On the Solutions of Partial Differential Equations of the First Order at the Singular Points. II.

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INTRODUCTION.

The homogeneous partial differential equation of the first order

$$Xf \equiv \xi_1(x)\frac{\partial f}{\partial x_1} + \xi_2(x)\frac{\partial f}{\partial x_2} + \ldots + \xi_n(x)\frac{\partial f}{\partial x_n} = o,$$

where $\xi_1(x)$, $\xi_2(x)$,...,$\xi_n(x)$ being holomorphic about the point $(o, o,\ldots o)$ and

$$\xi_i(x) = \lambda_i x_i + \ldots, \qquad i = 1,\ 2,\ldots n,$$

the dotted part being terms of higher degrees, has $n-1$ algebroidal solutions in the vicibity of the point $(o, o,\ldots o)$, provided λ_1, $\lambda_2,\ldots\lambda_n$ satisfy the following conditions:

1.° the relations

$$\lambda_1 p_1 + \lambda_2 p_2 + \ldots + \lambda_i p_i + \ldots + \lambda_n p_n = \lambda_i, \qquad i = 1,\ 2,\ldots n$$

are not satisfied by any positive integral values of p_1, $p_2,\ldots p_n$, provided $p_1 + p_2 + \ldots + p_n \geqq 2$;

2°. if we denote λ_1, $\lambda_2,\ldots\lambda_n$, by the points on a plane, then we can trace a convex polygon in which these n points lie but which does not contain the origin; or we may say that this is a straight line through the origin, on one side of which all the n points lie.

In the previous memoir[1] these are called *Poincaré's conditions*;

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1 T. Matsumoto, Memoir of the College of Science, Kyoto Imp. Univ., Vol. II, No. 5 (1917), On the solutions.... .

and the writer discussed the case where the coefficients $\xi_1(x), \xi_2(x), \ldots \xi_n(x)$ commence by general linear homogeneous functions of $x_1, x_2, \ldots x_n$; and the case of a complete system. In both cases Poincaré's conditions had a determinative power for the existence of solutions, i.e., for the existence of holomorphic solutions of the equations: $Xf = \lambda_i f$, $i = 1, 2, \ldots n$. In the first part of the present memoir, the author wishes to make, about this equation, some comments with respect to Poincaré's conditions and in the next part some considerations of the equation $Xf = \lambda_i f$ are extended over the partial homogeneous differential equations of a higher order in which we shall obtain a glance at the relations existing between our problem and the ordinary differential equations of Fuchs' class.

I.

1. When the system of numbers $\lambda_1, \lambda_2, \ldots \lambda_n$ satisfy the second part of Poincaré's conditions but not the first part i.e., if for positive integral values of $p_1, p_2, \ldots p_n$ a relation e.g., for $i = n$:

$$p_1\lambda_1 + p_2\lambda_2 \ldots\ldots + p_n\lambda_n = \lambda_n, \tag{1}$$

exist, we shall prove that *the total number of such relations is finite*. For when such relations as (1) would exist for infinitely many times, then the total sum of integers: $p_1 + p_2 + \ldots + p_n$ must increase indefinitely. But since $\lambda_1, \lambda_2, \ldots \lambda_n$ satisfy the second part of Poincaré's conditions, as usual the centre of masses

$$\frac{p_1\lambda_1 + p_2\lambda_2 + \ldots + p_n\lambda_n}{p_1 + p_2 + \ldots + p_n} \tag{2}$$

cannot be the origin, i.e., its absolute value must always be greater than a positive number ε. But when $p_1 + p_2 + \ldots + p_n$ become sufficiently great, the ratio

$$\frac{p_1\lambda_1 + p_2\lambda_2 + \ldots + p_n\lambda_n - \lambda_n}{p_1 + p_2 + \ldots + p_n - 1}$$

approaches (2). Therefore from certain value of $p_1 + p_2 + \ldots + p_n$, (1) can not be satisfied.

2. To test whether such a relation as (1) exist between λ's, let g be a straight line on one side of which the n points $\lambda_1, {}_2, \ldots \lambda_n$ lie.

