On the Solutions of Partial Differential Equations of the First Order at the Singular Points IV

By

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INTRODUCTION.

In continuing my study on the solutions of partial differential equations of the first order at the singular points, I have recently met with a valuable memoir by Mr. Dulac.[1] He has completed and developed, by his own methods, the researches of several mathematicians, on the system of the ordinary differential equations of the first order:

$$\frac{dx_1}{\xi_1}=\frac{dx_2}{\xi_2}=\dots=\frac{dx_n}{\xi_n},$$

where $\xi_1, \xi_2, \dots, \xi_n$ commence with terms of the first order in $x_1, x_2, \dots, x_n$. His memoir may be divided into three parts: 1°. The case where both of the Poincaré conditions[2] are satisfied; 2°. the case where the second of the conditions is valid; 3°. the case where neither condition is satisfied. For the second case he has devised a special transformation of the variables, somewhat resembling that which I used in my third paper.[3] In his memoir, sometimes such partial differential equations as

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1 Bull. soc. math. Fr., t. 40 (1912).

2 My first paper; these Memoirs, vol. II, no. 5 (1917), p. 257.

3 Third paper; these Memoirs. vol. III, no. 5 (1920).

$$Xf=\xi_1\frac{\partial f}{\partial x_1}+\xi_2\frac{\partial f}{\partial x_2}+\ldots+\xi_n\frac{\partial f}{\partial x_n}=\lambda_i f+\varphi$$

are employed and sometimes an auxiliary variable t. It is not his proper intention to discuss the partial differential equation $Xf=o$. I have treated in the first part of my first paper, the case where the characteristic equation

$$\begin{vmatrix} \lambda_{11}-\lambda & \lambda_{12} & \ldots\ldots \lambda_{1n} \\ \lambda_{21} & \lambda_{22}-\lambda & \ldots\ldots \lambda_{2n} \\ \ldots & \ldots & \ldots \\ \lambda_{n1} & \lambda_{n2} & \ldots\ldots \lambda_{nn}-\lambda \end{vmatrix}=o,$$

where

$$\xi_i=\lambda_{i1}x_1+\lambda_{i2}x_2+\ldots+\lambda_{in}x_n+\ldots,\quad i=1,\ 2,\ldots,n,$$

has the multiple roots $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ $(\nu<n)$ and they satisfy both of Poincaré's conditions. I have solved this case directly, considering upon Xf, not by aid of changes of the variables of the system of ordinary differential equations. But in my first paper, the case where the roots $\lambda_1, \lambda_2, \ldots, \lambda_\nu$, satisfy the second of Poincaré's conditions but not the first, is not discussed. This case is solved in the second part of Dulac's memoir. Since my considerations can easily be applied, I want here to discuss the case directly by my own method. After this discussion, I shall enter the discussion of the system of partial differential equations $Xf=o$, $Yf=o$ where $(XY)f\equiv o$.

1. We transform, as usual[1], the equation $Xf=o$ by a certain linear transformation of variables such that the determinant made by the coefficients of the first orders of $\xi_1, \xi_2, \ldots, \xi_n$ becomes

$$\begin{vmatrix} L_1 & O & \ldots\ldots O \\ O & L_2 & \ldots\ldots O \\ \ldots & \ldots & \ldots \\ O & O & \ldots\ldots L_\nu \end{vmatrix} \qquad (1)$$

where all the elements above the diagonal are zero and the diagonal elements of L_1 are the multiple root λ_1 of order n_1, those of L_2, the multiple root λ_2 of order $n_2, \ldots$, those of L_ν, the multiple root λ_ν of order n_ν. Moreover the multiple roots $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ may be arranged[2]

1 The first paper, *loc. cit.*, p. 268.

2 The second paper, these Memoirs, vol. IV, no. 3 (1919), p. 78.

in the increasing order of their real parts and the number of the relations such as

$$p_h\lambda_h+\ldots+p_i\lambda_i=\lambda_j, \quad 1\leq h\leq i<j, \tag{2}$$

are always finite. In the following we use the notations:

$$H\equiv n_1+n_2+\ldots+n_{h-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$I\equiv n_1+n_2+\ldots+n_{i-1},$$
$$J\equiv n_1+n_2+\ldots+n_{j-1},$$

and the like for other suffixes.

