On the Solutions of Partial Differential Equations of the First Order at the Singular Points. III.

By

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INTRODUCTION.

The problem of the partial differential equations is a never-ending one among mathematicians from ages ago. After the researches of Cauchy[1], Jacobi and Riemann[2]; during 1870, much remarkable progress was made: Mayer[3] found the method of the solution of Jacobi's system in 1872. The demonstration of Kowalewski[4] appeared in 1875. The first fundamental theorem of Lie's group was written in 1876. Among the many works of Darboux[5], the remarkable memoir on the singular solutions of partial differential equations of the first order was presented to the Academy of Paris in 1877. Poincaré, as we have said in the preceding volume of these memoirs[6], presented his thesis concerning the solutions of partial differential equations of the first order at the singular points in 1879. Cauchy established the existence theorem by the method of the calculations of limits. But in

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1 Existence theorem; C. R. (1842–43).

2 Lectures at Göttingen; (1860–62).

3 Math. Ann. 5.

4 Crelle J., 80.

5 C. R. 84.

6 Mem. Coll. Sci., Kyoto, 2, 255 (1917).

the remarkable memoir of Picard[1] in 1890, the method of successive approximations was applied. This method, as well as that of Cauchy, is used for the linear partial differential equations in the field of the real variables. The problem has been developed to the researches of Fredholm. On the other hand, the general problems of the partial differential equations were almost completed by Goursat in his books; especially the theory of characteristics of Cauchy. But the analytical problems of the solutions of the partial differential equations have been developed but slowly. We cite the rescarches of Le Roux in 1892[2] and 1898[3] and of Delassus[4] in 1895 about the linear homogeneous partial differential equations of the higher orders by aid of Picard's method. In these memoirs the singular points (lines) of the equations with two independent variables are determined and some singular solutions of Euler's equation are determined. In 1916 the thesis of Stoïlow was published which treats of the solutions of the same equations, being given the initial functions with the singular points on characteristics. In this research, Goursat's[5] note in 1906 is applied. But analytical researches into the solutions at the fixed singularities have not appeared, with the one exception of Poincaré's paper. While after Marotte[6], Darboux demonstrated that the equation

$$(x+\cdots)\frac{\partial f}{\partial x}+(\lambda_1 x_1+\cdots)\frac{\partial f}{\partial x_1}+(\lambda_2 x_2+\cdots)\frac{\partial f}{\partial x_2}=0,$$

has solutions of the form

$$u_1=\frac{(A_1 x_1+\cdots)^{\frac{1}{\lambda_1}}}{Ax+\cdots}, \qquad u_2=\frac{(A_2 x_2+\cdots)^{\frac{1}{\lambda_2}}}{Ax+\cdots},$$

(under certain conditions of λ_1, λ_2). On the other hand, Darboux read Poincaré's thesis[7], but he says nothing about this fact and I am not certain who was the first discoverer.

The present paper contains discussions of more general equations than those in the former two[8]; namely the equation

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1 Jordan J.; J. l'ecole poly.

2, 4 Ann. l'ecole nor. sup.

3 Jordan J.

5 Bull. Soc. Math. Fr.

6 C. R. (1896).

7 Oeuvres de Poincaré, II, xxi.

8 Mem. Coll. Sci., Kyoto., 2, (1917); 4, (1919).

$$\xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\cdots+\xi_n(x)\frac{\partial f}{\partial x_n}=0,$$

where $\xi_1(x)$, $\xi_2(x)$,...$\xi_n(x)$ are holomorphic about and vanish at $x_1=x_2=\cdots=x_n=0$. For this purpose I use a simple birational transformation plane to plane and the solutions are of a particular type of Laurent's expansions in a certain domain. After this, the method is extended to the discussions of the complete system.

1. Let us consider the transformation:

$$\left.\begin{aligned} x_1 &= y_1 y_n^m, \\ &\cdots\cdots\cdots\cdots \\ x_{n-1} &= y_{n-1} y_n^m, \\ x_n &= y_n, \end{aligned}\right\} \tag{1}$$

where m is a positive integer. By this transformation, for a system of values $(y_1,\ldots, y_{n-1}, y_n)$ there corresponds one and only one system of values $(x_1, \ldots x_{n-1}, x_n)$ and for $y_1=\cdots=y_{n-1}=y_n=0$, $x_1=\cdots=x_{n-1}=x_n=0$. A holomorphic function $f(x)$ of the variables $x_1,\ldots x_{n-1}, x_n$ becomes also a holomorphic function of $y_1,\ldots y_{n-1}, y_n$. Next, by $P_x(R_1; \ldots; R_n)$ we mean a domain consisting of circular domains with radii R_i, and centres at $x_i=0$, in the plane of x_i $(i=1, 2, \ldots n)$. Specially when $R_1=\cdots=R_n=R$, we express it by $P_x(R)$. If $f(x)$ be holomorphic in a domain containing $P_x(R)$, then we have an expansion in $P_x(R)$,

$$f(x)=\sum a_{p_1\cdots p_n} x_1^{p_1}\cdots x_n^{p_n},$$

then we have

$$\left| a_{p_1\cdots p_n} \right| < \frac{M}{R^{p_1+\cdots+p_n}}, \tag{2}$$

where M is the maximum modulus of the function $f(x)$ on the circles of $P_x(R)$. By the transformation (1), we have

$$f(x)=\sum a_{p_1\cdots p_{n-1}p_n} y_1^{p_1}\cdots y_{n-1}^{p_{n-1}} y_n^{m(p_1+\cdots+p_{n-1})+p_n},$$

and suppose this expansion has a common factor y_n^m, then writing

$$f(x)=y_n^m\,\varphi(y),$$

we have

$$\varphi(y)=\sum a_{p_1\cdots p_{n-1}p_n} y_1^{p_1}\cdots y_{n-1}^{p_{n-1}} y_n^{m(p_1+\cdots+p_{n-1})+p_n-m},$$

