddots & \\ \dots & & & f(z'\bar{z}') \\ \dots & & & \dots \end{vmatrix}$$

get also the same factors respectively. So we have the same value of hm (MM') after the transformation.

Without loss of generality, we may assume that the points (z'), (z''), ... are mutually orthogonal in our sense in K. In addition, we can assume that (x^i) and (y^i) $(i=1, 2\dots\rho)$ are the extreme points of the distances between M and M' and that $(x^\rho)\dots(x^{r-k})$ are the individual points in M and $(y^\rho)\dots(y^{r'-k})$ those in M'. Hence the determi-

$$\begin{vmatrix} f(x'\bar{x}') \dots & & & \\ & \ddots & & \\ \dots & f(y'\bar{y}') & & \\ & & \ddots & \\ \dots & & f(z'\bar{z}') & \\ & & & \dots \end{vmatrix}$$

is equal to the product

$$\begin{vmatrix} f(x'\bar{x}') & f(x'\bar{y}') \\ f(y'\bar{x}') & f(y'\bar{y}') \end{vmatrix} \dots \begin{vmatrix} f(x^\rho\bar{x}^\rho) & f(x^\rho\bar{y}^\rho) \\ f(y^\rho\bar{x}^\rho) & f(y^\rho\bar{y}^\rho) \end{vmatrix} f(x^{\rho+1}\bar{x}^{\rho+1})\dots f(y'\bar{y}')\dots f(z'\bar{z}')\dots .$$

So the given expression is reduced to

$$\frac{\begin{vmatrix} f(x'\bar{x}') & f(x'\bar{y}') \\ f(y'\bar{x}') & f(y'\bar{y}') \end{vmatrix}}{f(x'\bar{x}')f(y'\bar{y}')} \dots \frac{\begin{vmatrix} f(x^\rho\bar{x}^\rho) & f(x^\rho\bar{y}^\rho) \\ f(y^\rho\bar{x}^\rho) & f(y^\rho\bar{y}^\rho) \end{vmatrix}}{f(x^\rho\bar{x}^\rho)f(y^\rho\bar{y}^\rho)} .$$

But this is equal to

$$\operatorname{hsin}^2(x'y')\operatorname{hsin}^2(x''y'')\dots\operatorname{hsin}^2(x^{(\rho)}y^{(\rho)}).$$

which is to be proved.

This may be written in other forms

$$\operatorname{hm}^2(RR')=\frac{\begin{vmatrix} z'\ldots \\ \bar{z}'\ldots \end{vmatrix}_a \cdot \begin{vmatrix} x'\ldots y'\ldots z'\ldots \\ \bar{x}'\ldots \bar{y}'\ldots \bar{z}'\ldots \end{vmatrix}_a}{\begin{vmatrix} x'\ldots z'\ldots \\ \bar{x}'\ldots \bar{z}'\ldots \end{vmatrix}_a \cdot \begin{vmatrix} y'\ldots z'\ldots \\ \bar{y}'\ldots \bar{z}'\ldots \end{vmatrix}_a}$$

or

$$\operatorname{hm}^2(RR')=\frac{\left\{\left|z'z''\ldots\right|\cdot\left|\frac{\partial f}{\partial z'}\frac{\partial f}{\partial z''}\cdot\right|\right\}\left\{\left|x'\ldots y'\ldots z'\ldots\right|\cdot\left|\frac{\partial f}{\partial x'}\cdots\frac{\partial f}{\partial y'}\cdots\frac{\partial f}{\partial z''}\cdots\right|\right\}}{\left\{\left|x'\ldots z'\ldots\right|\cdot\left|\frac{\partial f}{\partial x'}\cdots\frac{\partial f}{\partial z'}\cdots\right|\right\}\left\{\left|y'\ldots z'\ldots\right|\left|\frac{\partial f}{\partial y'}\cdots\frac{\partial f}{\partial z'}\cdots\right|\right\}}.$$

As a special case, if R and R' have no common multi-point, then we obtain

$$\operatorname{hm}^2(RR')=\frac{\begin{vmatrix} x'x''\ldots y'y''\ldots \\ \bar{x}'\bar{x}''\ldots \bar{y}'\bar{y}''\ldots \end{vmatrix}_a}{\begin{vmatrix} x'\,x''\ldots \\ \bar{x}'\,\bar{x}''\ldots \end{vmatrix}_a \cdot \begin{vmatrix} y'\,y''\ldots \\ \bar{y}'\,\bar{y}''\ldots \end{vmatrix}_a}.$$

When $r+r\cdot=n$, we have:

$$\operatorname{hm}^2(RR')=\frac{\left|x'x''\ldots y'y''\ldots\right|\cdot\left|\bar{x}'\bar{x}''\ldots\bar{y}'\bar{y}''\ldots\right|}{\begin{vmatrix} x'\,x''\ldots \\ \bar{x}'\,\bar{x}''\ldots \end{vmatrix}_a \cdot \begin{vmatrix} y'\,y''\ldots \\ \bar{y}'\,\bar{y}''\ldots \end{vmatrix}_a}.$$

§ 2. Commoment of two multi-points.

The commoment of R and R' shall be defined as the product of the cosines of the Hermitian distances between R and R'. Therefore the commonent of R and R' is equal to the moment of R and R_1 or R_0 and R'.

Let $(u'), (u''), \dots (u^{l'})$ be l' individual points in L' and $(x'), (x''), (x')\dots(x^{r-l'})$ the other $r-l'$ points which specify R together with $(u'), (u')\dots$. Similarly $(v'), (v'')\dots(v^{l})$ be l individual points of L and $(y'), (y''), \dots (y^{r'-l})$ the other $r'-l$ points which specify R' together with $(v'), (v'')\dots$. And suppose

$$r-l'=r'-l=\rho+k.$$

Then we shall have

$$\mathrm{hcm}^2(RR') = \frac{\begin{vmatrix} f(u'\bar{u}') & f(u'\bar{u}'') & \cdots \\ f(u''\bar{u}') & f(u''\bar{u}'') & \\ \cdots & & \end{vmatrix} \cdot \begin{vmatrix} f(v'\bar{v}') & f(v'\bar{v}'') & \cdots \\ f(v''\bar{v}') & f(v''\bar{v}'') & \\ \cdots & & \end{vmatrix}}{\begin{vmatrix} f(x'\bar{x}') & \cdots\cdots & \\ \cdots & \cdots & \\ \cdots & f(u'\bar{u}') & \\ & & \cdots \end{vmatrix} \cdot \begin{vmatrix} f(y'\bar{y}') & & \\ & \cdots & \\ & & f(v'\bar{v}') \\ & & \cdots \end{vmatrix}}$$

$$\times \begin{vmatrix} f(x'\bar{y}') & f(x'\bar{y}'') & \cdots \\ f(x'\bar{y}') & f(x''\bar{y}'') & \\ \cdots & & \\ & & f(x^{r-l'}\bar{y}^{r-l'}) \end{vmatrix} \cdot \begin{vmatrix} f(y'\bar{x}') & f(y'\bar{x}'') & \cdots \\ f(y''\bar{x}') & f(y''\bar{x}'') & \cdots \\ \cdots & \cdots & \\ & & f(y^{r'-l}\bar{x}^{r'-l}) \end{vmatrix}.$$

These determinants in $u\ldots$ and $v\ldots$ in the above expression are in-invariants for individual points L' or L. Thus the second member remains invariable by this substitution.

Now substitute

$$\sum_{j=1}^{n}{}' \lambda_{ij}(x^j) + \sum_{j=1}^{n} \beta^j(u^j), \quad (i = 1, 2, 3 \ldots)$$

for $\quad (x^j)\,(j = 1, \ldots),\ (u^j)\ (j = 1,\ \ldots),$

Then we see that the determinant

$$\begin{vmatrix} f(x'\bar{y}') & \cdots\cdots\cdots & \\ \vdots & f(x''\bar{y}'') & \\ \vdots & & \cdots \end{vmatrix} \cdot \begin{vmatrix} f(y'\bar{x}') & \cdots\cdots\cdots & \\ \vdots & f(y''\bar{x}'') & \\ \vdots & & \cdots \end{vmatrix}$$

gets the factor $|\lambda_{ij}| \cdot |\bar{\lambda}_{ij}|$

In the same manner, the transformation multiplies the denominator by the same factor. We can suppose without loss of generality that the points $(u'), (u'') \ldots$ are orthogonal to each other in L' and $(v'), (v'') \ldots$ in L in our sense and $(x^i), (y^i)$ be ρ extreme points of the distance between R, R' and $(x^\rho), \ldots (x^{r'-l})$ be k individual points orthogonal to each other in K and $(y^\rho), \ldots (y^{r-l'})$ correspond to them. Then the given expression is reduced to

$$\frac{f(x'\bar{y}')f(y'\bar{x}')}{f(x'\bar{x}')f(y'\bar{y}')}\cdot\frac{f(x''\bar{y}'')f(y''\bar{x}'')}{f(x''\bar{x}'')f(y''\bar{y}'')}\cdot\ldots\frac{f(x^{\rho}\bar{y}^{\rho})f(y^{\rho}\bar{x}^{\rho})}{f(x^{\rho}\bar{x}^{\rho})f(y^{\rho-1}\bar{y}^{\rho}},$$

which is equal to

$$\mathrm{h}\cos^2(x'y')\,.\,\mathrm{h}\cos^2(x''y'')\ldots\mathrm{h}\cos^2(x^{\rho}y^{\rho}).$$

Therefore the given expression is equal to $hcm^2(RR')$.