Multiply these numbers by $e^{\alpha i}$, where α is a certain real quantity. Then all the real parts of $\lambda_1 e^{\alpha i}$, $\lambda_2 e^{\alpha i}, \ldots \lambda_n e^{\alpha i}$ may be made positive; for the multiplication by such a number is nothing but a rotation of amount α, of the straight line g about the origin, and hence g may be made to coincide with the y-axis. Therefore assume the transformation has already been done, then the real parts of $\lambda_1, \lambda_2, \ldots \lambda_n$ are all positive, Moreover suppose that the real parts of

$$\lambda_1, \lambda_2, \ldots \lambda_n \tag{3}$$

are of an increasing order. Then if a relation hold:

$$p_1\lambda_1 + p_2\lambda_2 + \ldots p_i\lambda_i + \ldots + p_n\lambda_n = \lambda_i,$$

then $p_i, \ldots p_n$ must be zero, or else after our arrangements, the real part of the sum of the terms of the left-hand side is greater than that of λ_i. This is absurd. Therefore it is only possible that

$$p_1\lambda_1 + p_2\lambda_2 + \ldots + p_{i-1}\lambda_{i-1} = \lambda_i. \tag{4}$$

Therefore we may determine all possible relations such as (4).

3. Given the equation

$$Xf \equiv (\lambda_1 x_1 + \ldots)\frac{\partial f}{\partial x_1} + (\lambda_2 x_2 + \ldots)\frac{\partial f}{\partial x_2} + \ldots + (\lambda_n x_n + \ldots)\frac{\partial f}{\partial x_n} = \lambda_j f, \tag{5}$$

where the dotted parts stand for terms of higher degrees, suppose, $\lambda_1, \lambda_2, \ldots \lambda_n$ being in the previous arrangement (3), that

$$p_1\lambda_1 + \ldots + p_i\lambda_i = \lambda_j, \qquad i<j, \tag{6}$$

where $p_1, \ldots p_i$ are positive integers whose total sum is not less than 2. We make a further assumption that in (6) none of $\lambda_1, \ldots \lambda_i$ can linearly be expressed by the other, i.e., they fulfil the first part of Poincaré's conditions. This assumption is possible, for if λ_i be not so, then we have only to eliminate it from (6). But as we have proved, since the total number of such a relation as (6) is finite, the case may occur where $p_1 + \ldots + p_i$ is maximum. Under these conditions the equation (5) has a holomorphic solution commencing with the term $x_1^{p_1} \ldots x_i^{p_i}$:

$$f_j = x_1^{p_1} \ldots x_i^{p_i} + \ldots, \tag{7}$$

By the method of calculations of limits, often used in the previous

Memoir, the *unique existence* of the solution f_j can easily be proved. It is our purpose to show that this solution is not helpful for solving $Xf=0$.

Since $\lambda_1,\dots\lambda_i$ satisfy both parts of Poincaré's conditions, the equations

$$Xf=\lambda_h f, \qquad h=1,\dots i, \tag{8}$$

have holomorphic solutions of the form

$$f_h=x_h+\dots, \qquad h=1,\dots i,$$

the dotted parts being terms of higher degrees. Now consider the product

$$F\equiv f_1^{p_1}\dots f_i^{p_i}=x_1^{p_1}\dots x_i^{p_i}+\dots .$$

Then clearly by (6),

$$XF=(p_1\lambda_1+\dots+p_i\lambda_i)F=\lambda_j F.$$

On the other hand since the solution f_j is unique, it must hold that

$$f_j\equiv F=f_1^{p_i}\dots f_i^{p_i} . \qquad Q.E.D.$$

II.

1. Now consider the equation

$$(ax^2+\dots)\frac{\partial^2 f}{\partial x^2}+(2bxy+\dots)\frac{\partial^2 f}{\partial x\partial y}+(cy^2+\dots)\frac{\partial^2 f}{\partial y^2}$$
$$+(hx+\dots)\frac{\partial f}{\partial x}+(ky+\dots)\frac{\partial f}{\partial y}+(\lambda+\dots)f=o, \tag{1}$$

where a, b, c, h, k, λ are constants not zero and the dotted parts stand for terms of higher degrees written before. All the functions being holomorphic about (o, o), the origin (o, o) is a singular points of this equation.