where all the powers of y_n are positive integers or zero. Since the series $f(x)$ is convergent in the domain $P_x(R)$, if $R>1$, it is clear that $y_n^m\varphi(y)$ is also convergent in the domain $P_y(R^{\frac{1}{m+1}})$, but it is not *a priori* clear that $\varphi(y)$ is convergent in the domain $P_y(R^{\frac{1}{m+1}})$, though it is clear for the case of one variable. By the inequality (2), we have

$$|\varphi(y)|<\sum\frac{M}{R^{p_1+\cdots+p_{n-1}+p_n}}\left|y_1^{p_1}\cdots y_{n-1}^{p_{n-1}} y_n^{m(p_1+\cdots+p_{n-1})+p_n-m}\right|$$

$$=\frac{M}{\rho^m}\sum\frac{\rho^{(m+1)(p_1+\cdots+p_{n-1})+p_n}}{R^{p_1+\cdots+p_{n-1}+p_n}}\left|\frac{y_1}{\rho}\right|^{p_1}\cdots\cdots\left|\frac{y_{n-1}}{\rho}\right|^{p_{n-1}}\left|\frac{y_n}{\rho}\right|^{m(p_1+\cdots+p_{n-1})+p_n-m},$$

where ρ is positive yet undetermined. If we give to ρ such *a* value that the first factors in Σ be less than unity, then we have

$$|\varphi(y)|<\frac{M}{\rho^m}\sum\left|\frac{y_1}{\rho}\right|^{p_1}\cdots\cdots\left|\frac{y_{n-1}}{\rho}\right|^{p_{n-1}}\left|\frac{y_n}{\rho}\right|^{m(p_1+\cdots+p_{n-1})+p_n-m}$$

$$<\frac{M}{\rho^m}\frac{1}{\left(1-\left|\frac{y_1}{\rho}\right|\right)\cdots\left(1-\left|\frac{y_{n-1}}{\rho}\right|\right)\left(1-\left|\frac{y_n}{\rho}\right|\right)}\,.$$

Hence $\varphi(y)$ is convergent in the domain $P_y(\rho)$. To determine ρ, if $R\leq 1$, we have only to take $\rho=R$. But when $R>1$, we must determine ρ such that

$$\rho^{(m+1)(p_1+\cdots+p_{n-1})+p_n}\leq R^{p_1+\cdots+p_{n-1}+p_n},\tag{3}$$

for any $p_1+\cdots+p_{n-1}+p_n$. But since

$$\frac{(m+1)(p_1+\cdots+p_{n-1})+p_n}{p_1+\cdots+p_{n-1}+p_n}=m\frac{p_1+\cdots+p_{n-1}}{p_1+\cdots+p_{n-1}+p_n}+1\leq m+1,$$

if we take ρ such that

$$\rho=R^{\frac{1}{m+1}},$$

the above inequality (3) is satisfied Hence we have the lemma:

Between two power-series $f(x)$ and $\varphi(y)$, by our transformation (1) *let such a relation as*

$$f(x)=y^m\varphi(y)$$

be established. If $f(x)$ be convergent in $P_x(R)$, then $\varphi(y)$ is convergent in $P_y(\rho)$, where

$$\rho=R, \text{ or } \rho=R^{\frac{1}{m+1}}$$

according as $R\leqq 1$ or $R>1$.

2. Let us consider the inverse transformation of (1), namely

$$\left.\begin{aligned} y_1&=x_1x_n^{-m}, \\ &\cdots\cdots\cdots\cdots \\ y_{n-1}&=x_{n-1}x_n^{-m}, \\ y_n&=x_n. \end{aligned}\right\} \tag{4}$$

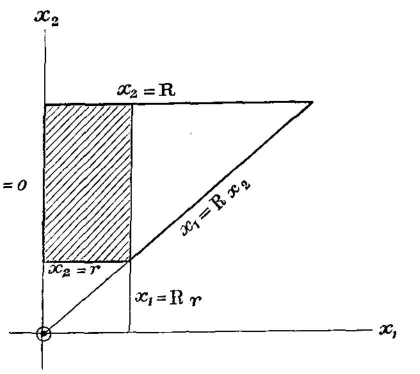
Since

$$\frac{\partial(x_1,\dots x_n)}{\partial(y_1,\dots y_n)}=y_n^{m(n-1)},$$

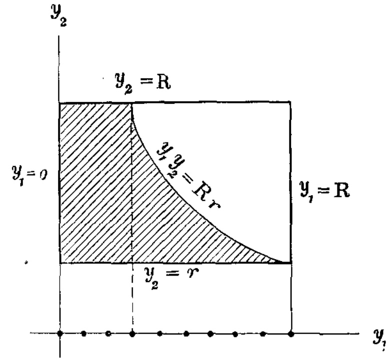
between $(x_1,\dots x_{n-1}, x_n)$ and $(y_1\dots y_{n-1}, y_n)$, there is one-one correspondence provided $y_n\neq o$. Now by $P_x(R_1;\dots;R_{n-1};R_n,r)$, we mean a domain consisting of $n-1$ circles with radii $R_1,\dots R_{n-1}$, with centres at the origins in the planes of $x_1, \dots x_{n-1}$ respectively, and of two concentric circles with radii R_n and r $(R>r)$ and centres at the origin of the plane of x_n. When $R_1=\dots=R_{n-1}=R_n=R$, we write it $P_x(R;R,r)$. Now by the transformation (4), any holomorphic function $\varphi(y)$ in the domain $P_y(R_1;\dots;R_{n-1};R_n,r)$ will be transformed into a holomorphic function $f(x)$ in the domain $P_x(R_1r^m;\dots;R_{n-1}r^m;R_n,r)$. When $R_1=\dots=R_{n-1}=R_n=R$, the domain will become $P_y(R;R,r)$ and $P_x(Rr^m;R,r)$. But we do not say that these domains have one-one correspondence by our transformation. Truly the domain $P_x(Rr^m;R,r)$ is contained in the domain transformed from $P_y(R;R,r)$. An illustration with two real variables is given in the following figures.