The expression for hcm can be written as follows when they have no orthogonality. In this case $l=0$ and $l'=r-r'$, therefore

$$\mathrm{hcm}^2(RR') = \frac{\begin{vmatrix} f(u'\bar{u}') & \ldots & \\ \ldots & \ldots & \\ & & f(u^{r-r'}\bar{u}^{r-r'} \end{vmatrix}}{\begin{vmatrix} f(x'\bar{x}') & \ldots & & \\ \ldots & \ldots & & \\ & & f(u'\bar{u}') & \\ & & & \ldots \end{vmatrix}\cdot\begin{vmatrix} f(y'\bar{y}') & \ldots \\ \ldots & \ldots \end{vmatrix}}\times$$

$$\begin{vmatrix} f(x'\bar{y}')f(x''\bar{y}') & \ldots \\ f(x'\bar{y}'')f(x''\bar{y}'') & \ldots \\ \vdots & \end{vmatrix}\cdot\begin{vmatrix} f(y'\bar{x}')f(y'\bar{x}'') & \ldots \\ f(y''\bar{x}')f(y'\bar{x}'') & \ldots \\ \vdots & \end{vmatrix}.$$

Further, if $r=r'$, then $l=r-r'=0$, so we have

$$\mathrm{hcm}^2(RR') = \frac{\begin{vmatrix} f(x'\bar{y}')\ f(x''\bar{y}') & \ldots \\ f(x'\bar{y}'')f(x''\bar{y}'') & \ldots \\ \ldots & \end{vmatrix}\cdot\begin{vmatrix} f(y'\bar{x}')f(y'\bar{x}'') & \ldots \\ f(y''\bar{x}')f(y''\bar{x}'') & \\ \ldots & \end{vmatrix}}{\begin{vmatrix} f(x'\bar{x}') & \ldots \\ & \ldots \end{vmatrix}\cdot\begin{vmatrix} f(y'\bar{y}') & \ldots \\ \ldots & \ldots \end{vmatrix}}.$$

Suppose that R is defined by the points $(x'), (x'') \ldots (x^r)$; L by the points $(v'), (v'') \ldots (v^l)$, R' by $(y') \ldots (y^{r'-l})$ together with $(v')\ (v'') \ldots (v^l)$, then we may write

$$\mathrm{hcm}^2(RR') = (-1)^{r'-l}\times$$

$$\frac{\begin{vmatrix} f(v'\bar{v}') & \ldots & \\ \vdots & f(v''\bar{v}'') & \\ & & \ldots \end{vmatrix}}{\begin{vmatrix} f(x'\bar{x}') & \ldots & \\ \vdots & f(x''\bar{x}'') & \\ & & \ldots \end{vmatrix}\cdot\begin{vmatrix} f(y'\bar{y}') & \ldots & \\ \ldots & f(y'\bar{y}'') & \\ & f(v'\bar{v}') & \ldots \end{vmatrix}}\begin{vmatrix} 0 \ldots\ldots\ldots\ldots 0 & f(y'\bar{x}')\ f(y'\bar{x}'') \ldots \\ \vdots\qquad\qquad\ \vdots & f(y''\bar{x}')f(y''\bar{x}') \\ 0 \ldots\ldots\ldots\ldots 0 & f(y^{r'-l}\,\bar{x}') \ \ldots \\ f(x'\bar{y}')\ldots f(x\bar{y}^{r'-l}) & f(x'\bar{x}')f(x'\bar{x}'')\ldots \\ \vdots\qquad\qquad\ \vdots & f(x''\bar{x}')f(x''\bar{x}'') \\ & \vdots \end{vmatrix}.$$

Similarly, if L' is defined by the points $(u'), (u'')\ldots(u^{l'})$, R by $(x')(x'')\ldots(x^{r-l'})$ together with $(u')(u'')\ldots(u^{l'})$, R' by $(y')(y'')\ldots(y^{r'})$, then we have

$$\mathrm{hcm}^2(RR') = (-1)^{r-l'} \times \frac{\begin{vmatrix} f(u'\bar{u}') & \cdots \\ \vdots & \end{vmatrix}}{\begin{vmatrix} f(x'\bar{x}') \cdots & & \\ & \cdots & \\ \vdots & f(u'\bar{u}') & \\ & & \cdots \end{vmatrix} \cdot \begin{vmatrix} f(y'\bar{y}') \cdots & \\ \vdots & \\ \vdots & \end{vmatrix}} \begin{vmatrix} 0 \ldots\ldots\ldots 0 & f(x'\bar{y}') & f(x'\bar{y}'') & \cdots \\ \vdots & \vdots & \vdots & \\ 0 \ldots\ldots\ldots 0 & f(x^{r'-l}\bar{y}') \cdots & & \\ f(y\bar{x}') \ldots f(y\bar{x}^{r-l'}) & f(y'\bar{y}') & f(y'\bar{y}'') & \cdots \\ \vdots & f(y''\bar{y}') & f(y''\bar{y}'') & \cdots \\ \vdots & \vdots & & \end{vmatrix}.$$

When R and R' have no Hermitian orthogonality, we have

$$\mathrm{hcm}^2(RR') = \frac{(-1)^{r'}}{\begin{vmatrix} f(x'\bar{x}') & \cdots \\ \vdots & \cdots \end{vmatrix} \cdot \begin{vmatrix} f(y'\bar{y}') & \cdots \\ \vdots & \cdots \end{vmatrix}} \begin{vmatrix} 0 \ldots 0 & f(y'\bar{x}') & f(y'\bar{x}'') \cdots \\ \vdots \; \vdots & \vdots & \\ 0 \ldots 0 & & \\ f(x'\bar{y}') \ldots & f(x'\bar{x}') \cdots & \\ \vdots & \vdots & \end{vmatrix}.$$

Let $\overset{r}{\mathfrak{X}}_{1\ldots r\ldots}$, be the coordinates of R and $\overset{r'}{\mathfrak{Y}}_{1\ldots r\ldots}$, those of R'. If $r+r' < n$, then they can not have common points, so the denominator of the right hand member is equal to

$$\sum a_{c\ldots d,\, g\ldots h\ldots} \begin{vmatrix} x_c & \cdots\cdots & \\ \vdots & \cdots & \\ \vdots & y_d & \\ \vdots & & \cdots \end{vmatrix} \cdot \begin{vmatrix} \bar{x}_g & \cdots\cdots & \\ \vdots & \cdots & \\ \vdots & \bar{y}_h & \\ \vdots & & \cdots \end{vmatrix},$$

where $c\ldots d\ldots, \ldots g\ldots h\ldots$ is a permutation of $1.2\ldots n$ taken $r+r'$ at once. Now expand the determinants in the expression

$$\sum a_{c\ldots d,\, g\ldots h\ldots} \begin{vmatrix} x_c & & \\ & \cdots & \\ & y_d & \\ & & \cdots \end{vmatrix} \cdot \begin{vmatrix} \bar{x}_g & & \\ & \cdots & \\ & \bar{y}_h & \\ & & \cdots \end{vmatrix},$$

then we obtain

$$\sum a_{c\ldots d\ldots,\, g\ldots h\ldots}\, \overset{r}{\mathfrak{X}}_{c\ldots}\, \overset{r'}{\mathfrak{Y}}_{d\ldots}\, \overset{r}{\bar{\mathfrak{X}}}_{g\ldots}\, \overset{r'}{\bar{\mathfrak{Y}}}_{h\ldots},$$

where $c\ldots d$ and $g\ldots h\ldots$ indicate the disposition of the class $r+r'$ of $1, 2, \ldots n$ and provided that $c\ldots$ and $g\ldots$ present no inversion and $d\ldots$, $h\ldots$ have the same property. Thus the denominator is reduced to

$$f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})\cdot f(\overset{r'}{\mathfrak{Y}}\overset{r'}{\bar{\mathfrak{Y}}}).$$

Hence we have

$$\mathrm{hm}^2(RR')=\frac{\sum a_{c\ldots d\ldots,\,g\ldots h\ldots}\overset{r}{\mathfrak{X}}_{e\ldots}\overset{r'}{\mathfrak{Y}}_{d\ldots}\overset{r}{\bar{\mathfrak{X}}}_{g\ldots}\overset{r'}{\bar{\mathfrak{Y}}}_{h\ldots}}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})\cdot(\overset{r'}{\mathfrak{Y}}\overset{r'}{\bar{\mathfrak{Y}}})}.$$