At the origin, we must have

$$\lambda(f)_0=o, \qquad \text{hence } (f)_0=o.$$

Differentiating (1) by x resp. y and putting $x=y=o$, we have

$$(h+\lambda)\left(\frac{\partial f}{\partial x}\right)_0=o, \qquad (k+\lambda)\left(\frac{\partial f}{\partial y}\right)_0=o.$$

If neither $h+\lambda$, nor $k+\lambda$ be zero, we must take

$$\left(\frac{\partial f}{\partial x}\right)_0 = \left(\frac{\partial f}{\partial y}\right)_0 = o.$$

Differentiating with respect to x twice, x and y, finally y twice, we have, for (o, o).

$$(2a+2h+\lambda)\left(\frac{\partial^2 f}{\partial x^2}\right)_0 = o,$$

$$(2b+h+k+\lambda)\left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 = o,$$

$$(2c+2k+\lambda)\left(\frac{\partial^2 f}{\partial y^2}\right)_0 = o,$$

Therefore suppose e.g., λ satisfy the equation

$$2a+2h+\lambda=o, \tag{2}$$

then the solution f of the equation (1) commences by x^2.

2. Now differentiate the equation (1), p-times with respect to x and q-times with respect to y, then for $x=y=o$, the term with differential quotient of highest order is found to be

$$[a\binom{p}{2}2+2bpq+c\binom{q}{2}2+hp+kq+\lambda]\left(\frac{\partial^{p+q} f}{\partial x^p \partial y^q}\right)_o$$
$$=[ap^2+2bpq+cq^2+(h-a)p+(k-b)q+\lambda]\left(\frac{\partial^{p+q} f}{\partial x^p \partial y^q}\right)_o,$$

the other being of lower order. If the coefficient

$$ap^2+2bpq+cq^2+(h-a)p+(k-b)q+\lambda \tag{3}$$

does not vanish for any positive integral values of p, q such as

$$p+q \geqq 3, \tag{4}$$

then we may calculate all the values of differential quotients at $x=y=o$, step by step.

3. Now put

$$f \equiv Ax^2+v, \tag{5}$$

A being a constant, the equation (1) may be written as follows:

$$ax^2\frac{\partial^2 v}{\partial x^2}+2bxy\frac{\partial^2 v}{\partial x \partial y}+cy^2\frac{\partial^2 v}{\partial y^2}+hx\frac{\partial v}{\partial x}+ky\frac{\partial v}{\partial y}+\lambda v$$
$$=\varphi_{20}\frac{\partial^2 v}{\partial x^2}+\varphi_{11}\frac{\partial^2 v}{\partial x \partial y}+\varphi_{02}\frac{\partial^2 v}{\partial y^2}+\psi_1\frac{\partial v}{\partial x}+\psi_2\frac{\partial v}{\partial y}+\theta v+\chi, \tag{6}$$

where φ_{20}, φ_{11}, φ_{02}, χ commence at least by terms of the third degree, ψ_1, ψ_2 by the second degree, θ by the first degree. Let M be the maximum modulus of these functions, r the smallest radius of convergent circles about (o, o), then we have, as fonctions majorantes,

$$\begin{matrix} |\varphi_{20}|, & |\varphi_{11}| \\ |\varphi_{02}|, & |\chi| \end{matrix} \leqq \frac{M\left(\frac{x+y}{r}\right)^3}{1-\frac{x+y}{r}}$$

$$|\psi_1|, \quad |\psi_2| \leqq \frac{M\left(\frac{x+y}{r}\right)^2}{1-\frac{x+y}{r}},$$

$$|\theta| \leqq \frac{M\frac{x+y}{r}}{1-\frac{x+y}{r}},$$

Hence consider the equation

$$\varepsilon\left(x^2\frac{\partial^2 V}{\partial x^2}+2xy\frac{\partial^2 V}{\partial x\partial y}+y^2\frac{\partial^2 V}{\partial y^2}+x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}-V\right)$$