Representation of the transformation for $m=1$:

$$\left.\begin{aligned} x_1&=y_1y_2 \\ x_2&=y_2 \end{aligned}\right\} \text{ or } \left.\begin{aligned} y_1&=x_1x_2^{-1} \\ y_2&=x_2 \end{aligned}\right\}$$



Domain corresponding to P_y.



$P_y(R; R, r)$

The shaded part of the left figure is $P_x(Rr; R, r)$ which corresponds to the shaded part of the domain $P_y(R; R, r)$ in the right figure.

The transformation (1) forms a group of one parameter m. Its infinitesimal transformation is given by

$$log\, x_n \cdot \left(x_1 \frac{\partial f}{\partial x_1} + \cdots + x_{n-1} \frac{\partial f}{\partial x_{n-1}}\right);$$

hence the invariants of the group are given by $\dfrac{x_i}{x_j}$=const. and x_n= const., where $i \neq j \neq n$. While the transformation (9) below forms a group of n parameters $m, l_1 \ldots l_{n-1}$ and hence is a transitive group. The infinitesimal transformations are

$$log\, x_n \cdot \left(x_1 \frac{\partial f}{\partial x_1} + \cdots + x_{n-1} \frac{\partial f}{\partial x_{n-1}}\right),\ x_n \frac{\partial f}{\partial x_1},\ \ldots\ldots,\ x_n \frac{\partial f}{\partial x_{n-1}}.$$

We remember that in each case the point $x_n = o$ belongs to the singular points of the transformations; but excepting the point, the transformations are birational and of plane to plane.

Next we shall consider the relations between the partial derivatives with respect to the transformation (1). Let f be a function of the variables $x_1 \ldots x_{n-1}, x_n$ and hence of the variables $y_1, \ldots y_{n-1}, y_n$, then we have

$$\left.\begin{aligned}
&\frac{\partial f}{\partial y_1}=y_n^m\,\frac{\partial f}{\partial x_1},\\
&\cdots\cdots\cdots\ \cdots\cdots\cdots\\
&\frac{\partial f}{\partial y_{n-1}}=y_n^m\,\frac{\partial f}{\partial x_{m-1}},\\
&\frac{\partial f}{\partial y_n}=m\left(y_1 y_n^{m-1}\frac{\partial f}{\partial x_1}+\cdots+y_{n-1}y_n^{m-1}\frac{\partial f}{\partial x_{n-1}}\right)+\frac{\partial f}{\partial x_n}.
\end{aligned}\right\}\tag{5}$$

Multiplying the first $n-1$ equations by $y_1, \dots y_{n-1}$ respectively, and the last by y_n, we have by aid of the transformation (1),

$$\left.\begin{aligned}
&y_1\frac{\partial f}{\partial y_1}=x_1\frac{\partial f}{\partial x_1},\\
&\cdots\cdots\cdots\cdots\cdots,\\
&y_{n-1}\frac{\partial f}{\partial y_{n-1}}=x_{n-1}\frac{\partial f}{\partial x_{n-1}},\\
&y_n\frac{\partial f}{\partial y_n}=m\left(y_1\frac{\partial f}{\partial y_1}+\cdots+y_{n-1}\frac{\partial f}{\partial y_{n-1}}\right)+x_n\frac{\partial f}{\partial x_n}.
\end{aligned}\right\}\tag{5'}$$

3. Now returning to our problem, consider as usual the partial differential equation

$$Xf\equiv\xi_1(x)\frac{\partial f}{\partial x_1}+\cdots+\xi_{n-1}(x)\frac{\partial f}{\partial x_{n-1}}+\xi_n(x)\frac{\partial f}{\partial x_n}=o, \tag{6}$$

where $\xi_1(x), \dots, \xi_{n-1}(x), \xi_n(x)$ are regular with respect to $x_1, \dots x_{n-1}, x_n$, in the domain $P_x(R)$ and vanish at $x_1=\cdots=x_{n-1}=x_n=o$. But *we do not expect that they necessarily begin with terms of the first orders of* $x_1, \dots x_{n-1}, x_n$.

It may occur that among the power-series $\xi_1(x), \dots, \xi_{n-1}(x)$, there is at least one which contain such term or terms as ax^m, a being a constant. Let m be the *smallest index*, not zero, among them and be contained in $\xi_1(x)$. Applying the transformation (1), we have

$$\xi_1(x)=y_n^m P_1(y),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\xi_{n-1}(x)=y_n^m P_{n-1}(y),$$

$$\xi_n(x)=y_n^{m'}P_n(y),$$

where $P_1(y), \ldots, P_{n-1}(y), P_n(y)$ are power-series and m' a positive integer and

$$P_1(o) = a \neq o.$$

By the lemma proved in § 1, these power-series are all convergent in the domain $P_y(\rho)$. Now by the relations (5) and (5'), we have

$$\begin{aligned} Xf = & P_1(y)\frac{\partial f}{\partial y_1} + \cdots + P_{n-1}(y)\frac{\partial f}{\partial y_{n-1}} \\ & + y_n^{m'} P_n(y)\frac{\partial f}{\partial y_n} - m y_n^{m'-1} P_n(y)\left(y_1 \frac{\partial f}{\partial y_1} + \cdots + y_{n-1}\frac{\partial f}{\partial y_{n-1}}\right) \\ = & \left\{P_1(y) - m y_1 y_n^{m'-1} P_n(y)\right\}\frac{\partial f}{\partial y_1} + \cdots + \left\{P_{n-1}(y) - m y_{n-1} y_n^{m'-1} P_n(y)\right\}\frac{\partial f}{\partial y_{n-1}} \\ & + y_n^{m'} P_n(y)\frac{\partial f}{\partial y_n}, \end{aligned}$$

where, of course, at the point $y_1 = \cdots = y_{n-1} = y_n = o$,

$$P_1(y) - m y_1 y_n^{m'-1} P_n(y) = a \neq o.$$

Hence our equation (6), divided by the first coefficient, will be transformed into the following

$$Yf \equiv \frac{\partial f}{\partial y_1} + \eta_2(y)\frac{\partial f}{\partial y_2} + \cdots + \eta_n(y)\frac{\partial f}{\partial y_n} = o, \tag{7}$$

where $\eta_2(y), \ldots, \eta_n(y)$ are holomorphic in a certain domain contained in $P_y(\rho)$.

