Some Properties of Analytic Elements On And Outside Their Circles of Convergence

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Introduction.

Consider a mathematical expression

$$F(z) \equiv c_0 + c_1 z + \dots + c_n z^n + \dots,$$

where z is a complex variable. We call it a *series* (more strictly a *power-series*). If $\lim\limits_{n \to \infty} S_n(z)$, where

$$S_n(z) \equiv c_0 + c_1 z + \dots + c_n z^n,$$

be finite and determinate, the series is convergent. When the limiting value is ∞ or indeterminate, the series is divergent. In the former case, put

$$S_n(z) \equiv A_n + iB_n.$$

Then the equality $\lim\limits_{n \to \infty} S_n(z) = \infty$ means that $\lim\limits_{n \to \infty} |A_n + iB_n| = \infty$; but not at the same time $\lim A_n = \infty$ and $\lim B_n = \infty$. Even when these limits are indeterminate, yet $S_n(z)$ may tend to ∞. Let us call the series *infinite* when both A_n and B_n tend to definite limiting values and $|A_n + B_n|$ tends to infinity. When at least one of A_n and B_n becomes indeterminate, we call the series *indeterminate*.

In the present paper, the intervals in which a power-series is convergent, infinite or indeterminate, are discussed. Next, it is proved that an analytic element outside its circle of convergence and in the part of the plane toward which the element may be continued, is indeterminate. Thirdly, the relation between Leibnitz's theory and Borel's generalised sums are considered. Finally, some special cases of the theorems of Fabry and of Hadamard concerning singular points are viewed from our standpoint.

1. Consider a power-series

$$F(z)=c_0+c_1z+\dots+c_nz^n+\dots,$$

whose radius of convergence may in general be supposed to be unity. The other case is now out of question. If the series be convergent at a point, say $z=1$, on its circle of convergence, then the power-series $F(z)$, when z converges to unity, takes the value $F(1)$. Thus the extension of Abel's theorem is due to Stolz.[1] In his book,[2] Mr. Stolz proved furthermore that if the series be infinite at $z=1$, then the coefficients being supposed real, along the real axis of z-plane, $\lim\limits_{x\to 1} F(x)=\infty$.

By aid of these theorems, we may easily conclude that if $F(z)$ may be continued over $z=1$, then the series $F(z)$ must be convergent or indeterminate at $z=1$. Put

$$z=x+iy, \qquad c_n=a_n+i\,b_n, \qquad n=0,\ 1,\ \dots\ n,\ \dots,$$

$$f(x)=a_0+a_1x+\dots+a_nx^n+\dots,$$

$$f_1(x)=b_0+b_1x+\dots+b_nx^n+\dots,$$

and let

$$F(z|z_0)=P+iQ,$$

be the continuation of $F(z)$ over $x=1$, where z_0 is a point within the interval $(0, 1)$. If z varies along the x-axis, then in the vicinity of $z=1$, $(x<1)$, we have $f(x)=P$, $f_1(x)=Q$. Therefore, neither $f(1)$ nor $f_1(1)$ can be infinite. Suppose, for example, $f(1)=\infty$, then by Stolz's theorem

$$\lim_{x\to 1} f(x)=\infty.$$

Hence for any given positive number G, we have

$$f(x)>G,$$

provided x is sufficiently near to unity. On the other hand, since $F(z|z_0)$ is in the vicinity of $z=1$, is convergent, we may suppose G such that

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1 Goursat, *Cour d'Analyse* II (1918), p. 22.

2 Stolz-Gemeiner, *Funktionentheorie* (1905), p. 289.

$$|F(z|z_0)| < G.$$

This is absurd, and the same may be said for $f_1(x)$. Thus, *any power-series must be convergent or indeterminate at the points on its circle of convergence, over which the series may be continued. Therefore if $F(1)$ be infinite, then $F(z)$ can not be continued over $z=1$.*

2. For example, consider the series

$$F(z)=1-z+z^2-\dots,$$

whose radius of convergence is unity and the series may be continued over $z=1$. But for $z=1$,

$$F(1)=1-1+1-\dots,$$

as is well known, is indeterminate and several objections[1] are made against Euler's assumption. I shall present one:

Since, n being a positive integer,

$$\frac{1}{2}\int_{n\pi}^{(n+1)\pi} \sin x dx=(-1)^{n+1},$$

we have

$$I_\infty=\frac{1}{2}\int_0^\infty \sin x dx=\frac{1}{2}\int_0^\pi+\frac{1}{2}\int_\pi^{2\pi}+\dots$$
$$=1-1+1-1+\dots.$$

On the other hand, we have

$$I_\alpha=\frac{1}{2}\int_0^\alpha \sin x dx$$
$$=1-1+\dots+(-1)^{n-1}+\frac{1}{2}\int_{n\pi}^\alpha \sin x dx.$$

Putting $\alpha=n\pi+A$, where A is an angle less than $\frac{\pi}{2}$, we have

$$I_\alpha=\frac{1}{2}(1 \pm \cos A).$$

So that for $\alpha\to\infty$, our series $F(1)$ would take any value in $(0, 1)$. Nevertheless for $|z|<1$,

$$F(z)=\frac{1}{1+z}.$$

1 Borel, *Séries divergentes* (1901).

Therefore the analytic function defined by the analytic element $F(z)$ takes $\frac{1}{2}$ at $z=1$ and Euler's assumption is to some degree not absurd. A complete discussion will be given later.