If $r+r'=n$ and R and R' have no common points, then we have

$$\mathrm{hm}^2(RR')=\frac{\begin{vmatrix} x_c & & \\ & \cdots & \\ & & y_d \\ & & \cdots \end{vmatrix}\cdot\begin{vmatrix} \bar{x}_c & & \\ & \cdots & \\ & & \bar{y}_d \\ & & \cdots \end{vmatrix}}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})f(\overset{r'}{\mathfrak{Y}}\overset{r}{\bar{\mathfrak{Y}}})}.$$

If $r+r'=n+k$, and R, R' have common k-points, then we get

$$\mathrm{hm}^2(RR')=\frac{\sum a_{b\ldots,f\ldots}\overset{r}{\mathfrak{X}}_{b\ldots c\ldots}\overset{r'}{\mathfrak{Y}}_{b\ldots d\ldots}\overset{r}{\bar{\mathfrak{X}}}_{f\ldots g\ldots}\overset{r'}{\bar{\mathfrak{Y}}}_{f\ldots h\ldots}}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})\cdot f(\overset{r}{\mathfrak{Y}}\overset{r'}{\bar{\mathfrak{Y}}})},$$

where $b\ldots c\ldots d\ldots, f\ldots g\ldots h\ldots$ are any permutations between $1, 2, \ldots n$ in which the group of k elements $b\ldots$ and $f\ldots$ have no inversion and $r'-k$ elements $c\ldots$ and $h\ldots$ have at least no inversion.

If $r+r'<n+k$ and R, R' have common k-point, then we can prove that

$$\mathrm{hm}^2(RR')=\frac{\sum a_{b\ldots,f\ldots}\,a_{b\,.\,c\,..d\ldots,f\ldots g\ldots h\ldots}\overset{r}{\mathfrak{X}}_{b\ldots c\ldots}\overset{r'}{\mathfrak{Y}}_{b\ldots d\ldots}\overset{r}{\bar{\mathfrak{X}}}_{f\ldots g\ldots}\overset{r'}{\bar{\mathfrak{Y}}}_{f\ldots h\ldots}}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})\cdot f(\overset{r'}{\mathfrak{Y}}\overset{r'}{\bar{\mathfrak{Y}}})},$$

where $b\ldots c\ldots d\ldots, f\ldots g\ldots h\ldots$ indicate such a disposition of the class $r+r'-k$ in which the first k suffices $b\ldots$ or $f\ldots$, the second $r-k$ suffices $c\ldots$ or $g\ldots$ and the remaining $r'-k$ suffices $d\ldots$ or $h\ldots$ with no inversion.

The expression for Hermitian commonent can be reduced from that of Hermitian moment of R and R_1. The coordinates of R are

$$\left(\frac{\partial f(\overset{n-r'}{\mathfrak{Y}}\overset{n-r'}{\bar{\mathfrak{Y}}})}{\partial\mathfrak{Y}_{1\cdots(n-r')}},\ \ldots,\ \ldots\right),$$

or

$$\left(\sum a_{1\cdots r',j\cdots}\overset{r'}{\bar{\mathfrak{Y}}}_{j\cdots},\ \ldots,\ \ldots\right).$$

When R and R' have no orthogonality, we have

$$\mathrm{hcm}^2(RR') = \frac{\sum a_{b\ldots,f\ldots}\, a_{c\ldots,p\ldots}\, a_{g\ldots,q\ldots}\, \overset{r}{\mathfrak{X}}_{b\ldots c\ldots}\, \overset{r'}{\bar{\mathfrak{Y}}}_{p\ldots}\, \overset{r}{\bar{\mathfrak{X}}}_{f\ldots g\ldots}\, \overset{r'}{\mathfrak{Y}}_{q\ldots}}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})\, f(\overset{r}{\mathfrak{Y}}\overset{r}{\bar{\mathfrak{Y}}})},$$

where $(b\ldots), (f\ldots, c\ldots), (g\ldots, p\ldots)$ and $(q\ldots)$ are the permutation of the class $r-r', r', r'$; group $b\ldots$ and $f\ldots$ have no common indices with $c\ldots$ and $g\ldots$ respectively.

When $r=r'$, the preceding expression is reduced to

$$\mathrm{hcm}^2(RR') = \frac{\left(\sum a_{c\ldots p\ldots}\, \overset{r}{\mathfrak{X}}_{c\ldots}\, \overset{r}{\bar{\mathfrak{Y}}}_{p\ldots}\right)\left(\sum a_{c\ldots p\ldots}\, \overset{r}{\bar{\mathfrak{X}}}_{c\ldots}\, \overset{r}{\mathfrak{Y}}_{p\ldots}\right)}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})\, f(\overset{r}{\mathfrak{Y}}\overset{r}{\bar{\mathfrak{Y}}})} \quad \ldots \text{(vi)},$$

where $c\ldots$ is a permutation of r figures taken from $1, 2, 3 \ldots n$ and $p\ldots$ is that of r figures taken from the remaining $n-r$ figures.

When R and R' have l-times orthogonal in our sense, we have the following relation:

$$\mathrm{hcm}^2(RR') = \frac{\sum a_{b\ldots f\ldots}\, a_{c\ldots d\ldots}\, a_{g\ldots l\ldots,q\ldots}\, a_{d\ldots h\ldots}\, \overset{r}{\mathfrak{X}}_{b\ldots c\ldots}\, \overset{r'}{\bar{\mathfrak{Y}}}_{p\ldots}\, \overset{r}{\bar{\mathfrak{X}}}_{f\ldots g\ldots}\, \overset{r'}{\mathfrak{Y}}_{q\ldots}}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}}) \cdot f(\overset{r}{\mathfrak{Y}}\overset{r}{\bar{\mathfrak{Y}}})},$$

where $(b\ldots)(f\ldots, c\ldots), (g\ldots, d\ldots), (h\ldots, p\ldots)$ and $(q\ldots)$ are the permutations of figures $12 \ldots n$ of the class $r-r'+l=l'$, $r'-l=r-l'$ $(=k+\rho)$, l, r' respectively; and $(b\ldots c\ldots d\ldots), (f\ldots g\ldots h\ldots)$ are the dispositions of the class $r+l=r'+l'$ of $12\ldots n$.

(vi) may also be written in the form

$$\mathrm{hcm}^2(RR') = \frac{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{Y}}}) \cdot f(\overset{r}{\mathfrak{Y}}\overset{r}{\bar{\mathfrak{X}}})}{f(\overset{r}{\mathfrak{X}}\overset{r}{\bar{\mathfrak{X}}})\, f(\overset{r}{\mathfrak{Y}}\overset{r}{\bar{\mathfrak{Y}}})}.$$

§ 3. Sine-amplitude.

We shall define the square root of the expression

$$\frac{\begin{vmatrix} f(x'\bar{x}') & f(x''\bar{x}') & f(x'''\bar{x}') & \ldots \\ f(x'\bar{x}'') & f(x''\bar{x}'') & & \ldots \\ \vdots & & \ldots & \\ \vdots & & & f(x^r\bar{x}^r) \end{vmatrix}}{f(x'\bar{x}')\, f(x''\bar{x}'') \ldots f(x^r\bar{x}^r)}$$

Hermitian sine-amplitude of r-point $(x')(x'')\ldots(x^r)$ and it may be denoted by the symbol

$$\mathrm{hsin}\,(x', x'' \ldots x^{(r)}).$$

As a special case if the Hermitian form of points be given by the equation

$$f(x\bar{x}) \equiv \varkappa^2 x'\bar{x}' + x''\bar{x}'' + \ldots + x_n\bar{x}_n \quad (\varkappa^2 = \pm 1),$$

then the expression for the sine-amplitude may be given in the form:

$$\sqrt{\frac{|\varkappa x' x'' \ldots x^{(r)}| \cdot |\varkappa \bar{x}' \bar{x}'' \ldots \bar{x}^{(r)}|}{f(x'\bar{x}')f(x''\bar{x}'')\ldots f(x^{(r)}\bar{x}^{(r)})}}.$$

Or, this expression may be written in the homogeneous form

$$\sqrt{\frac{|X'X''\ldots X^{(r)}| \cdot |\bar{X}'\bar{X}''\ldots\bar{X}^{(r)}|}{(X'\bar{X}')(X''\bar{X}'')\ldots X^{(r)}\bar{X}^{(r)})}},$$

where

$$X' = \varkappa x', \; X'' = x'' \ldots X^{(r)} = x^{(r)}.$$

In the real domain, the above expression becomes as follows:

$$\sqrt{\frac{|X'X''\ldots\ldots\ldots X^{(r)}|^2}{(X'X')(X''X'')\ldots(X^{(r)}X^{(r)})}}\ ^{(1)}$$

Specially if $r=2$, the above expression takes the form[(1)]

$$\sqrt{\frac{(X'\bar{X}')(X''\bar{X}'') - (X'\bar{X}'')(X''\bar{X}')}{(X'\bar{X}')(X''\bar{X}'')}}$$

which is equal to h sin $(x'x'')$. Hereafter we put $(X) \equiv (x)$, $(\varkappa\varkappa') = \varkappa^2$.