$$=\frac{M\left(\frac{x+y}{r}\right)^3}{1-\frac{x+y}{r}}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial x\partial y}+\frac{\partial^2 V}{\partial y^2}+1\right)$$

$$+\frac{M\left(\frac{x+y}{r}\right)^2}{1-\frac{x+y}{r}}\left(\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}\right)+\frac{M\frac{x+y}{r}}{1-\frac{x+y}{r}}V, \tag{7}$$

where ε is to be determined. Every term of each coefficient of the right-hand side of (6) is, in absolute value, less than the corresponding term of (7). Now differentiating the equation (7) p-times with respect to x and q-times with respect to y and put $x=y=o$, we get, from the left-hand side of (7), the differential quotient of highest order,

$$\varepsilon(p^2+2pq+q^2-1)\left(\frac{\partial^{p+q}V}{\partial x^p\partial y^q}\right)_o.$$

Therefore if for $p+q \geqq 3$,

$$\frac{|ap^2+2bpq+cq^2+(h-a)p+(k-b)q+\lambda|}{(p+q)^2-1} \geqq \varepsilon > 0, \tag{8}$$

the series calculated from (7) serves for *fonction majorante* of that of (6).

4. The fraction (8), for sufficiently great $p+q$, is nearly equal to

$$\frac{|ap^2+2bpq+cq|}{(p+q)^2}\ , \tag{9}$$

There if 1^0, the sum (3), *i.e.*, numerator of (8) do not vanish for $p+q \geqq 3$, and 2^0, the points a, b, c marked on Gauss' plane lie all on one side of a straight line through the origin, the inequality (8) may be fulfilled. These are nothing but Poincaré's conditions.

Under these conditions we shall prove the existence of a power-series commencing with terms of the third degree in x and y which satisfy the equation (7). Put

$$x+y \equiv u,$$

then we see, in the equation (7), that V is a function of u and it may be written as follows:

$$\varepsilon\left(u^2\frac{d^2V}{du^2}+u\frac{dV}{du}-V\right)$$

$$=\frac{M}{r^2}\frac{u^3}{r-u}\left(3\frac{d^2V}{du^2}+1\right)+\frac{M}{r}\frac{u^2}{r-u}\left(2\frac{dV}{du}\right)+M\frac{u}{r-u}V. \tag{10}$$

This equation, after dividing by ε, may be written in the following form:

$$u^2(1+\ldots)\frac{d^2V}{du^2}+u(1+\ldots)\frac{dV}{du}-(1+\ldots)V=mu^4+\ldots, \tag{11}$$

the dotted parts stand for the terms of higher degrees written before and m is a constant.

To solve this equation (11), consider the equation without the second member:

$$u^2(1+\ldots)\frac{d^2V}{du^2}+u(1+\ldots)\frac{dV}{du}-(1+\ldots)V=0. \tag{12}$$

This equation belongs to the equations of the second order of Fuchs' class. Its characteristic equation is

$$\varphi(r) \equiv r(r-1)+r-1=0.$$

The roots of this equation at $u=o$ are

$$r_1=1, \qquad r_2=-1,$$

where $r_2-r_1=-2$ which is not a positive integer. Therefore the equation (12) has an integral

$$V_1=u\,U(u), \tag{13}$$

where $U(u)$ is holomorphic and $U(o)\neq o$. To find the integral of (11), put

$$V=C.V_1=CuU(u),$$

where the function C is to be determined.

Putting

$$\frac{dC}{du}\equiv z,$$

z is found to satisfy the equation

$$u(U(o)+\ldots)\frac{dz}{du}+(3U(o)+\ldots)z=mu+\ldots\ldots .$$

Therefore z is of the form

$$z=\frac{m}{4U(o)}u(1+\ldots).$$

Hence

$$C=\int zdu=\frac{m}{8\,U(o)}u^2(1+\ldots),$$

$$V=\text{power-series commencing by the term } u^3.$$

Thus our equation (7) has holomorphic solution commencing by $(x+y)^3$, which is the required *fonction majorante* of the solution $v\ (x, y)$ of the equation (6). Hence, *the partial differential equation* (1) :