3. In the following we shall show some theorems concerning indeterminate series.

Consider a series

$$S=a_0+a_1+\ldots+a_n+\ldots,$$

which is indeterminate. Construct two series

$$S' = a_0'+a_1'+\ldots+a_n'+\ldots,$$
$$S'' = a_0''+a_1''+\ldots+a_n''+\ldots,$$

the first being made of all positive terms in S, taken in order, and the second, negative terms with signs changed. Then both series S' and S'' must diverge to infinity. Consequently *the number of variations of signs of each two successive terms of the series must be infinite.*

4. Next consider a power-series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots.$$

For simplicity we suppose this real. Put

$$S_n(x)=a_0+a_1x+\ldots+a_nx^n.$$

If the series be convergent in an interval, then in it

$$\lim_{n\to\infty} S_n(x)=\textit{finite and determinate},$$

which means that $S_n(x)$ tends to one and only one value, *independent of the mode of summation.* By our last expression, we mean to increase *n in any manner* and yet $S_n(x)$ converges to a number. But we do not mean to change the *order of the terms* of the series $f(x)$. On the contrary, if the series be indeterminate, then $\lim_{n\to\infty} S_n(x)$ depends upon the mode of summation.

Suppose for any value of n,

$$|S_n(x)|< G,$$

where G is a positive number, we write it as

$$|f(x)|< G,$$

and say that the series $f(x)$ is limited. *If $f(x)$ be limited at a point $x=x_0$, then the series $f(x)$ is convergent for $|x|<|x_0|$.* For the proof of this, consider the series

$$\sum_{n=0}^{\infty} S_n(x_0)\left(\frac{x}{x_0}\right)^n, \quad \text{for} \quad |x|<|x_0|.$$

Since by our assumption, all $S_n(x)$ are limited, the series is absolutely convergent. Therefore the product of the two series:

$$\left\{\sum_{n=0}^{\infty} S_n(x_0)\left(\frac{x}{x_0}\right)^n\right\}\left(1-\frac{x}{x_0}\right)$$

is also absolutely convergent, and hence the order of its terms may be changed. Hence it is equal to the series

$$\begin{aligned} S_0(x_0)+\left\{S_1(x_0)-S_0(x_0)\right\}\left(\frac{x}{x_0}\right)+\dots+\left\{S_n(x_0)-S_{n-1}(x_0)\right\}\left(\frac{x}{x_0}\right)^n+\dots \\ =a_0+a_1x+\dots+a_nx^n+\dots \\ =f(x). \end{aligned}$$

Thus $f(x)$ must be convergent. This is an example of the theorem that if all the terms of the series $f(x)$ be limited at $x=x_0$, then it is convergent for $|x|<|x_0|$.[1] Hence *if $f(x)$ be divergent in an interval (x_1, x_2) where $0<x_1<x_2$, then* it can not be limited within the interval. Moreover, since $f(x)$ can not be convergent for $x>x_2$, *it can not be limited in the interval* (x_1^*, x_2). x_1^* means the exclusion of the point $x=x_1$. The series $1-x+x^2-\dots$ is such an example for the interval $(1^*, \infty)$.

Moreover, we may prove that if $f(x)$ be indeterminate in an interval (x_1, x_2), it must not be *limited in one way*, namely neither

$$f(x)>G, \quad \textit{nor} \quad f(x)<G'$$

may be true, where G and G' are certain numbers. We shall prove *e.g.*, the case where $f(x)>G$. In this inequality we may suppose $G>0$. For this inequality means that for any value of n, $S_n(x)>G$. Hence if G be negative, we have only to add a certain positive number to a_0. Consequently, we may suppose that all $S_n(x)$ are positive in the interval.

[1] Goursat, *Cours d'analyse* I (1917). p. 446.

Now x_0 being a number in the interval (x_1, x_2),

$$S_n(x)=\sum_{h=0}^{n}\left\{S_h(x_0)-S_{h-1}(x_0)\right\}\left(\frac{x}{x_0}\right)^h$$
$$=\left\{\sum_{h=0}^{n-1}S_h(x_0)\left(\frac{x}{x_0}\right)^h\right\}\left(1-\frac{x}{x_0}\right)+S_n(x_0)\left(\frac{x}{x_0}\right)^n, \tag{1}$$

where $S_{-1}(x_0)\equiv 0$.

Let x take any value within the interval (x_1, x_0). To prove our proposition, three cases may occur: 1st. If the series

$$\sum_{h=0}^{\infty}S_h(x_0)\left(\frac{x}{x_0}\right)^h$$

be convergent, then we must have

$$\lim_{n\to\infty}S_n(x_0)\left(\frac{x}{x_0}\right)^n=0.$$

Therefore by aid of the equality (1), $f(x)$ must be convergent, and this is against our hypothesis that the series $f(x)$ is indeterminate. 2nd. If the series $\Sigma S_h(x_0)\left(\frac{x}{x_0}\right)^h$ be determinate and infinite, then since, by the preceding equality (1),

$$S_n(x)>\left\{\sum_{h=0}^{n-1}S_h(x_0)\left(\frac{x}{x_0}\right)^h\right\}\left(1-\frac{x}{x_0}\right). \tag{2}$$

Hence $f(x)$ must be divergent and infinite. This again contradicts our hypothesis. 3rd. The only remaining possible case is that the series

$$\sum_{h=0}^{\infty}S_h(x_0)\left(\frac{x}{x_0}\right)^h$$

should be indeterminate. But if so, by the theorem of §3, the variations of the signs of the series must be infinite. But this is contrary to the assumption that

$$\text{all } S_n(x)>G>0.$$

Hence this assumption cannot but be absurd. Since $x_2-x_0>0$ may be however small, $f(x)$, *in the interval, can not be limited in both ways.* The discussion is unvalid at $x=x_1$. Moreover, suppose that *at a point* $x=x_0$, $(x_1<x_0<x_2)$, $f(x_0)$ be limited in one way *e.g.*,

$f(x_0) > G > 0$, then as we have just proved, within the interval (x_1, x_0), it must give subintervals in which $f(x)$ is not limited in either way. Let x be such a point in one of them. Then by aid of the inequality (2), $f(x)$ must be limited in the negative way. This is absurd. Therefore we may conclude these as follows:

If the series

$$f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$$

be divergent and indeterminate in the interval (x_1, x_2), *then the series* $f(x)$ *can not be limited in either way at any point in the interval* (x_1^*, x_2^*), $x_1 > 0$. *Therefore if* $f(x)$ *be limited only in one way, then it must diverge to infinity.* This result may be extended to the field of complex numbers.