More generally, we shall define Hermitian sine-amplitude by the expression

$$\mathrm{hsin}^2(R_1'R''\ldots R^s) = \frac{\begin{vmatrix} f(\overset{r'}{\mathfrak{X}}{}'\overset{r'}{\bar{\mathfrak{X}}}{}') & f(\overset{r}{\mathfrak{X}}{}'\overset{r}{\bar{\mathfrak{X}}}{}'') & \ldots & f(\overset{r}{\mathfrak{X}}{}'\overset{r}{\bar{\mathfrak{X}}}{}^s) \\ f(\overset{p}{\mathfrak{X}}{}''\overset{r}{\bar{\mathfrak{X}}}{}') & f(\overset{r}{\mathfrak{X}}{}''\overset{r}{\bar{\mathfrak{X}}}{}'') & \ldots & f(\overset{r}{\mathfrak{X}}{}''\overset{r}{\bar{\mathfrak{X}}}{}^s) \\ \vdots & & & \\ f(\overset{r}{\mathfrak{X}}{}^s\overset{r}{\bar{\mathfrak{X}}}{}') & \ldots\ldots & \ldots & f(\overset{r}{\mathfrak{X}}{}^s\overset{r}{\bar{\mathfrak{X}}}{}^s) \end{vmatrix}}{f(\overset{r}{\mathfrak{X}}{}'\overset{r}{\bar{\mathfrak{X}}}{}')\, f(\overset{r}{\mathfrak{X}}{}''\overset{r}{\bar{\mathfrak{X}}}{}'')\ldots f(\overset{r}{\mathfrak{X}}{}^s\overset{r}{\bar{\mathfrak{X}}}{}^s)},$$

---

(1) See Fontené. L'hyperspace a $(n-1)$ dimensions.

(1) See Fubini. Sulle metriche definite da una forma Hermitiana. Atti del R-Instituto Veneto 63 (serie ottava (6) 1904). Also see Study. Kürzeste Wege im komplexen Raumes Math. Ann. Bd. 60 (1905).

where $R', ''R \dots R^s$ are all r-points and $(\overset{r}{\mathfrak{X}}{}^k)$ are the coordinates of R^k.

(1) Divide r-points into two groups $x' \dots x^k, x^{k+1} \dots x^r$, then we have

$$\text{hsin}\,(x'x''x \dots x^r) = \text{hsin}\,(x' \dots x^k)\,\text{hsin}\,(x^{k+1} \dots x^r)\,\text{hm}\,(x' \dots x^k, x^{k+1} x^r)\,.$$

(2) Choose k-points $x' \dots x^k$ in r-points $x' \dots x^r$ and form combinations of class s of $x' \dots x^r$, then we have

$$\left\{\text{hsin}\,(x' \dots x^k)\right\}^{\binom{r-k-1}{s}} \cdot \left\{\text{hsin}\,(x' \dots x^r)\right\}^{\binom{r-k-1}{s-1}}$$

$$= \text{hsin}\,(x \dots x^k x^{k+1})\,\text{hsin}\,(x' \dots x^k x^j \dots) \dots \text{hsin}\,(x' \dots x^k x^{k+1} x^j \dots, \dots)$$

by generalized Sylvester's theorem[1] of determinants. As a special case, we have

$$\frac{\text{hsin}\,(x'' \dots x^r)}{\text{hsin}\,(x'x'' \dots x^p, x'x'' \dots x^{r-1})} = \text{Constant.}$$

Let (Π^i) be a multi point formed by $x'x'' \dots x^{i-1} x^{i+1} \dots x^r$, then we have the relations

$$\frac{\text{hsin}\,(x'' \dots x^r)}{\text{hsin}\,(\Pi'' \dots \Pi^r)} = \frac{\text{hsin}\,(x' \dots x^r)}{\text{hsin}\,(\Pi'' \dots \Pi^r)} = \dots$$

$$= \frac{\text{h sin}\,(x'x'' \dots x^r)}{\text{h sin}\,(\Pi'\Pi'' \dots \Pi^r)}\,.$$

If the points $(x'), (x''), \dots (x^r)$ define r-point R and project this on an S-point $(r \leqq s)$ by the points $(z'), (z''), (z''') \dots (z^{(r)})$, then we have

$$\frac{\text{hsin}(z'z'' \dots z^{(r)})}{\text{hsin}\,(x'x'' \dots x^{(r)})} = \frac{\text{hcm}\,(R.\,S)}{\text{hcos}\,(x'z')\,\text{hcos}\,(x''z'') \dots \text{hcos}\,(x^{(r)}z^{(r)})}\,.$$

In fact, we see that the lines $x'z', z''z'' \dots, x^{(r)}z^{(r)}$ penetrate $(n-s)$-point S_1 (which is Hermitian conjugate to S) in the points $(u'), (u''), \dots (u^{(r)})$ which are the Hermitian projections of the points $(x'), (x''), \dots (x^{(r)})$ on S_1. Now we see that there is the relation:

$$\begin{aligned} &\text{hsin}\,(x'x'' \dots x^{(r)})\,\text{hsin}\,(u'u'' \dots u^{(r)})\,\text{hm}\,(RS_1) \\ &= \text{hsin}\,(x'x'' \dots x^{(r)} u'u'' \dots u^{(r)}) \\ &= \text{hsin}\,(z'z'' \dots z^{(r)} u'u'' \dots u^{(r)}) \\ &= \text{hsin}\,(x'u', x''u'', x'''u''', \dots x^{(r)}u^{(r)})\,\text{hsin}\,(x'u') \dots \text{hsin}\,(x^{(r)}u^{(r)}) \\ &= \text{hsin}\,(z'z'' \dots z^{(r)})\,\text{hsin}\,(u'u'' \dots u^{(r)})\,\text{hsin}\,(x'u') \dots \text{hsin}\,(x^{(r)}u^{(r)}) \end{aligned}$$

1 See Kowalewski. Einführung in der Determinanten-Theorie p. 110.

$$\begin{aligned} \text{hm}\,(RS_1) &= \text{hcm}\,(RS), \\ \text{hsin}\,(x'u') &= \text{hcos}\,(x'z'), \\ \text{hsin}\,(x''u'') &= \text{hcos}\,(x''z''). \end{aligned}$$

Therefore we have the following equation :—

$$\text{hsin}\,(x'x''\ldots)\,\text{hsin}\,(u'u''\ldots)\,cm\,\text{h}(RS)$$
$$= \text{hcos}\,(x'z')\,\text{hcos}\,(x''z'')\ldots\text{hsin}\,(z''z''\ldots)\,\text{hsin}\,(u'u''\ldots).$$

Therefore

$$\frac{\text{hsin}\,(z'z''\ldots)}{\text{hsin}\,(x'x''\ldots)} = \frac{\text{hcm}\,(RS)}{\text{hcos}\,(x'z')\,\text{hcos}\,(x''z'')\ldots}.$$

These formulae became equal to those given by D'Ovidio[1] when these points are all real.

§ 4. One dimensional chain in three dimensional space.

The manifoldness defined by the equations

$$x_i = x_i(t) \quad (t: \text{real}),$$
$$i = 1, 2 \ldots n,$$

shall be called a one dimensional chain.

Let P be a point in a one dimensional chain and P' be the next consecutive point in the chain. Draw the Hermitian perpendicular to the tangent at P to the chain. We shall call Q the intersection point of the perpendicular with the tangent at P. And we shall define the Hermitian curvature of the chain at the point P by the expression

$$\frac{1}{\rho} = \lim_{P'\to P} \left\{ \frac{2\,\text{hsin}\,\dfrac{P'Q}{\chi}}{\left(\text{hsin}\,\dfrac{PP'}{\chi}\right)^2} \right\}^2.$$

Now let the coordinates of P and P' be $(x), (x+dx)$ respectively, then we see that

$$\text{hsin}\,\frac{PP'}{\chi} = \frac{\sqrt{|xdx|\cdot|\bar{x}d\bar{x}|}^{\,1}}{\sqrt{(x\bar{x})(x+dx,\,\bar{x}+d\bar{x})}}$$

and

---

1 See D'Onidio. Le funzioni metriche fondamentali engli spazi di Guante si vogliano· dimensioni edi curvatura constante. Atti della Reale Accademia dei Lincei 1876-77. Serie Tera. Memorie. And also see D'Ovidio. Su varie questioni di metrica projettiva. Atti della Reale Accademia delle Scienze di Torino. Vol. xxviiii (1893).