4. When no one of the series $\xi_1(x), \ldots, \xi_{n-1}(x)$ contains such a term as ax_n^m, we take another variable *e.g.* x_i instead of x_n and the same process may be followed. But if all these cases fail, we consider the transformation:

$$\left.\begin{aligned} x_1 &= y_1 + l_1 y_n, \\ &\cdots\cdots\cdots\cdots\cdots, \\ x_{n-1} &= y_{n-1} + l_{n-1} y_n, \\ x_n &= y_n, \end{aligned}\right\} \tag{8}$$

where $l_1, \ldots l_{n-1}$ are constants, at least one of which is not zero. The inverse transformation is clearly

$$\left.\begin{array}{l} y_1 = x_1 - l_1 x_n, \\ \dots\dots\dots\dots\dots, \\ y_{n-1} = x_{n-1} - l_{n-1} x_n, \\ y_n = x_n, \end{array}\right\}$$

and hence between $(x_1, \dots x_{n-1}, x_n)$ and $(y_1, \dots y_{n-1}, y_n)$ there is one-one correspondence in all extents of the variables. By this transformation the equation (6) will become

$$Xf = \left\{\xi_1(x) - l_1 \xi_n(x)\right\}\frac{\partial f}{\partial y_1} + \dots + \left\{\xi_{n-1}(x) - l_{n-1}\xi_n(x)\right\}\frac{\partial f}{\partial y_{n-1}} + \xi_n(x)\frac{\partial f}{\partial y_n} = o.$$

In the first $n-1$ coefficients of this equation expressed in terms of $y_1, \dots y_{n-1}, y_n$, there must occur at least such a term as ay_n^m. Consequently we may proceed with the new equation just as before. In any case, by the composition of the transformations (1) and (8), namely, *by the transformation*

$$\left.\begin{array}{l} x_1 = y_1 y_n^m + l_1 y_n, \\ \dots\dots\dots\dots\dots\dots\dots, \\ x_{n-1} = y_{n-1} y_n^m + l_{n-1} y_n, \\ x_n = y_n, \end{array}\right\} \tag{9}$$

the equation (6) *will be transformed into the equation* (7). Therefore we assume, without loosing the generality, $l_1 = \dots = l_{n-1} = o$.

5. The equation (7) may be solved by the general method at once. Let $\varphi_1(y), \dots, \varphi_{n-1}(y)$ be the $n-1$ first integrals, then any arbitrary function

$$F(\varphi_1(y), \dots, \varphi_{n-1}(y))$$

is the general integral of the equation (7), the functions $\varphi_1(y), \dots \varphi_{n-1}(y)$ being regular about the point $y_1 = \dots = y_{n-1} = y_n = o$. Let $P_y(R; R, r)$ be the corresponding domain. By the transformation (4) the equation (7) is transformed into (6), while the first integrals $\varphi_1(y), \dots, \varphi_{n-1}(y)$ will become $f_1(x), \dots, f_{n-1}(x)$ which are the $n-1$ first integrals of the equation (6), and $F(f_1(x), \dots, f_{n-1}(x))$ is the required general integral of the equation (6). By aid of § 2 the integrals $f_1(x), \dots, f_{n-1}(x)$ are holomorphic in the domain $P_x(Rr^m; R, r)$ and truly they are expanded into the Laurent-series. Thus we have the theorem:

The partial differential equation

$$Xf=\xi_1(x)\frac{\partial f}{\partial x_1}+\cdots+\xi_{n-1}(x)\frac{\partial f}{\partial x_{n-1}}+\xi_n(x)\frac{\partial f}{\partial x_n}=o,$$

where $\xi_1(x), \ldots, \xi_{n-1}(x), \xi_n(x)$ *are regular analytic about the point* $x_1=\ldots=x_{n-1}=x_n=o$, *at which they vanish simultaneously, has the general integral*

$$F(f_1(x), \ldots, f_{n-1}(x))\,;$$

$f_1(x), \ldots, f_{n-1}(x)$ *are* $n-1$ *first integrals and have the form of the* Laurent-series *in such a domain as* $P_x(Rr^m;\, R, r)$.

6. It is not necessary to take the convergent circles of $\eta_2(y),\ldots \eta_n(y)$ to have the same radii R, and consequently the wider domain $P_x(R_1r^m;\, \ldots;\, R_{n-1}r^m;\, R_n, r)$ may be taken for the domain of the integrals $f_1(x), \ldots, f_{n-1}(x)$. This domain has a particular property. Since

$$R_i r^m, \quad (i=1, \ldots, n-1)$$

vanish at $r=o$, though we make the variable x_n an infinitesimal of the first order, yet the integrals $f_1(x), \ldots, f_{n-1}(x)$ remain finite and determinate, provided the orders of the infinitesimals of the other variables $x_1, \ldots x_{n-1}$ be not lower than m. We may extend the domain further by Goursat's theorem[1]. The first integrals $\varphi_1(y), \ldots, \varphi_{n-1}(y)$ are power-series. But if we use his theorem, they are holomorphic in a somewhat wider domain. Consequently the transformed functions $f_1(x), \ldots, f_{n-1}(x)$ will become holomorphic in a somewhat wider domain than $P_x(R_1r^m;\, \ldots;\, R_{n-1}r^m;\, R, r)$.

7. We have assumed that in § 4, $l_1, \ldots l_{n-1}$ are all zero, since by the transformation (8) the differential expression Xf becomes

$$Xf=\left\{\xi_1(x)-l_1\xi_n(x)\right\}\frac{\partial f}{\partial y_1}+\cdots+\left\{\xi_{n-1}(x)-l_{n-1}\xi_n(x)\right\}\frac{\partial f}{\partial y_{n-1}}+\xi_n(x)\frac{\partial f}{\partial y_n},$$

on which in general we may apply the transformation (1). But for a special case, it may occur that the transformed expression remains unchanged. In this case our method cannot be applied. Now under the assumption, we have at first

$$\xi_n(y_1+l_1y_n, \ldots, y_{n-1}+l_{n-1}y_n, y_n)=\xi_n(y_1, \ldots, y_{n-1}, y_n). \tag{10}$$

[1] Bull. Soc. Math. Fr. (1906).