Remark 1. As we have proved, an indeterminate power-series can not be limited, *i.e.*, $S_n(x)$ may take, at general points, values greater than any positive number, respectively any number less than any negative value. But for general functions, this is not true. For example $f(x) = \sin\frac{1}{x}$ is indeterminate for $x \to 0$, but it is not greater than unity, *i.e.*, limited in both ways.

Remark 2. Though an indeterminate power-series can not be limited in both ways, it may at certain points take finite determinate values for certain modes of summations. For example let $f(x)$ be

$$f(x) = x_0 - \left(1 + \frac{1}{1!}\right)x - x_0 x^2 + \left(1 + \frac{1}{3!}\right)x^3 + \dots$$
$$+ (-1)^n x_0 x^{2n} + (-1)^{n+1}\left(1 + \frac{1}{(2n+1)!}\right)x^{2n+1} \pm \dots,$$

where x_0 is a number greater than unity. The radius of convergence is unity. This series may be continued over $x = 1$; for if $|x| < 1$, then clearly

$$f(x) = \frac{x_0 - x}{1 + x^2} - \sin x.$$

Thus $f(x)$ is a sum of functions which may be continued over $x = 1$. Hence it may be continued. If at $x = x_0$, we sum up $f(x_0)$ by two successive terms from the beginning, we heve

$$f(x_0) = -\sin x_0.$$

On the contrary

$$\lim_{n \to \infty} S_{2n} = \pm \infty.$$

From the theorem of § 1, we naturally come to inquire whether a power-series be indeterminate outside the arc of its circle of convergence over which the series may be continued. For a special case it is clear. For example, the series

$$F(z) = 1 - z + z^2 - \dots$$

may be continued over $z=1$ and is indeterminate for $z \geqq 1$. We shall enter into the discussion of the general case where the answer is affirmative.

5. If a convergent power-series

$$f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$$

whose radius of convergence is unity, be determinate and infinite at $x=x_0$, $(x_0>1)$, then for $1<x\leqq x_0$ $f(x)$ must also be infinite. For proof of this, consider as usual the identity

$$\begin{aligned} S_n(x) &\equiv a_0 + a_1 x + \dots + a_n x^n \\ &\equiv \left\{ S_0(x_0) + S_1(x_0)\left(\frac{x}{x_0}\right) + \dots + S_{n-1}(x_0)\left(\frac{x}{x_0}\right)^{n-1} \right\} \\ &\qquad \times \left(1 - \frac{x}{x_0}\right) + S_n(x_0)\left(\frac{x}{x_0}\right)^n. \end{aligned}$$

1°. If the series $\Sigma S_h(x_0)\left(\frac{x}{x_0}\right)^h$ be convergent, then since

$$\lim_{n\to\infty} S_n(x_0)\left(\frac{x}{x_0}\right)^n = 0,$$

$S_n(x)$ must converge to a number, and this is impossible. 2°. The series $\Sigma S_h(x_0)\left(\frac{x}{x_0}\right)^h$ can not be indeterminate, since the variations of signs are finite. 3°. Consequently it must diverge to infinity. So that the series $f(x)$ is infinite in $1<x\leqq x_0$. This is true also for $f(x_0) = -\infty$.

From this result, it follows that if $f(x)$ be indeterminate for $x'>x_0$, then it must be so for any $x>x_0$. If not, it gives at least a point $x'>x_0$, for which $f(x')$ is ∞, (or $-\infty$). Hence for any $x<x'$, $f(x)$ must be ∞ (or $-\infty$) which is contrary to our assumption. Thus *the points $x \geqq 0$, are divided, if possible, into three parts: 1°. $0 \leqq x < 1$, 2°. $1<x<x_0$, 3°. $x_0<x$. In the first interval $f(x)$ is convergent;*

in the second it is infinite; and in the third, it is indeterminate (by the preceding theorem, it oscillates between $-\infty, +\infty$).

$$\begin{array}{cccc} & \textit{conv.} & \textit{inf.} & \textit{ind.} \\ \text{—}|\text{———}|\text{———}|\text{———}\rightarrow x \\ \text{o} & \text{I} & x_0 & \end{array}$$

6. Let us give an example. In the formula

$$S_n(x)=\left(1-\frac{x}{x_0}\right)\left\{S_0(x_0)+S_1(x_0)+\ldots+S_{n-1}(x_0)\left(\frac{x}{x_0}\right)^{n-1}\right\}+S_n(x_0)\left(\frac{x}{x_0}\right)^n,$$

put

$$S_h(x_0)\equiv s_h, \qquad h=0, 1, 2, \ldots, n.$$

Then for $x=2x_0$, we have

$$S_n(2x_0)=-(s_0+2s_1+\ldots+2^{n-1}s_{n-1})+2^n s_n.$$

Assume

$$s_{2h}=2^{2h+1}, \quad s_{2h+1}=2^{2h}, \quad h=0, 1, 2, \ldots, n.$$

Since

$$2^{2h}s_{2h}+2^{2h+1}s_{2h+1}=2\cdot 2^{4h+1},$$

we have for $n=2m$,

$$S_{2m}(2x_0)=-2(2+2^5+\ldots+4^{5m-3})+2^{4m+1}$$
$$=2^{4m+1}\left(1-\frac{2}{15}\right)+\frac{4}{15}>0.$$

For $n=2m+1$, we have

$$S_{2m+1}(2x_0)=-4\frac{2^{4m}-1}{15}<0.$$

Hence for $m\to\infty$,

$$\lim_{m\to\infty} S_{2m}(2x_0)=+\infty,$$
$$\lim_{m\to\infty} S_{2m+1}(2x_0)=-\infty,$$

and consequently our series $f(x)$ for $x=2x_0$ is indeterminate and not limited in both ways.