2 An analogous definition for curvature in non-Euclidian space is adopted by Bianchi. See Bianchi. Lezioni di geometria differenziale. 1902 Vol. I. p. 450.

$$\text{hsin}\,\frac{P'Q}{\kappa} = \sqrt{\frac{|xdxd^2x|\cdot|\bar{x}d\bar{x}d^2\bar{x}|}{(x\bar{x})(|xdx|\cdot|\bar{x}d\bar{x}|)}}.$$

Hence we have

$$\frac{1}{\rho} = \frac{\sqrt{\left|x\frac{dx}{dt}\frac{d^2x}{dt^2}\right|\cdot\left|\bar{x}\frac{d\bar{x}}{dt}\frac{d^2\bar{x}}{dt^2}\right|}}{\sqrt{\left|x\frac{dx}{dt}\right|\cdot\left|\bar{x}\frac{d\bar{x}}{dt}\right|}}\cdot\frac{2\kappa^{\frac{7}{2}}}{\left|x\frac{dx}{dt}\right|\cdot\left|\bar{x}\frac{d\bar{x}}{dt}\right|}.$$

The coordinates of the Hermitian orthogonal point to $P(x)$ on the tangent at P to the given one dimensional chain are

$$(t) = \left(\frac{\kappa^2}{\sqrt{|xx'|\cdot|\bar{x}\bar{x}'|}}\left\{\frac{x_i}{\kappa^2}-\frac{x_i'}{(x'\bar{x})}\right\}\right).$$

Next will we find the point (z) which is the Hermitian orthogonal point to the point (x) and (t) in the osculating plane. Since

$$z_i = \lambda x_i + \mu x_r' + \nu x_i'',$$

$$(z\bar{x}) = \lambda(x\bar{x}) + \mu(x'\bar{x}) + \nu(x''\bar{x}) = 0 \ldots\ldots\ldots\ldots \text{(vii)},$$

and

$$(z\bar{t}) = \lambda(x\bar{t}) + \mu(x'\bar{t}) + \nu(x''\bar{t}) = 0 \ldots\ldots\ldots \text{(viii)},$$

i.e.

$$\lambda(xx) + \lambda(x'\bar{x}) + \nu(x''\bar{x}) - (\lambda(x\bar{x}') + \mu(x'\bar{x}') + \nu(x''\bar{x}'))\frac{\kappa^2}{(\bar{x}'x)} = 0,$$

we have

$$\lambda(x\bar{x}') + \mu(x'\bar{x}') + \nu(x''\bar{x}') = 0 \ldots\ldots\ldots\ldots\ldots \text{(ix)}.$$

From (vii), (ix), we have

$$\lambda : \mu : \nu$$

$$= |x'x''|\cdot|\bar{x}\bar{x}'| : |xx''|\cdot|\bar{x}\bar{x}'| : |xx'|\cdot|\bar{z}\bar{x}'|.$$

Therefore

$$z_i = \frac{|x'x''|\cdot|\bar{x}\bar{x}'|}{s}x_i + \frac{|xx''|\cdot|\bar{x}\bar{x}'|}{s}x_i' + \frac{|xx'|\cdot|\bar{x}\bar{x}'|}{s}x_i''.$$

We shall define the Hermitian torsion of a one-dimensional chain at a point on it by the ratio of the chain elements of the given chain and its osculating chain. By osculating chain, we mean the chain defined by

$$(y) \equiv \|x\,x'x''\|.$$

So

$$\frac{.\iota\sigma}{dt} = \frac{\sqrt{|ydy||\bar{y}d\bar{y}|}}{\sqrt{(y\bar{y})(y\bar{y})}} : \frac{\sqrt{|xdx|\cdot|\bar{x}d\bar{x}|}}{\sqrt{(x\bar{x})(x\bar{x})}}$$

$$
= \frac{\sqrt{\begin{vmatrix} |xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''| & |xx'x'''|\cdot|\bar{x}\bar{x}'\bar{x}''| \\ |xx'x'|\cdot|\bar{x}\bar{x}'\bar{x}'''| & |xx'x'''|\cdot|\bar{x}\bar{x}'\bar{x}'''| \end{vmatrix}}}{\sqrt{(|xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''|)^2}} : \frac{\sqrt{|xx'|\cdot|\bar{x}\bar{x}'|}}{\sqrt{(x\bar{x})^2}}
$$

$$
= \frac{\sqrt{\begin{vmatrix} |xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''| & |xx'x'''|\cdot|\bar{x}\bar{x}'\bar{x}''| \\ |xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}'''| & |xx'x'''|\cdot|\bar{x}\bar{x}'\bar{x}'''| \end{vmatrix}}}{\sqrt{(|xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''|)}} : \frac{\sqrt{|xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''|)}}{\sqrt{|xx'|\cdot|\bar{x}\bar{x}'|}} .
$$

$$
: \frac{\sqrt{|xx'|\cdot|\bar{x}\bar{x}'|}}{(x\bar{x})} .
$$

Now

$$
\begin{vmatrix} |xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''| & |xx'x'''|\cdot|\bar{x}\bar{x}'\bar{x}''| \\ |xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}'''| & |xx'x'''|\cdot|\bar{x}\bar{x}'\bar{x}'''| \end{vmatrix}
$$

$$
\begin{aligned}
= & \left(|xx'|\cdot|\bar{x}\bar{x}'|\cdot(x''\bar{x}'') - |xx'|\cdot|\bar{x}\bar{x}'|'\cdot(x''\bar{x}') + xx'|\cdot|\bar{x}'\bar{x}''\cdot|(x''\bar{x}) \right) \\
& \times \left(|xx'|\cdot|\bar{x}\bar{x}'|\cdot(x'''\bar{x}''') - |xx'|\cdot|\bar{x}\bar{x}'''|\cdot(x'''\bar{x}') + |xx'|\cdot|\bar{x}'\bar{x}'''|\cdot(x'''\bar{x}) \right) \\
& - \left(|xx'|\cdot|\bar{x}\bar{x}'|(x''\bar{x}''') - |xx'|\cdot|\bar{x}\bar{x}'''|\cdot(x''\bar{x}) + |xx'|\cdot|\bar{x}'\bar{x}'''|\cdot(x''\bar{x}) \right) \\
& \times \left(|xx'|\cdot|\bar{x}\bar{x}'|(x'''\bar{x}'') - |xx'|\cdot|\bar{x}\bar{x}''|\cdot(x'''\bar{x}') + |xx'|\cdot|\bar{x}'\bar{x}''|\cdot(x''\bar{x}) \right)
\end{aligned}
$$

$$
\begin{aligned}
= & (|xx'|\cdot|\bar{x}\bar{x}'|)^2(|x''x'''|\cdot|\bar{x}''\bar{x}'''|) - (|xx'|\cdot|\bar{x}\bar{x}'|)(|xx'|\cdot|\bar{x}\ \bar{x}'''|)(|x''x'''|\cdot|\bar{x}'\ \bar{x}''|) \\
& - (|xx'|\cdot|\bar{x}\bar{x}'|)(|xx'|\cdot|\bar{x}\ \bar{x}''|)(|x''x'''|\cdot|\bar{x}'\ \bar{x}'''|) \\
& - (|xx'|\cdot|\bar{x}\bar{x}'|)(|xx'|\cdot|\bar{x}'\ \bar{x}'''|)(|x''x'''|\cdot|\bar{x}'\ \bar{x}''|) \\
& - (|xx'|\cdot|\bar{x}\bar{x}'|)(|xx'|\cdot|\bar{x}'\ \bar{x}'''|)(|x\ x'''|\cdot|\bar{x}''\bar{x}'''|) \\
& - (|xx'|\cdot|\bar{x}\bar{x}'|)(|xx'|\cdot|\bar{x}''\bar{x}'''|)(|x''x'''|\cdot|\bar{x}\ \bar{x}'|)
\end{aligned}
$$

$$
= (|xx'|\cdot|\bar{x}\bar{x}'|)\cdot \begin{vmatrix} (x\ \bar{x}) & (x\ \bar{x}') & (x\ \bar{x}'') & (x\ \bar{x}''') \\ (x'\ \bar{x}) & (x'\ \bar{x}') & (x'\ \bar{x}'') & (x'\ \bar{x}''') \\ (x''\ \bar{x}) & (x''\ \bar{x}') & (x''\ \bar{x}'') & (x''\ \bar{x}''') \\ (x'''\bar{x}) & (x'''\bar{x}') & (x'''\bar{x}''') & (x'''\bar{x}''') \end{vmatrix}
$$

$$
= (|xx'|\cdot|\bar{x}\bar{x}'|)(|xx'x''x'''|\cdot|\bar{x}\bar{x}'\bar{x}''\bar{x}'''|).
$$

Therefore

$$
\frac{d\sigma}{ds} = (x\bar{x}) \frac{\sqrt{|xx'x''x'''|\cdot|\bar{x}\bar{x}'\bar{x}''\bar{x}'''|}}{\sqrt{(|xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''|)^2}}
$$

$$
= \sqrt{\frac{|xx'x''x'''|\cdot|\bar{x}\bar{x}'\bar{x}''\bar{x}'''|}{|xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''|}} : \sqrt{\frac{|xx'x''|\cdot|\bar{x}\bar{x}'\bar{x}''|}{|xx'|\cdot|\bar{x}\bar{x}'|}}
$$

$$
: \sqrt{\frac{|xx'|\cdot|\bar{x}\bar{x}'|}{(x\bar{x})(x\bar{x})}} .
$$

§ 5. r-Dimensional chain.