If λ_i can not be expressed linearly by $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}$ with some positive integral coefficients (zero inclusive), then as I have proved, we may obtain n_i holomorphic solutions about $(o, o, \ldots, o)$:

$$\begin{aligned} &f_{I+1}=x_{I+1}+\ldots, \\ &\ldots\ldots\ldots\ldots \\ &f_{I+n_i}=x_{I+n_i}+\ldots, \end{aligned} \tag{3}$$

the dotted parts standing for terms of higher orders, such that

$$Xf_{I+1}=\lambda_i f_{I+1},$$
$$\ldots\ldots\ldots\ldots$$
$$Xf_{I+n_i}=\lambda_{I+n_i1}f_{I+1}+\lambda_{I+n_i2}f_{I+2}+\ldots+\lambda_i f_{I+n_i},$$

where

$$\xi_{I+1}=\lambda_i\, x_{I+1}+\ldots,$$
$$\ldots\ldots\ldots\ldots$$
$$\xi_{I+n_i}=\lambda_{I+n_i1}\, x_{I+1}+\lambda_{I+n_i2}\, x_{I+2}+\ldots\lambda_i\, x_{I+n_i}+\ldots$$

But for the equation $Xf=\lambda_j f$, the same considerations are fatal. Suppose λ_j the first coefficient which satisfies some number of relations such as (2). Differentiate the equation

$$Xf=\lambda_j f, \tag{4}$$

p_h times with respect to $x_{H+1}, \ldots, p_i$ times with respect to x_{I+1} and put $x_1=x_2=\ldots=x_n=o$, then we obtain the equation

$$(p_h\lambda_h+\dots+p_i x_i)\ (\overset{p_h}{x}_{H+1}\dots\overset{p_i}{x}_{I+1})+\dots\dots$$

$$=\lambda_j(\overset{p_h}{x}_{H+1}\dots\overset{p_i}{x}_{I+1}).$$

The dotted part stands for a linear expression of quotients of lower stages and of lower orders. Since by the relation

$$p_h\lambda_h+\dots+p_i\lambda_i=\lambda_j,$$

we can not determine the quotient $(\overset{p_h}{x}_{H+1}\dots\overset{p_i}{x}_{I+1})$. But if we consider the equation

$$Xf=\lambda_j f+A\overset{p_h}{x}_{H+1}\dots\overset{p}{x}_{I+1},$$

where A is yet undetermined, we obtain the relation

$$(p_h\lambda_h+\dots+p_i\lambda_i)\ (\overset{p_h}{x}_{H+1}\dots\overset{p_i}{x}_{I+1})+\dots$$

$$=\lambda_j(\overset{p_h}{x}_{H+1}\dots\overset{p_i}{x}_{I+1})+Ap_h!\dots p_i!$$

The dotted part has the same meaning. Now we may take the indeterminate coefficient A such that the equation is valid.

Since $(\overset{p_h}{x}_{H+1}\dots\overset{p_i}{x}_{I+1})$ is arbitrary, we fix

$$(\overset{p_h}{x}_{H+1}\dots\overset{p_i}{x}_{I+1})=0.$$

2. Since the multiplicity of the root λ_h is n_h, such circumstances will also occur when we require the quotients

$$(\overset{q}{x}_{H+1}\overset{r}{x}_{H+2}\dots\overset{s}{x}_{H+n_h}\dots\overset{p_i}{x}_{I+1}),\quad q+r+\dots+s=p_h,$$

for the coefficient of x_{H+t} $(1\leq t\leq n_h)$ in ξ_{H+t}, the coefficient of $\dfrac{\partial f}{\partial x_{H+t}}$, is λ_h. This is the same for the remaining multiple roots e.g., for λ_i. Therefore we must add to the right hand side of (4), every possible term, each multiplied by an indeterminate coefficient, such that

$$A\overset{q}{x}_{H+1}\overset{r}{x}_{H+2}\dots\overset{s}{x}_{H+n_h}\dots\overset{q'}{x}_{I+1}\overset{r'}{x}_{I+2}\dots\overset{s'}{x}_{I+n_i},$$

$$q+r+\dots+s=p_1,\ \dots,\ q'+r'+\dots+s'=p_i.$$

Observing these, we consider therefore the equation

$$Xf=\lambda_j f+\sum A f_{H+1}^{q} f_{H+2}^{r} \dots f_{H+n_h}^{s} \dots f_{I+1}^{q'} f_{I+2}^{r'} \dots f_{I+n_i}^{s'}. \tag{5}$$

$\sum$ denotes the sum of all possible terms such that

$$(q+r+\dots+s)\lambda_h+\dots+(q'+r'+\dots+s')\lambda_i=\lambda_j. \tag{6}$$

The total number of such terms, as we have said, is finite. From quotients of the lower stages to the higher, we calculate successively by means of equation (5) and determine all the indeterminate coefficients A where we take all

$$(x_{H+1}^{q}\, x_{H+2}^{r} \dots x_{H+n_h}^{s} \dots x_{I+1}^{q'}\, x_{I+2}^{r'} \dots x_{I+n_i}^{s'})=0.$$

Thus we may calculate all the quotients of f which satisfy (5). As the initial condition we take as usual.

$$(x_1)=\dots=(x_J)=(x_{J+1})-1=(x_{J+2})=\dots=(x_n)=0,$$

and the solution will be

$$f=f_{J+1}=x_{J+1}+\dots,$$

the dotted part being terms of higher orders.