Now to find the concrete form of the series, we must solve the following equations with an infinite number of unknowns:

$$s_0=a_0,$$
$$s_1=a_0+a_1x_0,$$
$$s_2=a_0+a_1x_0+a_2x_0^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$s_n = a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n,$$

$$\dots\dots\dots\dots$$

This is a special case of the systems which are treated in Volterra's equations.[1] For our case, the treatment is easy.

We have

$$a_0 = s_0,$$

$$a_1 = \frac{s_1 - s_0}{x_0},$$

$$\dots\dots\dots\dots$$

$$a_n = \frac{s_n - s_{n-1}}{x_0^n},$$

$$\dots\dots\dots\dots$$

Therefore substituting the values of s_n, we have

$$a_0 = 2,$$

$$a_{2h} = \frac{2^{2h+1} - 2^{2(h-1)}}{x_0^{2h}} = \frac{7}{4} \frac{2^{2h}}{x_0^{2h}},$$

$$a_{2h+1} = -\frac{1}{2} \frac{2^{2h+1}}{x_0^{2h+1}}.$$

The required series is

$$f(x) = 2 - \frac{1}{2}\left(\frac{2}{x_0}\right)x + \frac{7}{4}\left(\frac{2}{x_0}\right)^2 x^2 - \frac{1}{2}\left(\frac{2}{x_0}\right)^3 x^3 + \dots .$$

Specially for $x_0 = 2$, we have

$$f(x) = 2 - \frac{1}{2}x + \frac{7}{4}x^2 - \frac{1}{2}x^3 + \dots, \qquad (1)$$

whose radius of convergence is unity and

$$f(1) = \infty, \qquad f(2 \times 2) = \pm\infty.$$

The point which divides the intervals where the series $f(x)$ becomes infinite, respectively, indeterminate is found to be $\frac{7}{2}$. For at this point

$$a_{2h}x^{2h} + a_{2h+1}x^{2h+1} = \left(\frac{7}{4} - \frac{1}{2}\frac{7}{2}\right)\left(\frac{7}{2}\right)^{2h} = 0,$$

$$a_{2h-1}x^{2h-1} + a_{2h}x^{2h} = \left(-\frac{1}{2} + \frac{7}{4}\frac{7}{2}\right)\left(\frac{7}{2}\right)^{2h-1} > 0.$$

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[1] Volterra, *Leçons sur les Équations intégrales et* ..., (1913), p. 40.

Thus $S_n\left(\frac{7}{2}\right)$ is limited in one way. Therefore by the theorem of § 5, for x just less than $\frac{7}{2}$, the series must be determinately infinite. Concluding these, for our series (1), the whole interval of x is divided into four parts:

$+$ *inf.*	*conv.*	$+$ *inb.*	*ind.*
$x \leq -1$	$< x <$	$+1 \leq x < 3{\cdot}5$	$< x$

At $x = 3{\cdot}5$, $f(x)$ oscillates between 0 and ∞.

7. In § 1, we have proved, by aid of Stolz's theorem, that a power-series $F(z)$ which may be continued over $z=1$, must be convergent or indeterminate at $z=1$. If it be indeterminate at $z=1$, then by § 5, it must be indeterminate for $z>1$ along the real axis of the z-plane. But if it be convergent at $z=1$, whether it is indeterminate for $z>1$ is not known. In the following we shall prove it. As before, instead of considering $F(z)$, we discuss its real or imaginary part for real values of z which is divergent for $z>1$, namely

$$f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots.$$

Since $F(z)$, by our assumption, may be continued over $z=1$, its real and imaginary parts may also be continued over $x=1$. Therefore $f(x)$ may be continued over $x=1$.

8. For the general proof, we take a quite different method which is nothing but to connect three theorems proved by Borel, Hardy and Vivanti. In the following the process is briefly stated.

Since the series

$$f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$$

may be continued over $x=1$, the analytic function defined by $f(x)$ is holomorph about $x=1$. Now draw two concentric circles, center at the point $x=\frac{1}{2}$. Let the radius of the inner circle be $\frac{1}{2}$. Make that of the outer one just greater than this; so that in and on the circle, the analytic function may be holomorph. This circle intersects the real axis at O' and A. Let the circle be called C. By Cauchy's theorem for the analytic function $f(x)$, we have

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz, \qquad n = 0, 1, 2, \cdots$$

Consider the associated function of $f(x)$:

$$A(t)=a_0+a_1x\frac{t}{1\,!}+\ldots+a_nx^n\frac{t^n}{n\,!}+\ldots,$$

where t is a real variable. This function is clearly an integral function. Putting the values of a_n, we have for any value of t,

$$A(t)=\frac{1}{2\pi i}\int_C\frac{f(x)}{z}e^{\frac{xt}{z}}dz.$$

So we have

$$e^{-t}A(t)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z}e^{t\left(\frac{x}{z}-1\right)}dz,$$

$$e^{-t}A^{(h)}(t)=\frac{1}{2\pi i}\int_C\frac{f(z)x^h}{z^{h+1}}e^{t\left(\frac{x}{z}-1\right)}dz.$$

Therefore for any x of a certain ellipse whose major axis is $O'A$, our series $f(x)$, is absolutely summarble. Thus the polygon of summability may be determined. These are due to Mr. Borel.[1]

Now after Mr. Hardy[2], writing for simplicity,

$$u_n=a_nx^n,\ s_n=u_0+u_1+\cdots+u_n,\ n=0,\ 1.\ 2,\ \ldots,$$

we have

$$e^{-t}\sum_{n=0}^{\infty}s_n\frac{t^n}{n\,!}-u_0=\int_0^t e^{-t}\sum_{n=0}^{\infty}\frac{u_{n+1}t^n}{n\,!}dt.$$

But the series $u_0+u_1+\ldots+u_n+\ldots$ is, as we have seen, summable. Therefore

$$\lim_{t\to\infty}e^{-t}\sum_{n=0}^{\infty}s_n\frac{t^n}{n\,!}$$

is finite and determinate; so that our series Σu_n admits both of Borel's definitions of summability.