Next we have

$$\xi_i(y_1+l_1y_n, \ldots, y_{n-1}+l_{n-1}y_n, y_n)-l_i\xi_n(y_1+l_1y_n, \ldots, y_{n-1}+l_{n-1}y_n, y_n)$$
$$=\xi_i(y_1, \ldots, y_{n-1}, y_n).$$

Hence by aid of (10),

$$\begin{aligned}\xi_i(y_1+l_1y_n, \ldots, y_{n-1}+l_{n-1}y_n, y_n)-\xi_i(y_1, \ldots, y_{n-1}, y_n)\\ =l_i\xi_n(y_1, \ldots, y_{n-1}, y_n).\end{aligned} \tag{11}$$

Therefore we have

$$\xi_i(y_1+2l_1y_n, \ldots, y_{n-1}+2l_{n-1}y_n, y_n)-\xi_i(y_1+l_1y_n, \ldots, y_{n-1}+l_{n-1}y_n, y_n)$$
$$=l_i\xi_n(y_1, \cdots, y_{n-1}, y_n).$$

Hence we have by equating the left-hand sides

$$\xi_i(y_1+2l_1y_n, \ldots, y_{n-1}+2l_{n-1}y_n, y_n)+\xi_i(y_1, \ldots, y_{n-1}, y_n)$$
$$=2\xi_i(y_1+l_1y_n, \ldots, y_{n-1}+l_{n-1}y_n, y_n).$$

This equation being identical with respect to $l_1, \ldots l_{n-1}$ we must have, expanding both sides into Taylor's series, the relations

$$\frac{\partial^2\xi_i(y)}{\partial y_h \partial y_k}=o, \qquad {h \atop k}=1, \ldots, n-1.$$

So that we have

$$\xi_i(y)=A_{i1}(y_n)y_1+\ldots+A_{in-1}(y_n)y_{n-1}+A_i(y_n).$$

But since $\xi_i(y)$ vanishes for $y_1=\ldots=y_{n-1}=y_n=o$, we have rather

$$\xi_i(y)=A_{i1}(y_n)y_1+\ldots+A_{in-1}(y_n)y_{n-1}+A_{in}(y_n)y_n, \quad i=1, \ldots n-1,$$

where $A_{i1}(y_n), \ldots A_{in-1}(y_n), A_{in}(y_n)$ are regular with respect to y_n. Substituting this expression in (11),

$$\sum_{j=1}^{n-1} A_{ij}(y_n)l_jy_n=l_i\xi_n(y_1, \ldots, y_{n-1}, y_n).$$

Hence we have

$$A_{ii}(y_n)=\frac{\xi_n(y)}{y_n}\equiv M(y), \qquad i, j=1, 2, \ldots n-1,$$

$$A_{ij}(y_n)=o, \qquad i\neq j,$$

that is

$$\xi_i(y)=y_iM(y), \qquad i=1, 2, \ldots n.$$

Our differential expression is therefore

$$Xf=M(x)\left(x_1\frac{\partial f}{\partial x_1}+\cdots+x_{n-1}\frac{\partial f}{\partial x_{n-1}}+x_n\frac{\partial f}{\partial x_n}\right).$$

In this case the first integrals can at once be found or we may consider a little more complex transformation than (8).

We shall make a remark upon this equation. Suppose $M(x)=1$, then consider the equation

$$x_1\frac{\partial f}{\partial x_1}+\cdots+x_{n-1}\frac{\partial f}{\partial x_{n-1}}+x_n\frac{\partial f}{\partial x_n}=\lambda f.$$

This equation shows that the function f is homogeneous of the λth order with respect to the variables. Especially the equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=-2f$$

is satisfied by Weierstrass' $\wp$-function with two periods x and y. This function is uniform about the point $x=y=z=o$.

8. We proceed to the problem of a complete system of partial differential equations. This case may partly be solved by a method similar to that by which I succeeded before as shown in the first paper[1].

Let a complete system

$$\left.\begin{aligned}X_1f&\equiv\xi_{11}(x)\frac{\partial f}{\partial x_1}+\xi_{12}(x)\frac{\partial f}{\partial x_2}+\cdots+\xi_{1n}(x)\frac{\partial f}{\partial x_n}=o,\\X_2f&\equiv\xi_{21}(x)\frac{\partial f}{\partial x_1}+\xi_{22}(x)\frac{\partial f}{\partial x_2}+\cdots+\xi_{2n}(x)\frac{\partial f}{\partial x_n}=o,\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\X_rf&\equiv\xi_{r1}(x)\frac{\partial f}{\partial x_1}+\xi_{r2}(x)\frac{\partial f}{\partial x_2}+\cdots+\xi_{rn}(x)\frac{\partial f}{\partial x_n}=o,\end{aligned}\right\}\qquad(12)$$

be given, where $r<n$, and $\xi_{11}(x), \xi_{12}(x), \ldots, \xi_{rn}(x)$ are regular about and vanish at the point $x_1=x_2=\cdots=x_n=o$ and among $X_1f, X_2f, \ldots, X_rf$, we have the relations

[1] loc. cit. (1917).