3. Next consider the equation

$$Xf=\lambda_j f+\lambda_{J+2,J+1} f_{J+1}+\sum A' f_{H+1}^{q} f_{H+2}^{r} \dots f_{H+n_h}^{s} \dots f_{I+1}^{q'} f_{I+2}^{r'} \dots f_{I+n_i}^{s'}$$

Quite in the same way, we obtain the solution

$$f=f_{J+2}=x_{J+2}+\dots,$$

the dotted part being terms of higher orders. We continue the process up to the solution

$$f=f_{J+n_j}=x_{J+n_j}+\dots$$

of the equation

$$Xf\equiv\lambda_j f+\lambda_{J+n_j,\,J+1} f_{J+1}+\lambda_{J+n_j,\,J+2} f_{J+2}+\dots+\sum A''\dots.$$

The proof of the convergencies of these solutions may be done quite in the same way as stated in the first paper. For we have only to add to θ[1] the *série majorante* of such series as

$$\sum A f_{H+1}^{q} f_{H+2}^{r} \cdots f_{H+n_h}^{s} \cdots f_{I+1}^{q'} f_{I+2}^{r'} \cdots f_{I+n_i}^{s'}$$

For the other multiple roots which may be expressed linearly by the others, we may treat quite in the same way and obtain n independent holomorphic solutions.

$$f_1 = x_1 + \dots,$$
$$\dots\dots\dots\dots,$$
$$f_n = x_n + \dots.$$

The dotted parts stand for terms of higher orders.
Now by the transformation

$$y_i = f_i, \quad i = 1, 2, \dots n,$$

Xf will become

$$Yf \equiv \eta_1 \frac{\partial f}{\partial y_1} + \eta_2 \frac{\partial f}{\partial y_2} + \dots + \eta_n \frac{\partial f}{\partial y_n},$$

where

$$(7)\begin{cases} \eta_1(y) = \lambda_1 y_1, \\ \eta_2(y) = \lambda_{21} y_1 + \lambda_1 y_2, \\ \dots\dots\dots\dots\dots\dots \\ \eta_{n_1}(y) = \lambda_{n_1 1} y_1 + \lambda_{n_1 2} y_2 + \dots + \lambda_1 y_{n_1}, \end{cases}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$(8)\begin{cases} \eta_{J+1}(y) = \lambda_j y_{J+1} + \sum A y_{H+1}^{q} y_{H+2}^{r} \cdots y_{H+n_h}^{s} \cdots y_{I+1}^{q'} y_{I+2}^{r'} \cdots y_{I+n_i}^{s'}, \\ \eta_{J+2}(y) = \lambda_{J+2\ J+1} y_{J+1} + \lambda_j y_{J+2} + \sum A' \dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \eta_{J+n_j}(y) = \lambda_{J+n_j\ J+1} y_{J+1} + \lambda_{J+n_j\ J+2} y_{J+2} + \dots + \lambda_j y_{J+n_j} + \sum A'' \dots, \end{cases}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

The coefficients not written here belong to the one or the other type and the functions under the sign $\sum$ are *polynomials*. We notice that

1 The first paper, *loc. cit.*, p. 276.

the polynomials in (8) are the same, except their coefficients A, A', $\ldots$, A''.

4. To solve the equation $Xf=o$ is thus reduced to solving the equation $Yf=o$. For this purpose consider, as it is classical, the system of the differential equations.

$$\frac{dy_1}{\eta_1}=\frac{dy_2}{\eta_2}=\ldots=\frac{dy_{n_1}}{\eta_{n_1}}=\ldots$$

$$=\frac{dy_{J+1}}{\eta_{J+1}}=\frac{dy_{J+2}}{\eta_{J+2}}=\ldots=\frac{dy_{J+n_j}}{\eta_{J+n_j}}=\ldots=dt.$$

First consider the system

$$\frac{dy_1}{\eta_1}=\frac{dy_2}{\eta_2}=\ldots=\frac{dy_{n_1}}{\eta_{n_1}}=dt.$$