Next suppose our series $f(x)$ *i.e.*, Σu_n be ∞, then we can determine a positive number N, such that given a number however great G,

$$s_n>G,\quad \text{for any}\quad n>N.$$

Therefore

$$e^{-t}\sum_{n=0}^{\infty}s_n\frac{t^n}{n\,!}>e^{-t}\sum_{n=0}^{N-1}(s_n-G)\frac{t^n}{n\,!}+G,$$

and hence

1 Borel, *loc. cit.*, p. 123.

2 Hardy, *Quarterly J.* (1904), p. 34.

$$\lim_{t \to \infty} e^{-t} \sum_{n=0}^{\infty} s_n \frac{t^n}{n!} = \infty.$$

This proof is due to Mr. Vivanti[1]. This result is contradictory. Thus our series must be indeterminate. Now by aid of the theorem of § 5, our proposition is true:

When a power-series, whose radius of convergence is unity, $f(x) = a_0 + a_1 x + \cdots + a_n x^n + \ldots$ *may be continued over* $x=1$, *then it must be indeterminate for all points* $x > 1$. *For* $x=1$, *the series may sometimes be convergent or sometimes be indeterminate.* Moreover when $f(x)$ may be continued over $x=1$, this is also the case for any derivatives of $f(x)$, Therefore, *at the same time, all derivatives of* $f(x)$ *are indeterminate, for* $x > 1$.

9. The above theorem is a necessary condition of the analytic continuation, but not clearly sufficient. Or, take, for example, Mr. Fredholm's series, changing a into $-a$,

$$f(x) = 1 - ax + a^2 x^{2^2} - \ldots + (-1)^n a^n x^{n^2} \pm \ldots,$$

where a is a positive number. The circle of convergence is the natural limit of the function. But for $x > 1$,

$$\frac{a^{n+1} x^{(n+1)^2}}{a^n x^{n^2}} = a x^{2n+1} > 1,$$

provided n is sufficiently great. Therefore

$$(1 - ax) + (a^2 x^{2^2} - a^3 x^{3^2}) + \ldots = -\infty,$$
$$1 - (ax - a^2 x^{2^2}) - (a^3 x^{3^2} - a^4 x^{4^2}) - \ldots = +\infty,$$

so that $f(x)$ is indeterminate for $x > 1$; likewise for its derivatives.

10. In remark 2, § 4, we have given a power-series which takes a value by a certain mode of summation. From that series, we obtain a new one,

$$f(x) = \frac{x_0 - x}{1 + x^2}$$

$$= x_0 - x - x_0 x^2 + x^3 + \ldots + (-1)^n x_0 x^{2n} + (-1)^{n+1} x^{2n+1} \pm \ldots.$$

We see that.

$$\lim_{n \to \infty} S_{2n+1}(x_0) = S_{2n+1}(x_0) = 0.$$

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[1] Vivanti, *Theorie der Funktionen* (1906), p. 329.

Here we notice that o is the value of the analytic function $f(x)$. Therefore we naturally come to conjecture that such would be a general property. But this is at once proved to be wrong. For, consider the series

$$f(x)=1-x+x^2-\dots+(-1)^n x^n \pm \dots .$$

For this series $S_{2n+1}(1)=0$. But the value of the analytic function at $x=1$ is $\frac{1}{2}$, different from zero. We notice that o is the lower limit of $S_n(1)$. This is not the case for the preceding series. The second conjecture is that when the limit of $S_n(x_0)$ at some point $x=x_0$ for a certain mode of summation converges to a number different from the upper respectively lower limit of $S_n(x_0)$, that limit would be the value of the analytic function at $x=x_0$. This conjecture is also destroyed by the consideration of the following series;

$$1+\cos\theta+\cos 2\theta+\dots+\cos n\theta+\dots .$$

This series is divergent and θ being not equal to $2m\pi$, its Borel's integral sum is easily found to be $\frac{1}{2}$.[1] Let us consider a power-series

$$f(x)=1+x\cos\theta+x^2\cos 2\theta+\dots+x^n\cos n\theta+\dots,$$

which is convergent within the circle of radius unity. At the point $x=1$, it is divergent. Now put

$$S_n(1)=1+\cos\theta+\cos 2\theta+\dots+\cos n\theta,$$

Then

$$2\,S_n(1)=1+\frac{\sin\frac{2n+1}{2}\theta}{\sin\frac{\theta}{2}}.$$

Specially consider the case $\theta=\frac{\pi}{4}$, then

$$\begin{aligned} 2\,S_n(1) &= 1+\frac{\sin(2n+1)\frac{\pi}{8}}{\sin\frac{\pi}{8}}, \\ &= 1+1, && n\equiv 0,\ 3 \pmod{8}, \\ &= 1+\cot\frac{\pi}{8}, && \equiv 2,\ 1 \quad ,, \\ &= 1-1, && \equiv 4,\ 7 \quad ,, \\ &= 1-\cot\frac{\pi}{8}, && \equiv 6,\ 5 \quad ,, . \end{aligned}$$

[1] Bromwich, *Theory of infinite series* (1908), p. 275.