$$(X_iX_j)f=\sum_{s=1}^{r} C_{ijs}X_sf, \qquad i,j=1,2,\ldots\ldots r.$$

We *assume* that all the functions C_{ijs} are holomorphic about the point $x_1=x_2=\ldots x_n=o$. Among $\xi_{i1}(x), \xi_{i2}(x), \ldots, \xi_{in-1}(x), (i=1,2,\ldots r)$, let $\xi_{11}(x)$ be one which contains such terms as ax_n^m where m is the smallest positive integer. [If this assumption be absurd, we have only to consider the transformation (8)]. Then applying the transformation (1), we have the following complete system

$$\left.\begin{array}{l} Y_1f\equiv\dfrac{\partial f}{\partial y_1}+\eta_{12}(y)\dfrac{\partial f}{\partial y_2}+\cdots+\eta_{1n}(y)\dfrac{\partial f}{\partial y_n}=o, \\ Y_2f\equiv\eta_{21}(y)\dfrac{\partial f}{\partial y_1}+\eta_{22}(\eta)\dfrac{\partial f}{\partial y_2}+\cdots+\eta_{2n}(y)\dfrac{\partial f}{\partial y_n}=o, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_rf\equiv\eta_{r1}(y)\dfrac{\partial f}{\partial y_1}+\eta_{r2}(y)\dfrac{\partial f}{\partial y_2}+\cdots+\eta_{rn}(y)\dfrac{\partial f}{\partial y_n}=o, \end{array}\right\} \qquad (13)$$

where all the coefficients $\eta_{12}(y), \ldots\eta_{1n}(y), \ldots\eta_{rn}(y)$ are holomorphic about the point $y_1=y_2=\cdots=y_n=o$, and such that

$$(Y_iY_j)f=\sum_{s=1}^{r} D_{ijs}Y_sf, \qquad i,j=1,2,\ldots\ldots r.$$

It can easily be shown that all the functions D_{ijs} are also holomorphic about the point $y_1=y_2=\cdots=y_n=o$.

Now put

$$Y_i'f\equiv Y_if+\sum_{j=1}^{r} R_{ij}(y)Y_jf,$$

where $R_{ij}(y)$, $(j=1,2,\ldots r)$ are yet unknown functions; then for the permutability of the infinitesimal transformations Y_1f and $Y_i'f$, we must have these functions such that

$$Y_1R_{is}+D_{1is}+\sum_{j=1}^{r} D_{1js}R_{ij}=o, \qquad s=1,2,\ldots\ldots r.^1$$

1 See the first Memoir (pp. 290—1).

But this is a system of r simultaneous partial differential equations of the first order with respect to the unknowns R_{ij}, $(j=1, 2,\dots r)$. All the coefficients are holomorphic about the point $x_1=x_2=\cdots=x_n=o$ and those of $\frac{\partial R_{is}}{\partial y_1}$, $(s=1, 2,\dots r)$ are unity. Hence by the general theorem, with respect to the ordinary points, there exists a system of holomorphic solutions $R_{is}(y)$, $(s=1, 2,\dots r)$ about the point $x_1=x_2=\cdots=x_n=o$, with the mere initial conditions that all these functions vanish at the point $y_1=y_2=\cdots=y_n=o$. Following this process we obtain a complete system

$$Y_1'f\equiv Y_1f,\ Y_2'f,\ \dots\dots,\ Y_r'f, \tag{14}$$

such that

$$\begin{aligned} &(Y_1'Y_i')f\equiv o, \qquad i,\ j=2,\ 3,\dots\dots r, \\ &(Y_i'Y_j')f\equiv\sum_{s=1}^{r} D'_{ijs}Y_s'f, \end{aligned} \tag{15}$$

where all the composition-functions such as D'_{ijs} are holomorphic about the point $y_1=y_2=\cdots=y_n=o$. Moreover we may conclude that

$$Y_1D'_{ijt}=o, \qquad i,j=2,\dots\dots r,\ t=1,\ 2,\dots\dots r. \tag{16}$$

9. Consider the equation

$$Y_1'f=\frac{\partial f}{\partial y_1}+\eta_{12}(y)\frac{\partial f}{\partial y_2}+\cdots+\eta_{1n}(y)\frac{\partial f}{\partial y_n}=o.$$

By the general existence-theorem, we can find $n-1$ first integrals $\varphi_1(y),\dots\varphi_{n-1}(y)$ which are regular about the point $y_1=y_2=\dots=y_n=o$, satisfying the initial conditions

$$\begin{gathered} \varphi_1(y)=y_2+\cdots\cdots, \\ \dots\dots\dots\dots\dots\dots\dots, \\ \varphi_{n-1}(y)=y_n+\cdots\cdots, \end{gathered}$$

at $y_1=o$. The dotted parts stand for terms of higher degrees with respect to the variables $y_2,\dots y_n$, (or we may give up these terms). Let us consider the transformation

$$\left.\begin{aligned} &z_1=y_1, \\ &z_2=\varphi_1(y)=y_2+\cdots\cdots, \\ &\dots\dots\dots\dots\dots\dots\dots\dots, \\ &z_n=\varphi_{n-1}(y)=y_n+\cdots\cdots, \end{aligned}\right\} \tag{17}$$

the dotted parts stand for terms of higher degrees with respect to the variables $y_1, y_2, \dots y_n$. This transformation is reversibly holomorphic about the point $y_1 = y_2 = \cdots = y_n = o$. Applying this to the complete system (14), then we obtain a system such as

$$1°. \qquad Z_1 f \equiv \frac{\partial f}{\partial z_1},$$

$$Z_2 f \equiv \zeta_{21}(z)\frac{\partial f}{\partial z_1} + \zeta_{22}(z)\frac{\partial f}{\partial z_2} + \cdots + \zeta_{2n}(z)\frac{\partial f}{\partial z_n},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$Z_r f \equiv \zeta_{r1}(z)\frac{\partial f}{\partial z_1} + \zeta_{r2}(z)\frac{\partial f}{\partial z_2} + \cdots + \zeta_{rn}(z)\frac{\partial f}{\partial z_n},$$

where the coefficients $\zeta_{21}(z), \zeta_{22}(z), \dots, \zeta_{rn}(z)$ are holomorphic about the point $z_1 = z_2 = \cdots = z_n = o$ By aid of (15), these differential expressions satisfy the relations

$$2°. \qquad (Z_1 Z_i) f \equiv o,$$

$$3°. \qquad (Z_i Z_j) f \equiv \sum_{s=1}^{r} E_{ijs} Z_s f, \qquad i, j = 2, \dots\dots r,$$

where all the composition-functions E_{ijs} are holomorphic about the point $z_1 = z_2 = \cdots = z_n = o$. Now by aid of 2° we have

$$Z_1 \zeta_{is} \equiv o, \qquad i = 2, \dots\dots r, \quad s = 1, 2, \dots\dots n,$$

and by (16) we have

$$Z_1 E_{ijt} \equiv o, \qquad i, j = 2, \dots\dots r, \quad t = 1, 2, \dots\dots r.$$