Let us consider ∞^r manifoldness of complex points defined by the equations of the form

$$x_i = x_i(u_1, u_2 \dots u_r)(i = 1, 2, 3 \dots n),$$

where u_i are all real parameters. We shall call such an assemblage of complex points an r-dimensional chain.

One dimensional chain-element at any point P in the given r-dimensional chain will be

$$ds = \varkappa \sqrt{\lim_{P' \to P} \text{hsin}^2\left(\frac{PP'}{\varkappa}\right)},$$

where P' is a point in the r-dimensional chain which approaches without limit to P. The expression for $\overline{ds}|^2$ in terms of u_i can easily be found. Since

$$| x dx | \cdot | \bar{x} d\bar{x} | = \sum_{ij=1}^{r} E_{ij}\, du_i\, du_j,$$

where

$$E_{ij} \equiv \left| x \frac{\partial x}{\partial u_i} \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial u_j} \right|,$$

so we see that

$$\varkappa^2 \overline{ds}|^2 = \sum_{i=j}^{r} E_{ij}\, du_i, du_j.$$

Further, we add that there are the relations:

$$E_{ij} = \bar{E}_{ji}.$$

Now form a determinant with $E_{ij}\,(i, j = 1, 2, \dots r)$ i.e.

$$\triangle \equiv | E_{ij} |,$$

then we have the relation

$$\left| E_{ij} \right| = \left| x \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \dots \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial u_1} \frac{\partial \bar{x}}{\partial u_2} \dots \right|.$$

The totality of the lines joining the point (x) with the next consecutive points $(x + dx)$ forms a plane whose equation is

$$\left| \xi\, x \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} \dots \right| = 0,$$

where (ξ) is the current coordinate of the point of the plane. The absolute Hermitian pole of the plane has the coordinates given by the ratio of the determinants of the matrix

$$\left| \bar{x} \frac{\partial \bar{x}}{\partial u_1} \frac{\partial \bar{x}}{\partial u} \cdots \right|.$$

Hence we have the relation

$$(x\bar{y}) = 0,$$

where (y) denotes the coordinates of the absolute Hermitian pole of the plane. But (y) depends upon the real parameters $u_1, u_1, \dots u_r$, as we can consider the locus of (y) when (x) moves in the given chain. Therefore if we denote the element of a one-dimensional chain in the chain (y) passing through the point (y) by ds', then we have

$$x^2 \overline{ds'}|^2 = \sum_{ij=1}^{r} E'_i \, du_i \, du ,$$

where

$$E_{ij}' \equiv \left| y \frac{\partial y}{\partial u_i} \right| \cdot \left| \bar{y} \frac{\partial \bar{y}}{\partial u_j} \right|$$

$$= \left| x \frac{\partial x}{\partial u_1} \cdots \frac{\partial^2 x}{\partial u_i \partial u_j} \cdots \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial u_1} \cdots \frac{\partial^2 \bar{x}}{\partial u_i \partial u_j} \cdots \right|.$$

Here we see that

$$\triangle' \equiv \left| E_{ij}' \right| = \left| y \frac{\partial y}{\partial u_1} \cdots \right| \cdot \left| \bar{y} \frac{\partial \bar{y}}{\partial u_1} \cdots \right|.$$

Then the Hermitian commoment of the two tri-points $(x), (y), (x+dx)$ and $(x), (y) (x+\delta x)$ is equal to

$$\sqrt{\frac{\{| xydx |\cdot| \bar{x}\bar{y}\delta\bar{x} |\}\{| \bar{x}\bar{y}d\bar{x} |\cdot| xy\delta x |\}}{\{| xydx |\cdot| \bar{x}\bar{y}d\bar{x} |\}\{| xy\delta x |\cdot| \bar{x}\bar{y}\delta\bar{x} |\}}}.$$

But, since $(y\bar{y}) = x^2$, we have

$$| xydx |\cdot| \bar{x}\bar{y}\delta\bar{x} |$$

$$= x^2 \, | xdx |\cdot| \bar{x}\delta\bar{x} |.$$

$$= \left(\sum_{i,j=1}^{r} \left| x \frac{\partial x}{\partial u_i} \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial u_i} \right| \cdot du_i \, . \, \delta u \right)$$

$$= x^2 \left\{ \sum_{i,j=1}^{r} E_i \, du_i \, \delta u_j \right\},$$

And

$$|\bar{x}\bar{y}d\bar{x}|\cdot|xy\delta x| = x^2\left\{\sum_{i,j=1}^{r} E_{i,j}\, du_i \,.\, \delta u_j\right\}.$$

And

$$|xydx|\cdot|\bar{x}\bar{y}\delta\bar{x}| = x^2 \overline{ds}|^2.$$

$$|xydx|\cdot|\bar{x}\bar{y}\delta\bar{x}| = x^2 \overline{\delta s}|^2.$$

Thus the Hermitian commoment is equal to

$$\frac{\sqrt{\left(\sum_{i\,j=1}^{r} E_{ij}\, du_i\, \delta u_j\right)\left(\sum_{i\,j=1}^{r} E_{ji}\, da_i\, \delta u_j\right)}}{ds\,\delta s\,.}.$$

This is equal to zero when and only when

$$\sum_{i,j=1}^{r} E_{ij}\, du_i \delta u_j = 0.$$

Therefore the two multi-points are orthogonal to each other in our sense when and only when

$$\sum_{i,j=1}^{r} E_{ij}\, du_i \,.\, \delta u_i = 0.$$

In three-dimensional space, this condition takes the form

$$E\, du\delta u + F\, du\delta v + \bar{F}\,\delta u dv + G\, dv\delta v = 0.$$

Therefore the condition of Hermitian orthogonality of parameter one-dimensional chain is

$$F = 0.$$

Now consider three points $P(x), P'\left(x+\frac{\partial x}{\partial u}du\right)$, $P''\left(x+\frac{\partial x}{\partial v}dv\right)$, then the sine-amplitude in the Hermitian space formed by the three points will be

$$\text{h}\sin(PP'P'') = \sqrt{\frac{\left|x\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right|\cdot\left|\bar{x}\frac{\partial \bar{x}}{\partial u}\frac{\partial \bar{x}}{\partial v}\right|}{(x\bar{x})^3}}\, dudv$$

$$= \frac{\sqrt{EG-F\bar{F}}}{(x)^3}\, dudv.$$

We shall define the area of the two-dimensional chain by the integral

$$\frac{x^3}{\underline{|2}}\int \text{h}\sin(PP'P'').$$

Thus the Hermitian area is equal to

$$\frac{1}{2}\int\sqrt{EG-F\bar{F}}\,dudv.$$

Generally, the Hermitian volume of a polytope in an n-dimensional space shall be defined by the expression

$$\frac{x^r}{\underline{|r-1}}\ \mathrm{h}\sin(P'P''\ldots P^r).$$

The tangent plane at the point $(x+dx)$ to the chain has the equation:

$$\left| X\, x+dx\, \frac{\partial}{\partial u_1}(x+dx)\, \frac{\partial}{\partial u_2}(x+dx)\, \ldots\, \frac{\partial}{\partial u_r}(x+dx)\right| = 0.$$

$$= \left| X\, x+\sum\frac{\partial x}{\partial u_i}du_i\, \frac{\partial x}{\partial u_1}+\sum\frac{\partial^2 x}{\partial u_1\partial u_i}du_i \ldots\right|$$

$$= \left| X\, x\, \frac{\partial x}{\partial u_1}\ldots\frac{\partial x}{\partial u_r}\right| + \left\{\left| X\, x\, \frac{\partial^2 x}{\partial u_1^2}\frac{\partial x}{\partial u_2}\ldots\right| + \left| X\, x\, \frac{\partial x}{\partial u_1}\frac{\partial^2 x}{\partial n_1\partial u_2}\ldots\right| + \right\} du_1$$

$$+\ldots = 0.$$

The intersection of the plane with the tangent plane at the point (x) may be given by the equations

$$\left| X x \frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}\ldots\right| = 0,$$

$$\sum\left\{\left| X x \frac{\partial^2 x}{\partial u_1^2}\frac{\partial x}{\partial u_2}\ldots\right| + \left| X x \frac{\partial x}{\partial u_1}\frac{\partial^2 x}{\partial u_1\partial u_2}\ldots\right| + \ldots\right\} du_1 = 0.$$

The condition for which the point $(x+\delta x)$ should be included in the above multi-points may be given by the equation:

$$\sum\left\{\left| \delta x\, x \frac{\partial^2 x}{\partial u_1^2}\frac{\partial x}{\partial u_2}\ldots\right| + \left| \delta x\, x \frac{\partial x}{\partial u_1}\frac{\partial x_2}{\partial u_1\partial u_2}+\ldots\right| + \ldots\right\} du_1 = 0.$$

Or

$$\sum D_{ij}\, du_i\, \delta u_j = 0,$$

where

$$D_{ij} = \left(\frac{\partial x}{\partial u_i}\frac{\partial \bar{y}}{\partial u_j}\right) = \frac{\left| x \frac{\partial x}{\partial u_i}\ldots\frac{\partial^2 x}{\partial u_i\partial u_j}\ldots\right|}{\sqrt{|E_{ij}|}}.$$

We shall say that the directions $(du, du\ldots)$ and $(\delta u_1 \delta u_2 \ldots)$ are conjugate if the condition

$$\sum D_{ij}\, du_i\, \delta u = 0$$

will be satisfied.