Therefore for $n=8m+4$,

$$S_n=\lim S_n=0.$$

Since $2<\cot\frac{\pi}{8}<3$, this value lies between the upper and the lower limit of the sums.

The a nalytic function can easily be found to be

$$f(x)=\frac{\sqrt{2}+x-x^3}{\sqrt{2}(1+x^4)},\quad f(1)=\frac{1}{2}.$$

Hence for our mode of summation

$$f(1)\neq\lim_{n\to\infty}S_n(1).$$

The third conjecture is that when S_n tend to several values, among them there should occur the value of the analytic function. This is also fatal. For by aid of the last example, lf $S_n(1)$ tend to $\frac{1}{2}$, by a certain mode of summation, then

$$\lim_{n\to\infty}\sin(2n+1)\frac{\pi}{8}=0.$$

And this is impossible. All the conjectures are proved to be wrong on the circle of convergence.

11. Now we notice in the preceding problem that

$$\lim_{k\to\infty}S_{8k+h}(1)=\textit{finite and determinate},\ h=0,1,2,\ldots,7,$$

and their arithmetical mean $\frac{1}{2}$ is nothing bnt the value of the analytic function at $x=1$. Consequently we come to the fourth *conjecture*. Namely, *given a divergent series*

$$S=a_0+a_1+\ldots+a_n+\ldots,$$

if

$$\lim_{k\to\infty}S_{km+h}=S^{(h)},\qquad h=0,1,\ldots,m-1,$$

be all finite and determinate, then

$$\frac{1}{m}\left(S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}\right)$$

is equal to the generalised sum of the given series. This supposition is true:

At first we shall consider Cèsaro's generalisation. Since the sequence of numbers

$$S_h,\ S_{m+h},\ \ldots,\ S^{km+h},\ \ldots$$

converges to $S^{(h)}$, we have

$$\lim_{k\to\infty}\frac{S_h+S_{m+h}+\ldots+S_{km+h}}{k}=S^{(h)}$$

$$=\lim_{k\to\infty}\frac{S_h+S_{m+h}+\ldots+S_{km+h}}{km+m-1}\cdot\frac{km+m-1}{k}.$$

$$\therefore\ \lim_{k\to\infty}\frac{S_h+S_{m+h}+\ldots+S_{km+h}}{km+m-1}=\frac{S^{(h)}}{m},\quad h=0,\ 1,\ \ldots,\ m-1.$$

Adding these m equations side by side, we may conclude that

$$\lim_{k\to\infty}\frac{S_0+S_1+\ldots+S_{km+m-1}}{km+n-1}=\frac{1}{m}(S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}).$$

Hence, in general,

$$\lim_{n\to\infty}\frac{S_0+S_1+\ldots+S_n}{n}=\frac{1}{m}(S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}).\quad (1)$$

Q. E. D.

And our series is *simply indeterminate.*

Next, we consider Borel's generalisation. For this purpose consider the limits

$$\lim_{t\to\infty}\frac{E_h(t)}{e^t},\qquad h=0,\ 1,\ \ldots,\ m-1,$$

where

$$E_h(t)\equiv\frac{t^h}{h\,!}+\frac{t^{m+h}}{(m+h)!}+\ldots+\frac{t^{km+h}}{(km+h)\,!}+\ldots.$$

The functions $E_n(t)$ satisfy the differential equation

$$\frac{d^mE}{dt^m}=E.$$

Therefore let $r_0(=1),\ r_1,\ \ldots,\ r_{m-1}$ be the roots of the characteristic equation

$$r^m=1,$$

then we have

$$E_h(t)=C_{h0}e^{r_0t}+C_{h1}e^{r_1t}+\ldots+C_{hm-1}e^{r_{m-1}t},\qquad (2)$$

$$h=0,\ 1,\ \ldots,\ m-1,$$

where $C's$ are the integration-constants. These constants are to be so determined that

$$C_{h0}+C_{h1}+\ldots+C_{hm-1}=0,$$

$$C_{h0}r_0+C_{h1}r_1+\ldots+C_{hm-1}r_{m-1}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$C_{h0}r_0^h + C_{h1}r_1^h + \ldots + C_{hm-1}r_{m-1}^h \quad = 1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$C_{h0}r_0^{m-1} + C_{h1}r_1^{m-1} + \ldots + C_{hm-1}r_{m-1}^{m-1} = 0.$$

To find C_{h0}, its denominator is equal to

$$\Delta \equiv \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_0 & r_1 & \ldots & r_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ r_0^{m-1} & r_1^{m-1} & \ldots & r_{m-1}^{m-1} \end{vmatrix} \begin{aligned} = \pm (r_0 - r_1)(r_0 - r_2) \;\cdots\cdots\; & (r_0 - r_{m-1}) \\ (r_1 - r_2) \;\cdots\cdots\; & (r_1 - r_{m-1}) \\ \cdots\cdots \quad & \cdots\cdots \\ & (r_{m-2} - r_{m-1}). \end{aligned}$$