These results mean that all the functions $\zeta_{is}(z)$ and E_{ijt} are independent of the variable z_1. Now put

$$\bar{Z}_i f \equiv Z_i f - \zeta_{i1}(z) Z_1 f \tag{18}$$

$$= \zeta_{i2}(z)\frac{\partial f}{\partial z_2} + \cdots + \zeta_{in}(z)\frac{\partial f}{\partial z_n}, \qquad i = 2, \dots\dots r.$$

Then still we have

$$(Z_1 \bar{Z}_i) f \equiv o, \qquad i = 2, \dots\dots r.$$

Moreover we have

$$(\bar{Z}_i\bar{Z}_j)f \equiv \sum_{s=1}^{r} E_{ijs}\bar{Z}_s f, \qquad i, j=2,\ldots\ldots r.$$

The newly deduced complete system consisting of $r-1$ equations with respect to $n-1$ independent variables:

$$\left.\begin{array}{l} \bar{Z}_2 f \equiv \zeta_{22}(z)\dfrac{\partial f}{\partial z_2}+\cdots+\zeta_{2n}(z)\dfrac{\partial f}{\partial z_n}=o, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \bar{Z}_r f \equiv \zeta_{r2}(z)\dfrac{\partial f}{\partial z_2}+\cdot\;+\zeta_{rn}(z)\dfrac{\partial f}{\partial z_n}=o \end{array}\right\} \qquad (19)$$

may be treated in a similar way. (It may occur that among the coefficients $\zeta_{22}(z), \ldots, \zeta_{rn}(z)$, some number may exist which does not vanish at the point $z_2=\cdots=z_n=o$. In this case the next transformation of the variables is unnecessary.) Therefore, by mathematical induction we arrive at the result: *The $n-r$ integrals of the complete system of the partial differential equations* (12) *at the singular point $x_1=x_2=\cdots=x_n=o$, can be obtained by aid of some number of successively combined applications of the transformations such as* (1) *and* (17). *The forms of the integrals about the point $x_1=x_2=\cdots=x_n=o$ are very complex, but they are* uniform *in the domain surrounding the point $x_1=x_2=\cdots=x_n=o$,* (the point being excluded).

In the preceding discussions the complete system (12) is supposed as any of the coefficients $\xi_{11}(x), \xi_{12}(x), \ldots, \xi_{rn}(x)$ were not zero. But as we can easily verify, this restriction is not essential.

10. In § 9, we have used the solutions of the equation of the form

$$Yf=\frac{\partial f}{\partial y_1}+\eta_2(y)\frac{\partial f}{\partial y_2}+\cdots+\eta_n(y)\frac{\partial f}{\partial y_n}=o,$$

having been given that initially the solutions shall become the given functions for $y_1=o$. The given functions being regular about the point $y_2=\cdots=y_n=o$ and the integrals are also regular. Now for simplicity let an initial function be y_2 for $y_1=o$; this function will become by the transformation (1) $x_2x_n^{-m}$. For this function, the point $x_1=x_2=\cdots=x_n=o$ is an unessential singularity. Therefore by our transformation, we can find such a solution as will become $x_2x_n^{-m}$ for $x_1=o$ of the equation:

$$Xf\equiv\xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_2}+\cdots+\xi_n(x)\frac{\partial f}{\partial x_n}=o$$

in the domain $P_x(Rr^m; R, r)$.

The general problem of the integral at the singular point, being given initially a function with singularity at a point, is very difficult. But when the singular point $x_1 = x_2 = \cdots = x_n = o$ be an unessential singularity of the given function, *i.e.*, when we may write it

$$\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are regular about and vanish at the point $x_1 = x_2 = \cdots = x_n = o$, by our transformation (1), if this function become regular about the point, we can also find the solution of the equation $Xf = o$ with this initial function. This property may be extended to such an equation as

$$Xf = F(x, f) \equiv u(x)f + v(x),$$

where $u(x)$ and $v(x)$ are holomorphic about the point $x_1 = x_2 = \cdots = x_n = o$. When the functions $\xi_1(x), \xi_2(x), \ldots, \xi_n(x)$; $u(x), v(x)$ have the point $x_1 = x_2 = \cdots = x_n = o$ as an unessential singularity, our method may still be applied; for by the multiplication of a certain regular function of $x_1, x_2, \ldots x_n$ into the equation, it should be transformed into the form already discussed. But such function, equated to zero, is nothing but the singular manifoldness which is fixed. Thus *our method may be applied to the fixed singularity of the higher class.* For the homogeneous partial differential equation of the *higher order,* Mr. Stoïlow found singular solutions at the moving singularity, the initial function being quasi-uniform, but not at the fixed singularity.

11. After this extension we can easily give up the assumption made in § 8. We have assumed that all the composition-functions C_{ijs} were holomorphic about the point $x_1 = x_2 = \cdots = x_n = o$. We shall first prove that these functions, if not holomorphic, must have only an *unessential* singularity. From the relations

$$\begin{aligned}(X_i X_j) f &= \sum_{s=1}^{n} (X_i \xi_{js} - X_j \xi_{is}) \frac{\partial f}{\partial x_s} \\ &= \sum_{t=1}^{r} C_{ijt} X_t f \\ &= \sum_{s=1}^{n} \sum_{t=1}^{r} C_{ijt} \xi_{ts} \frac{\partial f}{\partial x_s}, \qquad i, j = 1, 2, \ldots\ldots r,\end{aligned}$$