The conditions for which the parameter chains defined by (u_j= constant $j \neq i$), $i = 1, 2 \ldots r$ are Hermitian conjugates are

$$-(d\bar{y}\, dx) = \sum D_{ij} \cdot du_i\, du_j.$$

From this equality, we can determine the sign of D_{ij}.

Now consider a three-dimensional Hermitian space and we will discuss the chain of curvature in it. The necessary and sufficient condition for which four points $(x), (y), (x+dy), (y+dy)$ should coplanar is

$$| y x\, dx\, dy | = 0,$$

i.e.

$$\left| \bar{x} \frac{\partial \bar{x}}{\partial u} \frac{\partial \bar{x}}{\partial v} \right| \cdot \left| x\, dx\, dy \right| = 0.$$

Or

$$\begin{vmatrix} (x\bar{x}) & (\bar{x}\, dx) & (\bar{x} dy) \\ \left(x \dfrac{\partial \bar{x}}{\partial u}\right) & \left(\dfrac{\partial \bar{x}}{\partial u}\, dx\right) & \left(\dfrac{\partial \bar{x}}{\partial u}\, dy\right) \\ \left(x \dfrac{\partial \bar{x}}{\partial v}\right) & \left(\dfrac{\partial \bar{x}}{\partial v}\, dx\right) & \left(\dfrac{\partial \bar{x}}{\partial v}\, dy\right) \end{vmatrix} = 0.$$

Or

$$\begin{vmatrix} (x\bar{x}) & \left(\bar{x} \dfrac{\partial x}{\partial u}\right) du + \left(\bar{x} \dfrac{\partial x}{\partial v}\right) du & \left(\bar{x} \dfrac{\partial y}{\partial u}\right) du + \left(\bar{x} \dfrac{\partial y}{\partial v}\right) dv \\ \left(x \dfrac{\partial \bar{x}}{\partial u}\right) & \left(\dfrac{\partial \bar{x}}{\partial u} \dfrac{\partial x}{\partial u}\right) du + \left(\dfrac{\partial \bar{x}}{\partial u} \dfrac{\partial x}{\partial v}\right) dv & \left(\dfrac{\partial \bar{x}}{\partial u} \dfrac{\partial y}{\partial u}\right) du + \left(\dfrac{\partial \bar{x}}{\partial u} \dfrac{\partial y}{\partial v}\right) dv \\ \left(x \dfrac{\partial \bar{x}}{\partial v}\right) & \left(\dfrac{\partial \bar{x}}{\partial v} \dfrac{\partial x}{\partial u}\right) du + \left(\dfrac{\partial \bar{x}}{\partial v} \dfrac{\partial x}{\partial v}\right) dv & \left(\dfrac{\partial \bar{x}}{\partial v} \dfrac{\partial y}{\partial u}\right) du + \left(\dfrac{\partial \bar{x}}{\partial v} \dfrac{\partial y}{\partial v}\right) dv \end{vmatrix} = 0.$$

For the sake of brevity, we put

$$D = \left(\frac{\partial \bar{x}}{\partial u} \frac{\partial y}{\partial u}\right), \quad E = \left(\frac{\partial \bar{x}}{\partial u} \frac{\partial x}{\partial u}\right),$$

$$D' = \left(\frac{\partial \bar{x}}{\partial v} \frac{\partial y}{\partial v}\right), \quad F = \left(\frac{\partial \bar{x}}{\partial u} \frac{\partial x}{\partial v}\right),$$

$$D'' = \left(\frac{\partial \bar{x}}{\partial v} \frac{\partial y}{\partial v}\right), \quad G = \left(\frac{\partial \bar{x}}{\partial v} \frac{\partial x}{\partial v}\right),$$

then the above equation may be written as follows:

$$\begin{vmatrix} E\, du + F\, dv & D\, du + D' dv \\ \bar{F}\, cu + G\, du & \bar{D}' du + D'' dv \end{vmatrix} = 0.$$

The equation must be satisfied for every value of the ratio $du:dv$, so we have

$$\frac{E}{D}=\frac{\bar{F}}{\bar{D}'},\ \frac{F}{D'}=\frac{G}{D''}.$$

Therefore this is the condition in order that two consecutive normals to the chain should intersect. The totality of such points shall be called chain of curvature.

Let D be the distance from the point (x) to the intersection point of the normal chain with the adjacent chain. The coordinates of the point shall be represented by the coordinates (z). Then we see

$$z_i = x_i \cos D' - y_i \sin D'$$

$$\frac{dz_i}{ds}=\frac{dx_i}{ds}\cos D'-\frac{dy_i}{ds}\sin D'$$
$$-[x_i \sin D' + y_i \cos D']\frac{dD'}{ds}.$$

But, by hypothesis $\left(\frac{dz}{ds}\right)$ is linearly dependent on (x) and (y), so we have

$$dx_i \cos D' + dy_i \sin D' = \lambda x_i + \mu y_i.$$

Further,

$$\begin{aligned}(xd\bar{x})+(\bar{x}dx) &= (yd\bar{y})+(\bar{y}dy)\\ &= (x\bar{y}) = (\bar{x}\bar{y})\\ &= (\bar{x}dy) = (xd\bar{y})\\ &= (yd\bar{x}) = (\bar{y}dx)\\ &= 0.\end{aligned}$$

Therefore it must be

$$\lambda = \mu = 0.$$

Hence

$$dx_i = dy_i \tan D'.$$

or

$$\sum_j \frac{\partial x_i}{\partial u_j}du_i = \sum_j \tan D' \frac{\partial y_i}{\partial u_j}du_j.$$

In particular, let us take as parameter-chain the chain of curvature, then we have

$$(dxd\bar{y}) = \sum \frac{E_i}{\tan D_i'}\overline{du}|^2.$$

$$(dyd\bar{y}) = \sum \frac{E_i}{\tan^2 D_i'}\overline{du}|^2.$$

In general

$$\sum_i E_i\, du = -\tan D' \sum D_{ij}\, du_i.$$
$$(j = 1, 2 \ldots).$$

Eliminate the ratio $du_1 : du_2 : \ldots$ from these equations, then we have

$$\begin{vmatrix} E_{11}+\tan D'D_{11} & E_{12}+\tan D'.D_{12} \ldots \\ E_{21}+\tan D'D_{21} & E_{22}+\tan D'.D_{22} \ldots \\ \ldots & \end{vmatrix} = 0 \ldots\ldots\ldots (\mathrm{x}).$$

We shall define the expression

$$\sqrt{\frac{1}{\tan D_1' \overline{\tan D_1'}, \tan D_2' \overline{\tan D_2'}, \ldots \tan D_r' \overline{\tan D_r'}}}^{\,1}$$

total Hermitian curvature of this space at the point (x) in analogy of non-Euclidean geometry. From the equation (x), the Hermitian curvature may be expressed in the form

$$\frac{\sqrt{|D_{ij}| \cdot |\bar{D}_{ij}|}}{|E_{ij}|}.$$

Let us consider a manifoldness M defined by a pair of two orthogonal points (x) and (y) in our sense. Then a congruence of M may be represented by the parametric equations of (x) and (y), i.e.

$$x_i = x_i(uv),\ y_i = y_i(uv),$$

with the conditions

$$(x\bar{x}) = (y\bar{y}) = \varkappa^2,$$
$$(x\bar{y}) = (y\bar{x}) = 0,$$

where u, v are all real parameters.