The numerator is equal to

$$\Delta_{h0} \equiv \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 0 & r_1 & \ldots & r_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & r_1^h & \ldots & r_{m-1}^h \\ \cdots & \cdots & \cdots & \cdots \\ 0 & r_1^{m-1} & \ldots & r_{m-1}^{m-1} \end{vmatrix} = \pm \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{h-1} & r_2^{h-1} & \ldots & r_{m-1}^{h-1} \\ r_1^{h+1} & r_2^{h+1} & \ldots & r_{m-1}^{h+1} \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{m-1} & r_2^{m-1} & \ldots & r_{m-1}^{m-1} \end{vmatrix}$$

$$\begin{aligned} = M(r_1 - r_2)(r_1 - r_3) \;\ldots\; & (r_1 - r_{m-1}) \\ (r_2 - r_3) \;\ldots\; & (r_2 - r_{m-1}) \\ & \cdots\cdots\cdots \\ & (r_{m-2} - r_{m-1}), \end{aligned}$$

where M in a certain factor to be determined. Since

$$\begin{aligned} r^m - 1 &= (r-1)(r^{m-1} + r^{m-2} + \ldots + 1) \\ &= (r-1)(r-r_1)(r-r_2) \;\ldots\; (r - r_{m-1}), \end{aligned}$$

for $r = r_0 (=1)$, we have

$$(r_0 - r_1)(r_0 - r_2) \;\ldots\; (r_0 - r_{m-1}) = m. \tag{3}$$

On the other hand, with respect to the equation

$$1 + r + \ldots + r^h + \ldots + r^{m-1} = 0, \tag{4}$$

we have the relations

$$1 + r_1 + \ldots + r_1^h + \ldots + r_1^{m-1} = 0,$$

$$\dots\dots\dots\dots\dots\dots$$

$$1+r_{m-1}+\dots+r_{m-1}^h+\dots+r_{m-1}^{m-1}=0.$$

Hence we have

$$\begin{vmatrix} 1 & r & \dots & r^h & \dots & r^{m-1} \\ 1 & r_1 & \dots & r_1^h & \dots & r_1^{m-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & r_{m-1} & \dots & r_{m-1}^h & \dots & r_{m-1}^{m-1} \end{vmatrix}=0.$$

This equation must be identical with (4). The coefficients of r^{m-1} is

$$\begin{aligned}\pm(r_1-r_2)(r_1-r_3)\dots(r_1-r_{m-1})&\\ (r_2-r_3)\dots(r_2-r_{m-1})&\\ \dots\dots\dots\dots&\\ (r_{m-2}-r_{m-1})&,\end{aligned}$$

That of r^h is nothing but $\pm\Delta_{h0}$. Hence, noticing that the coefficients of the equation (4) are all unity, we have

$$M=\pm 1.$$

Hence by (3) we have

$$C_{h0}=\frac{\Delta_{h0}}{\Delta}=\pm\frac{1}{m}, \qquad h=0, 1, \dots, m-1.$$

On the other hand, since the real parts of r's except $r_0=1$ are all less than unity, we have

$$\lim_{t\to\infty}\frac{e^{r_h t}}{e^t}=0, \qquad h=1, 2, \dots, n-1.$$

Therefore we have from (2)

$$\lim_{t\to\infty}\frac{E_h(t)}{e^t}=C_{h0}=\frac{1}{m}, \qquad h=0, 1, 2, \dots, m-1. \tag{5}$$

(This is clear, since we must take the + sign.)
Now since the sequence $S_h, S_{m+h}, \dots, S_{km+h}, \dots$ converges to $S^{(h)}$, we have

$$\lim_{t\to\infty}\frac{S_h\dfrac{t^h}{h!}+S_{m+h}\dfrac{t^{m+h}}{(m+h)!}+\dots+S_{km+h}\dfrac{t^{km+h}}{(km+h)!}+\dots}{E_h(t)}=S^{(h)},$$

$$h=0, 1, \dots, m-1.$$

Whence by aid of (5),

$$\lim_{t\to\infty} \frac{S_h \dfrac{t^h}{h\,!}+S_{m+h}\dfrac{t^{m+h}}{(m+h)\,!}+\ldots+S_{km+h}\dfrac{t^{km+h}}{(km+h)\,!}+\ldots}{e}$$

$$=\lim_{t\to\infty} S^{(h)}\frac{E_h(t)}{e^t}=\frac{S^{(h)}}{m}.$$

So we reach the result

$$\lim_{t\to\infty}\frac{S_0+S_1\dfrac{t}{1\,!}+\ldots+S_n\dfrac{t^n}{n\,!}+\ldots}{e^t}$$

$$=\frac{S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}}{m}. \qquad Q.\,E.\,D. \tag{6}$$

Applying this result to the analytic function defined by

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots$$

which as often said may be continued over $x=1$, *the value of the analytic function* $f(x)$ *at* $=1$ *is equal to the arithmetical mean of* $S^{(0)}+S^{(1)}+\cdots+S^{(m-1)}$, where

$$\lim_{k\to\infty} S_{km+h}=S^{(h)}, \qquad h=0,1,\ldots,m-1,$$

are all supposed to be finite and determinate.

Outside the circle of convergence $x>1$, these limits can not all be finite and the theorem is clearly untrue.

12. After the fourth conjecture has been proved, we may give a complete discussion according to the series

$$S=1-1+1-1+\ldots. \tag{1}$$

After Mr. Borel[1], Euler and others took $\frac{1}{2}$ as the value of the series. Now, by aid of § 11, $\frac{1}{2}$ the arithmetical mean is equal to the generalised sums. If we consider the function defined by the series.

$$f(x)=1-x+x^2-\ldots,$$

the arithmetical mean $\frac{1}{2}$ is equal to the value of Borel's integral sum at $x=1$, which is the value of the analytic function defined by

[1] Borel, *loc. cit.* pp. 4–7.

$f(x)$. That Leibnitz[1] gave $\frac{1}{2}$ as the value of the series from the *standpoints of the theory of probability*, corresponds by our side, to the equality (1) of the last article.

For Lagrange's serier[1].

$$f(x)=1+0+0-x^3+0+x^5+0+0-x^8+0+x^{10}+0+\dots,$$

using the same notations as before for $x=1$,

$$S_{5k+h}=1,\ 1,\ 1,\ 0,\ 0, \qquad \text{according as } h=0,\ 1,\ 2,\ 3,\ 4.$$

Hence by our theorem, $\frac{3}{5}$ is the value of the analytic function

$$f(x)=\frac{1-x^3}{1-x^5}=\frac{x^2+x+1}{x^4+x^3+x^2+x+1}$$

at $x=1$. To give another value does not rest upon any theoretical standpoint.