From the nature of the functions $\eta_1, \eta_2, \ldots, \eta_{n_1}$, we have the solutions

$$\begin{aligned} &y_1=e^{\lambda_1 t} \\ &y_\mu=[C_\mu+poly\,(t, C_2, C_3, \ldots C_{\mu-1})]e^{\lambda_1 t}, \ \mu=2, \ldots, n_1, \end{aligned} \tag{9}$$

where C_2, C_3, $\ldots$, C_μ are constants. These processes are followed for all the multiple roots $\lambda_h, \ldots, \lambda_i$ and will obtain similar solutions. Next to solve the system

$$\frac{dy_{J+1}}{\eta_{J+1}}=\frac{dy_{J+2}}{\eta_{J+2}}=\ldots=\frac{dy_{J+n_j}}{\eta_{J+n_j}}=dt,$$

at first we notice that by the relations (5) and (6), putting the values of $y_{H+1}, \ldots, y_{I+n_i}$ such as (7), we have

$$y_{H+1}^{q}\,y_{H+2}^{r}\cdots y_{H+n_h}^{s}\cdots y_{I+1}^{q'}\,y_{I+2}^{r'}\cdots y_{I+n_i}^{s'}=poly\,(t, C_{H+2}, C_{H+2}, \ldots, C_{H+n_h}, \ldots,$$

$$C_{I+1}, C_{I+2}, \ldots, C_{I+n_i})e^{\lambda_j t}$$

Therefore the solutions are

$$y_{J+\mu}=[C_{J+\mu}+poly\,(t, C_{H+1},\ldots, C_{I+n_i};\, C_{J+1}, \ldots, C_{J+\mu-1})]e^{\lambda_j t}, \ \mu=1, 2, \ldots n_j.$$

Hence the solutions of the partial differential equation $Yf=o$ are :

$$y_{M+\mu}\, y_1^{-\frac{\lambda_1}{\lambda_m}} + poly\,(\log y_1, y_{M+1}\, y_1^{-\frac{\lambda_1}{\lambda_m}}, \ldots, y_{M+\mu-1}\, y_1^{-\frac{\lambda_1}{\lambda_m}}),$$
$$M=o, \ldots, H. \ldots, I, \ldots;\ \mu=1, 2, \ldots, n_m,$$

where for $M=o$, we commence μ by 2 and for the other case, we cancel the expression *poly*, provided $\mu=1$; and

$$y_{J+\mu}\, y_1^{-\frac{\lambda_1}{\lambda_j}}$$
$$+poly\,(\log y_1, y_{H+1}\, y_1^{-\frac{\lambda_1}{\lambda_h}}, \ldots, y_{I+n_i}\, y_1^{-\frac{\lambda_1}{\lambda_i}};\, y_{J+1} y_1^{-\frac{\lambda_1}{\lambda_j}}, \ldots, y_{J+\mu-1}\, y_1^{-\frac{\lambda_1}{\lambda_j}}),$$
$$J=J, \ldots;\ \mu=1, 2, \ldots, n_j.$$

When λ_l is not the first coefficient which may be expressed linearly, yet the solutions may be given by the latter forms. Thus *our proposition is fully solved* and we see that the transformation of Mr. Dulac is unnecessary. The forms of the solutions written above are allowed as the extensions of the forms of the solutions given in my first paper.[1]

5. We shall enter into the discussions of two simultaneous partial differential equations

$$Xf \equiv \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \ldots + \xi_n \frac{\partial f}{\partial x_n} = o,$$

$$Yf \equiv \eta_1 \frac{\partial f}{\partial x_1} + \eta_2 \frac{\partial f}{\partial x_2} + \ldots + \eta_n \frac{\partial f}{\partial x_n} = o,$$

where Xf, Yf, are permutable to each other, i.e., $(XY)f \equiv o$ and all the coefficients $\xi_1, \ldots, \eta_n$ commence by terms of the first orders ln x_1, $x_2, \ldots, x_n$. Since Xf and Yf are permutable, the matrices formed by the coefficients of the first orders in the coefficients $\xi_1, \ldots, \eta_n$ are permutable and hence they may be transformed to become normal simultaneously, by a certain transformation of the variables.[2] Therefore we suppose from the beginning:

$$Xf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \ldots + \xi_n \frac{\partial f}{\partial x_n}$$