$$(ax^2+\ldots)\frac{\partial^2 f}{\partial x^2}+(2bxy+\ldots)\frac{\partial^2 f}{\partial x\partial y}+(cy^2+\ldots)\frac{\partial^2 f}{\partial y^2}$$
$$+(hx+\ldots)\frac{\partial f}{\partial x}+(ky+\ldots)\frac{\partial f}{\partial y}+(\lambda+\ldots)f=o,$$

where the dotted parts mean terms of higher degrees than the foregoing terms, under Poincaré's conditions has a holomorphic solution of the form

$$f=Ax^2+\ldots,$$

provided the equation (2) *be fulfilled:*

$$2a+2h+\lambda=o.$$

5. The result can easily be extended to the case where the first part of Poincaré's conditions is not satisfied, namely when

$$ap^2+2bpq+cq^2+(h-a)p+(k-b)q+\lambda=o. \tag{3'}$$

Suppose this relation is the last one where $p+q$ is a maximum integer. Then

$$\left(\frac{\partial^{p+q}f}{\partial x^p \partial y^q}\right)_o$$

is arbitrary. Hence put, instead of (5),

$$f\equiv Ax^p y^q+v,$$

then in (6) the function χ will commence with terms of $(p+q+1)$ *th* degree, the other remaining the same. Consequently the second member of (11) will be of the form

$$mu^{p+q+1}+\ldots.$$

Moreover C will commence with u^{p+q}. Therefore the lowest power in V is $p+q+1$, hence v will commence by the power $p+q+1$. Thus, *when a, b, c lie on one side of a straight line through the origin, our eqnation* (1) *has still a holomorphic solution in the vicinity of* (o, o).

6. Our considerations may be extended easily for cases of equations of higher order or for cases of equations depending upon variables more than two, For the equation of higher order a little modification is necessary. Let us consider the equation of the *nth* order with two variables:

$$\begin{aligned}
&(a_{n,o}x^n+\ldots)\frac{\partial^n f}{\partial x^n}+\left(\tbinom{n}{1}a_{n-1,1}x^{n-1}y+\ldots\right)\frac{\partial^n f}{\partial x^{n-1}\partial y}+\ldots+(a_{o,n}y^n+\ldots)\frac{\partial^n f}{\partial y^n}\\
&+(b_{n-1,0}x^{n-1}+\ldots)\frac{\partial^{n-1}f}{\partial x^{n-1}}+\left(\tbinom{n-1}{1}b_{n-2,1}x^{n-2}y+\ldots\right)\frac{\partial^{n-1}f}{\partial x^{n-2}\partial y}\ldots\\
&+(b_{o,n-1}y^{n-1}+\ldots)\frac{\partial^{n-1}f}{\partial y^{n-1}}\\
&+\ldots\ldots\ldots\\
&+(h_{1,o}x+\ldots)\frac{\partial f}{\partial y}+(h_{o,1}y+\ldots)\frac{\partial f}{\partial y}+\lambda f=o
\end{aligned} \tag{14}$$

where in general the term of lowest degree in the coefficient of $\dfrac{\partial^m f}{\partial x^{m-i}y^i}$ is $\dbinom{m}{i}x^{m-}\,y^i$ multiplied by a constant.

For this epuation, the equation corresponding to (11) is of the form:

$$u^n(1+\dots)\frac{d^nV}{du^n}+u^{n-1}(1+\dots)\frac{d^{n-1}V}{du^{n-1}}+\dots+u(1+\dots)\frac{dV}{du}$$
$$-(1+\dots)V=mu^h, \tag{15}$$

the dotted part being as usual terms of higher degrees. The characteristic equation of (15) without the second member is

$$\varphi(r)\equiv r(r-1)(r-2)\dots(r-n+1)+\dots\quad +r(r-1)+r-1=0. \tag{16}$$

This equation has a root $r_1=1$, and the other roots cannot be a positive integer greater than unity. Therefore when r_i runs the series of the other roots, r_i-r_1 cannot be a positive integer. Hence our Fucks' equation, the equation (15) without the second member, has a holomorphic integral commencing with the term u, while the other integral cannot be holomorphic at $u=0$.