we have

$$\left.\begin{array}{l} X_i\xi_{j1}-X_j\xi_{i1}=C_{ij1}\xi_{11}+C_{ij2}\xi_{21}+\cdots+C_{ijr}\xi_{r1}, \\ X_i\xi_{j2}-X_j\xi_{i2}=C_{ij1}\xi_{12}+C_{ij2}\xi_{22}+\cdots+C_{ijr}\xi_{r2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ X_i\xi_{jn}-X_j\xi_{in}=C_{ij1}\xi_{1n}+C_{ij2}\xi_{2n}+\cdots+C_{ijr}\xi_{rn}, \end{array}\right\} \qquad (20)$$

$$i,\ j=1,\ 2,\ldots\ldots r.$$

Since $X_1f, X_2f,\ldots X_rf$ are linearly independent, we may assume without loosing the generality that

$$\Delta_{ij}(x)\equiv\begin{vmatrix} \xi_{11} & \xi_{12} & \cdots\cdots & \xi_{1r} \\ \xi_{21} & \xi_{22} & \cdots\cdots & \xi_{2r} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \xi_{r1} & \xi_{r2} & \cdots\cdots & \xi_{rr} \end{vmatrix}\neq o.$$

Therefore from the first r equations of (20), we can find all the functions $C_{ij1}, C_{ij2},\ldots C_{ijr}$. The functions $\xi_{11}(x), \xi_{12}(x),\ldots, \xi_{rn}(x)$ being all holomorphic about the point $x_1=x_2=\cdots=x_n=o$, we see that all the functions C_{ij1}, C_{ij2}, C_{ijr} may have only unessential singularities. But if the point $x_1=x_2=\cdots=x_n=o$ belong to the unessential singularity, then we put $\Delta(x)$ the least common multiple of all the functions $\Delta_{ij}(x)$ and

$$\Delta(x)X_if\equiv\bar{X}_if, \qquad i=1,\ 2,\ldots\ldots r,$$

so that

$$(\bar{X}_i\bar{X}_j)f=(\Delta X_i,\ \Delta X_j)f$$

$$=X_i\Delta\bar{X}_jf-X_j\Delta\bar{X}_if+\sum_{s=1}^{r}\Delta C_{ijs}\bar{X}_sf$$

Sinee all the products ΔC_{ijs} are regular, $\bar{X}_1f, \bar{X}_2f,\ldots\bar{X}_rf$ constructs a complete system whose composition-functions are all regular about the point $x_1=x_2=\cdots=x_n=o$. With this system we may go as shown in §§ 8, 9 and arrive at the same result. Thus *the discussion of the complete system of the partial differential equations is completed.*

12. In the following we shall give some applications of the preceding theory to a class of linear homogeneous partial differential equations of the second order.

Let

$$\left.\begin{aligned} Xf &\equiv \xi_1(x)\frac{\partial f}{\partial x_1}+\xi_2(x)\frac{\partial f}{\partial x_r}+\cdots+\xi_n(x)\frac{\partial f}{\partial x_n}, \\ X'f &\equiv \xi_1'(x)\frac{\partial f}{\partial x_1}+\xi_2'(x)\frac{\partial f}{\partial x_2}+\cdots+\xi_n'(x)\frac{\partial f}{\partial x_n}, \end{aligned}\right\} \qquad (21)$$

where $\xi_1(x), \ldots, \xi_n(x), \ldots \xi_n'(x)$ are holomorphic about and vanish at the point $x_1=x_2=\cdots=x_n=o$. Moreover suppose that among the expansions of $\xi_1(x), \ldots, \xi_{n-1}(x)$; $\xi_1(x)$ contains such a term as ax_n^m where m is a least positive integer and it be the same for $\xi_1'(x)$. We consider the following linear homogenous partial differential equation of the second order:

$$\begin{aligned} X'Xf=&\left\{\xi_1'(x)\frac{\partial}{\partial x_1}+\xi_2'(x)\frac{\partial}{\partial x_2}+\cdots+\xi_n'(x)\frac{\partial}{\partial x_n}\right\}\left\{\xi_1(x)\frac{\partial}{\partial x_1}\right. \\ &\left.+\xi_2(x)\frac{\partial}{\partial x_r}+\cdots+\xi_n(x)\frac{\partial}{\partial x_n}\right\}f = u(x)f+u'(x) \end{aligned} \qquad (22)$$

where $u(x)$ and $u'(x)$ are holomorphic about the point $x_1=x_2=\cdots=x_n=o$ which is a fixed singularity of the equatlon. This equation is equivalent to the linear simultaneous equations:

$$\left.\begin{aligned} Xf &= \varphi, \\ X'\varphi &= u(x)f+u'(x). \end{aligned}\right\}$$

Applying our transformation (1), this system is transformed into

$$\left.\begin{aligned} Yf &\equiv \eta_1(y)\frac{\partial f}{\partial y_1}+\eta_2(y)\frac{\partial f}{\partial y_2}+\cdots+\eta_n(y)\frac{\partial f}{\partial y_n}=\varphi, \\ Y'f &\equiv \eta_1'(y)\frac{\partial f}{\partial y_1}+\eta_2'(y)\frac{\partial f}{\partial y_2}+\cdots+\eta_n'(y)\frac{\partial f}{\partial y_n}=v(y)f+v'(y), \end{aligned}\right\}$$

where all the functions $\eta_i(y), \eta_i'(y)$ and $v(y), v'(y)$ are regular about the point $y_1=y_2=\cdots=y_n=o$ and $\eta_1(y), \eta_1'(y)$ do not vanish at the point. The equation (22) then becomes

$$Y'Yf=v(y)f+v'(y).$$

The coefficient $\eta_1(y)\eta_1'(y)$ of $\dfrac{\partial^2 f}{\partial y_1^2}$ does not vanish at the point $y_1=y_2=\cdots=y_n=o$. Therefore, by the general theorem of the existence of Cauchy-Kowalewski, we may prove the existence of the singular solution as will become, for $y_1=o$

$$f=\text{regular function of } y_2,\dots y_n,$$
$$\frac{\partial f}{\partial y_1}= \qquad \text{"} \qquad .$$

Thus our equation (22) has solutions expanded into a special Laurent-series. These considerations may be extended further, the point $x_1=x_2=\cdots=x_n=o$ being a fixed singularity of each characteristic function of the partial differential equation.