We will use notations analogous to Kummer's notations[2]. Now consider the product of two matrices $|xdx|$ and $|\bar{y}d\bar{x}|$, and we will use the notations

$$|ydx| \cdot |\bar{y}d\bar{x}|$$
$$= E\,du|^2 + (F + F)\,dudv + G\,dv|^2,$$

where

$$E = \left|y\frac{\partial x}{\partial u}\right| \cdot \left|\bar{y}\frac{\partial \bar{x}}{\partial u}\right| = \varkappa^2\left(\frac{\partial x}{\partial u}\frac{\partial \bar{x}}{\partial u}\right) - \left(\frac{\partial x}{\partial u}\bar{y}\right)\left(\frac{\partial \bar{x}}{\partial u}y\right),$$

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1 See Killing. Nicht-euclidichen Raumformen 1885. p. 210.

2 See Kummer. Allgemeine Theorie der geradlinigen Strahlensysteme (Crelle's Journal Bd. 57). Also see Fibbi's paper. Annali della R. scuola Normale superiore di Pisa, 1891.

$$F = \left| y \frac{\partial x}{\partial u} \right| \cdot \left| y \frac{\partial \bar{x}}{\partial v} \right| = x^2 \left(\frac{\partial x}{\partial u} \frac{\partial \bar{x}}{\partial v} \right) - \left(\frac{\partial x}{\partial u} \bar{y} \right) \left(\frac{\partial \bar{x}}{\partial v} y \right),$$

$$G = \left| y \frac{\partial x}{\partial v} \right| \cdot \left| \bar{y} \frac{\partial \bar{x}}{\partial v} \right| = x^2 \left(\frac{\partial x}{\partial v} \frac{\partial \bar{x}}{\partial v} \right) - \left(\frac{\partial x}{\partial v} \bar{y} \right) \left(\frac{\partial \bar{x}}{\partial v} y \right).$$

Let us also consider another product of similar matrices $|xdy|$, $|\bar{x}d\bar{y}|$, then we have

$$|xdy| \cdot |\bar{x}d\bar{y}| = E'|\overline{du}|^2 + (F' + \bar{F}')\, dudv + G'|\overline{dv}|^2,$$

where

$$E' = \left| x \frac{\partial y}{\partial u} \right| \cdot \left| \bar{x} \frac{\partial \bar{y}}{\partial u} \right|,$$

$$F' = \left| x \frac{\partial y}{\partial v} \right| \cdot \left| \bar{x} \frac{\partial \bar{y}}{\partial v} \right|,$$

$$G' = \left| x \frac{\partial y}{\partial v} \right| \cdot \left| \bar{x} \frac{\partial \bar{y}}{\partial v} \right|.$$

At last, let us consider the sum of the products of two matrices

$$|ydx| \cdot |\bar{x}d\bar{x}|, \quad |ydy| \cdot |\bar{y}d\bar{x}|,$$

then we obtain

$$|ydy| \cdot |\bar{x}d\bar{x}| + |ydy| \cdot |\bar{y}d\bar{x}|$$
$$= e|\overline{du}|^2 + (f + f')\, dudv + g|\overline{dv}|^2,$$

where

$$e = \left| y \frac{\partial x}{\partial u} \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial u} \right| + \left| y \frac{\partial y}{\partial u} \right| \cdot \left| \bar{y} \frac{\partial \bar{x}}{\partial u} \right|,$$

$$f = \left| y \frac{\partial x}{\partial u} \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial v} \right| + \left| y \frac{\partial y}{\partial u} \right| \cdot \left| \bar{y} \frac{\partial \bar{x}}{\partial v} \right|,$$

$$f' = \left| y \frac{\partial x}{\partial v} \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial u} \right| + \left| y \frac{\partial y}{\partial v} \right| \cdot \left| \bar{y} \frac{\partial \bar{x}}{\partial u} \right|,$$

$$g = \left| y \frac{\partial x}{\partial v} \right| \cdot \left| \bar{x} \frac{\partial \bar{x}}{\partial v} \right| + \left| y \frac{\partial y}{\partial v} \right| \cdot \left| \bar{y} \frac{\partial \bar{x}}{\partial v} \right|.$$

Now we see the relation

$$\Delta \equiv \begin{vmatrix} E & F \\ \bar{F} & G \end{vmatrix} = x^2 \left| y \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} \right| \cdot \left| \bar{y} \frac{\partial \bar{x}}{\partial u} \frac{\partial \bar{x}}{\partial v} \right|,$$

$$\Delta' \equiv \begin{vmatrix} E' F' \\ \bar{F}' G' \end{vmatrix} = x^2 \left| x \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \right| \cdot \left| \bar{x} \frac{\partial \bar{y}}{\partial u} \frac{\partial \bar{y}}{\partial v} \right|.$$

Now assume $x^2 = +1$.

Consider the chain defined by the points (x), (y), then the chain is represented by

$$\lambda(x)+\mu(y),$$

where λ and μ are of course complex. The orthogonal point to the point $\lambda(x)+\mu(y)$ in the chain may be represented by the coordinates

$$\cos D'(x)+\sin D'(y),$$

where D' is the distance between the point (x) and $\lambda(x)+\mu(y)$ in the elliptic space. Let the intersection-point of the line $\lambda(x)+\mu(y)$ with the common perpendicular of the lines $\lambda(x+dx)+\mu(y+dy)$ and $\lambda(x)+\mu(y)$ be given by the coordinates

$$\cos D'(x)+\sin D'(y) \qquad \text{(xi)}$$

and the corresponding point of the line defined by the coordinates

$$(x+dx),\ (y+dy),$$

be

$$\lambda(x+dx)+\mu(y+dy) \qquad \text{(xii)},$$

then we see that the orthogonal points to the points (xi), (xii) in these two lines in our sense can be represented by the coordinates

$$-\overline{\sin D'}\,(\mathrm{u})+\overline{\cos D'}\,(y),$$
$$-\bar{\mu}(x+dx)+\bar{\lambda}(y+dy)$$

respectively.

By the property of common Hermitian orthogonality, we have the relations :

$$(\cos D'(x)+\sin D'(y),\ -\mu(\bar{x}+d\bar{x})+\lambda(\bar{y}+d\bar{y}))=0$$

and

$$(-\sin D'(\bar{x})+\cos D'(\bar{y}),\ \lambda(x+dx)+u(y+dy))=0.$$

Elliminate $\lambda:\mu$ from these equations and substitute in the result

$$(xd\bar{y})+(\bar{y}dx)=-(dx,d\bar{y}),$$
$$(\bar{x}dx)+(yd\bar{x})=-(d\bar{x},dy),$$
$$(xd\bar{x})+(\bar{x}dx)=-(dx,d\bar{x}),$$
$$(yd\bar{y})+(\bar{y}dy)=-(dy,d\bar{y}),$$

then we have

$$\sin^2 D'\{|ydx|\cdot|\bar{x}d\bar{x}|+|ydy|\cdot|\bar{y}dx|\}$$
$$-\cos^2 D'\{|\bar{y}d\bar{x}|\cdot|\mathrm{u}d\mathrm{u}|+|\bar{y}d\bar{y}|\cdot|yd\mathrm{u}|\}$$
$$+\sin D'\cos D'\{|ydx|\cdot|\bar{y}d\bar{x}|-|xdy|\cdot|\bar{x}d\bar{y}|\}=0.$$

Or, this may be written as follows :

$$\cos^2 D'\{e\,\overline{du}|^2+(f+f')\,dudv+g\overline{dv}|^2\}$$
$$-\sin^2 D'\,\{\bar{e}\,\overline{du}|^2+(\bar{f}+\bar{f}')\,dudv+\bar{g}\,\overline{dv}|^2\}$$
$$+\cos D'\sin D'\{(E-E')\,\overline{du}|^2+[F+\bar{F}-(F'+\bar{F}')]\,dudv$$
$$+(F-F')\,\overline{dv}|^2\}=0.$$

Therefore

$$(E'F'F'G')-(EF\bar{F}G)$$
$$=\frac{-\sqrt{\{(E'E'E'G')-(EF\bar{F}G)\}^2-4(eff'g).(\bar{e}\bar{f}\bar{f}'\bar{g})}}{(eff'g)}$$
$$=\frac{M}{(eff'g)},$$

where

$$E\overline{du}|^2+(F+\bar{F})\,dudv+G\,\overline{dv}|^2=(E,F,F,G),$$
$$e\,\overline{du}|^2+(f+f')\,dudv+g\,\overline{dv}|^2=(ef,f',g).$$

Thus

$$\sqrt{\frac{\cos\partial\,.\,\overline{\cos\partial}}{\cos\left(\frac{\Pi}{2}-\partial\right)\,\overline{\cos}\left(\frac{\Pi}{2}-\partial\right)}}=\frac{M}{\sqrt{(eff'g)(\bar{e}\bar{f}\bar{f}'\bar{g})}}.$$

Therefore

$$\frac{\mathrm{hcos}\,\partial}{\mathrm{hcos}\,\partial'}=\frac{M}{\sqrt{(eff'g)(\bar{e}\bar{f}\bar{f}'\bar{g})}},$$

where ∂ and ∂' mean the Hermitian distance between the points (x) and (y) from the point $\cos D'(x)+\sin D'(y)$.