7. To prove the existence of a holomorphic solution of the equation (15), consider in general the equation

$$F(V)\equiv p_0\frac{d^nV}{du^n}+p_1\frac{d^{n-1}V}{du^{n-1}}+\dots+p_i\frac{d^{n-i}V}{du^{n-i}}+\dots+p_{n-1}\frac{dV}{du}+p_nV=p.$$

Then

$$p\frac{dF}{du}-F\frac{dp}{du}=pp_0\frac{d^{n+1}V}{du^{n+1}}+(pp_0'+pp_1-p'p_0)\frac{d^nV}{du^n}+\dots\dots$$
$$+(pp_i'+pp_{i+1}-p'p_i)\frac{d^{n-i}V}{du^{n-i}}+\dots+(pp'_{n-1}+pp'_n-p'p_{n-1})\frac{dV}{du}$$
$$+(pp_n'-p'p_n)V=0.$$

In our case

$$p_i=u^{n-i}+\dots,\ i=0,\ 1,\dots\ n-1,$$
$$p_n=-1+\dots,$$
$$p=mu^h+\dots,$$
$$p'=mhu^{h-1}+\dots,$$
$$pp_i'+pp_{i+1}-p'p_i=(mu^h+\dots)\,((n-i)u^{n-i-1}+\dots)+(mu^h+\dots)\,(u^{n-i-1}$$
$$+\dots)-(mhu^{h-1}+\dots)\,(u^{n-i}+\dots)$$
$$=mu^{h-1}(\,(n+1-i-h)u^{n-i}+\dots).$$
$$pp_0=mu^{h-1}(u^{n+1}+\dots),$$
$$pp'_{n-1}+pp_n-p'p_{n-1}=mu^{h-1}(-hu+\dots),$$
$$pp_n'-p'p_n=mu^{h-1}(h+\dots).$$

Hence the equation

$$p\frac{dF}{du}-F\frac{dp}{du}=o, \tag{17}$$

will be, after dividing by mu^{h-1},

$$\begin{aligned}&(u^{n+1}+\ldots)\frac{d^{n+1}V}{du^{n+1}}+((n+1-h)u^n+\ldots)\frac{d^nV}{du^n}+\ldots\\&+((n+1-i-h)u^{n-i}+\ldots)\frac{d^{n-i}V}{du^{n-i}}+\ldots\\&+(-hu+\ldots)\frac{dV}{du}+(h+\ldots)V=0.\end{aligned} \tag{18}$$

The characteristic equation is, therefore,

$$\begin{aligned}\varphi(r)\equiv r(r-1)(r-2)\ldots(r-n)+(n+1-h)r(r-1)(r-2)\ldots(r-n+1)+\ldots\\+(n+1-i-h)r(r-1)(r-2)\ldots(r-n+i+1)+\ldots+(-h)r+h=0.\end{aligned}$$

But since

$$\begin{aligned}&h(h-1)\ldots(h-n+i)+(n+1-i-h)h(h-1)\ldots(h-n+i+1)\\&=h(h-1)\ldots(h-n+i+1),\end{aligned}$$

we have by successive calculation

$$\varphi(h)=h(h-1)-h^2+h=0.$$

Thus h is a root of the characteristic equation $\varphi(r)=0$. Moreover if this equation has an integral root, then it must be a divisor of h; therefore when r_i runs the series of the other root, r_i-h cannot be a positive integer. Therefore the equation (18) has a holomorphic integral such as

$$V=Ku^h+\ldots,$$

K being a constant. As we have said in the preceding paragraph, this cannot be an integral of the equation (15) without the second member. Hence from (17),

$$F(V)=cp,$$

c being a constant. Therefore $\frac{V}{c}$ is an integral of the equation (15). Thus our equation (15) has a holomorphic solution commencing with u^h which serves as the *fonction majorante* of the equation (14). Thus we may conclude as follows: *The equation* (14), *has always a holomorphic solution in the vicinity of* (o, o) *provided* $a_{n,0}\ldots a_{0,n}$ *marked as points of a Gauss' plane, lie on one side of a straight line through the origin.* It is of-course necessary that λ satisfy such an equation as (3').