Returning to the series (1) the discovery of Borel's sum verifies the fact that Leibnitz's theory is not incorrect. But whether this is generally true or not, is not known. By our proof, *the gap between Leibnitz's theory and Borel's generalisation has been crossed.* At the same time, Pringsheim's objection looses any value as Mr. Borel has already declared.[2]

We shall add another example.

$$S=0+\sin\theta+\dots+\sin n\theta+\dots,$$

$$2S=\cot\frac{\theta}{2}-\frac{\cos\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}.$$

For simplicity put $\theta=\frac{\pi}{4}$, then

$$\begin{aligned} 2S_n&=0, && n\equiv 0,\ 7 \pmod{8},\\ &=\cot\frac{\pi}{8}-1, && \ \equiv 1,\ 6 \quad ,,\\ &=\cot\frac{\pi}{8}+1, && \ \equiv 2,\ 5 \quad ,, \end{aligned}$$

1 Borel, *loc. cit.*, p.p. 4–7.

2 Borel, *loc. cit.*, p. 7.

$$=2\cot\frac{\pi}{8}, \qquad \equiv 3,\ 4 \quad ,, \quad .$$

Hence the generalised sum is equal to $\frac{1}{2}\cot\frac{\pi}{8}$, which can easily be calculated by Borel's integral.[1]

13. Let us return to the discussion of singular points on the circle of convergence.

If the series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots,$$

may be continued over $x=1$, then as we have proved in § 8, $f(x)$, for $x>1$, (sometimes $x=1$ inclusive) must be indeterminate. Therefore by § 3, the number of variations of signs in $f(x)$ must be infinite. Consequently we have the following result:

Given a series of real coefficients

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots,$$

if between the coefficients

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1,$$

then the point $x=1$ must be a singular point of the analytic function defined by $f(x)$. If the radius of the convergence be unity, and if

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}>0,$$

the point $x=1$ is a singular point of the function. If $a_n=0$, suppose a_{n+1} to be the consecutive of a_{n-1}. This is a special case of Fabry's theorem.[2] A special case of his theorem is that given a power-series

$$F(x)=c_0+c_1z+\ldots+c_nz^n+\ldots,$$

if between the coefficients

$$\lim_{n\to\infty}\frac{c_n}{c_{n+1}}=1,$$

hen $z=1$ is a singular point of the analytic function defined by $F(z)$. But our result is wider than this special case. For consider the series

$$F(x)=(1+i)+(1-i)z+\ldots+(1+(-1)^ni)z^n+\ldots.$$

[1] Bromwich, *loc. cit.*, p. 275.

[2] Hadamard, *La Série de Taylor*, (1901) p. 25.

Then

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \pm i.$$

Yet

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1,$$

and $z=1$ is a singular point.

If, moreover,

$$|a_n - A| < \varepsilon^n,$$

where $\varepsilon < 1$, then at the point $x=1$, $f(x)$ becomes infinity. This point is a pole of the first order. For

$$\left| \sum_{n=m}^{m+h} a_n x^n - A \sum_{n=m}^{m+h} x^n \right| < \sum_{n=m}^{m+h} (\varepsilon x)^n,$$

in the vicinity of the point $x=1$. This implies the special case of Mr. Hadamard's theorem.[1]

From these considerations, we notice that several researches are intimately related to the *simple* property that the analytic element $F(z)$ must be indeterminate outside the circle of convergence to which it may be continued.

14. If in the preceding conditions, A be zero, *i.e.*, when

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1, \qquad |a_n| < \varepsilon_n,$$

where $\Sigma\varepsilon_n$ is convergent, then the point $x=1$ is not to be a pole and is either an *essential singularity* or a *branch-point* of the analytic function defined by $f(x)$. For by the second condition $f(1)$ is convergent. While by the first condition, the point must be a singular point. Examples are these:

$$1^\circ \qquad f(x) = 1 + \frac{x^2}{1^2} + \frac{x^{2^2}}{2^2} + \dots + \frac{x^{2^n}}{n^2} + \dots.$$

$$2^\circ \qquad f(x) = 1 - \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^2 - \dots - \frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\left(\frac{x}{2}\right)^n - \dots.$$

The first series has its circle of convergence as its natural limits.

[1] Hadamard, *loc. cit.*, p. 39.

[2] Hadamard, *loc. cit.*, p. 32.

For the second which is an element of $(1-x)^{\frac{1}{2}}$, $x=1$ is the branch-point. Both are convergent at $x=1$.

15. At the end we shall shortly consider some divergent double series.

Consider as usual the series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots.$$

Suppose it may be continued over $x=1$, then taking a point $x=a$ within the interval $(0, 1)$, the series

$$f(x|a)=f(a)+\frac{(x-a)}{1!}f'(a)+\ldots+\frac{(x-x)^n}{n!}f^{(n)}(a)+\ldots$$

is convergent for certain values of x. This may be stated as follows

The double-series, writing $x-a\equiv\xi$,

$$\begin{aligned} S= \;& a_0 \\ & +a_1a+a_1\xi \\ & +a_2a^2+2a_2a\xi+\xi^2 \\ & +\ldots\ldots\ldots \\ & +a_na^n+na_na^{n-1}\xi+\ldots+\xi^n \\ & +\ldots\ldots\ldots \end{aligned}$$

is convergent, if we sum up first by columns and then by rows. But if we sum up first by rows and then by columns, the sum is indeterminate. Pushing such considerations further, we may give several cases.