1 First paper, *loc. cit.*, p. 279.

2 First paper, *loc. cit.*, p. 280.

where using the same notations,

$$\left.\begin{aligned}
&\xi_{M+1}=\lambda_m\, x_{M+1},\\
&\xi_{M+2}=\lambda_{M+2\ M+1}\, x_{M+1}+\lambda_m\, x_{M+2},\\
&\dots\dots\dots\dots\dots\dots\\
&\xi_{M+n_m}=\lambda_{M+n_m\ M+1}\, x_{M+1}+\lambda_{M+n_m\ M+2}\, x_{M+2}+\dots+\lambda_m\, x_{M+n_m},
\end{aligned}\right\}\qquad(10)$$

provided λ_m can not be expressed linearly by the others, and

$$\left.\begin{aligned}
&\xi_{J+1}=\lambda_j\, x_{J+1}+\sum C\, x_{H+1}^{p_h'}\dots x_{H+n_h}^{p_h''}\dots x_{I+1}^{p_i'}\dots x_{I+n_i}^{p_i''},\\
&\xi_{J+2}=\lambda_{J+2\ J+1}\, x_{J+1}+\lambda_j\, x_{J+2}+\sum C'\dots,\\
&\dots\dots\dots\dots\dots\dots\\
&\xi_{J+n_j}=\lambda_{J+n_j\ J+1}\, x_{J+1}+\lambda_{J+n_j\ J+2}\, x_{J+2}+\dots+\lambda_j\, x_{J+n_j}+\sum C''\dots,
\end{aligned}\right\}\qquad(11)$$

provided λ_j satisfies the relations (2). $\sum$ denotes the same polynomial, with the exception of the coefficients $C, C',\dots, C'',\dots$. By our transformation Yf does not change its form. So we have

$$Yf=\eta_1\frac{\partial f}{\partial x_1}+\eta_2\frac{\partial f}{\partial x_2}+\dots+\eta_n\frac{\partial f}{\partial x_n},$$

$$\eta_l=\mu_{l1}\, x_1+\mu_{l2}\, x_2+\dots+\mu_{ll}\, x_l+\dots,\quad l=1, 2,\dots, n. \qquad (12)$$

The dotted part stands for terms of higher orders. We shall determine the forms of the coefficients of Yf more closely.

6. Since Xf and Yf are permutable, corresponding to the root λ_m we have the relations

$$X\eta_{M+1}=\lambda_m\, \eta_{M+1},$$

$$X\eta_{M+2}=\lambda_m\, \eta_{M+2}+\lambda_{M+2\ M+1}\, \eta_{M+1},$$

$$\dots\dots\dots\dots\dots\dots$$

$$X\eta_{M+n_m}=\lambda_m\, \eta_{M+n_m}+\lambda_{M+n_m\ M+1}\, \eta_{M+1}+\dots+\lambda_{M+n_m\ M+n_m-1}\, \eta_{M+n_m-1}.$$

First we consider the equation

$$X\eta_{M+1}=\lambda_m\, \eta_{M+1}.$$

The variables belonging to λ_a, i.e., the variables which have λ_a as their coefficients of the first orders in $\xi_{A+1}, \xi_{A+2}, \dots, \xi_{A+n_a}$ respectively, are $x_{A+1}, x_{A+2}, \dots, x_{A+n_a}$. In the other coefficients, λ_a does not occur. Therefore between the quotients of the first orders, we have the relations

$$\lambda_a (x_{A+n_a}) = \lambda_m (x_{A+n_a}),$$

$$\lambda_a (x_{A+r}) + \dots = (\lambda_m x_{A+r}), \qquad 1 \leq r < n_a.$$

The dotted part stands for a linear form of quotients of lower stages. But if we notice the form of the determinant (1), the quotients contained in the form are only

$$(x_{A+1}), (x_{A+2}), \dots, (x_{A+r-1}).$$

Since $\lambda_a \neq \lambda_m$, by the mathematical induction, we have

$$(x_{A+1}) = (x_{A+2}) = \dots = (x_{A+n_a}) = 0.$$

Since by (12) the terms of first orders of η_{M+1} is a linear conbination of x_{M+1} and the other variables whose suffixes are less than $M+1$, we conclude that

$$\eta_{M+1} = \mu x_{M+1} + \dots,$$

the dotted part being terms of higher orders.

Next consider the quotients of the second orders. They are all zero. For put

$$H \equiv \eta_{M+1} - \mu x_{M+1},$$

then H commences at least by terms of second orders. Hence we have the relations

$$2\lambda_{\nu} (x_n^2) = \lambda_m (x_n^2), \qquad 1 \leq r \leq n_a,$$

$$(\lambda_a + \lambda_b)(x_{A+r} x_{B+s}) + \dots = \lambda_m (x_{A+r} x_{B+s}), \qquad 1 \leq s \leq n_b.$$

The dotted part stands for a linear form of quotients of lower stages. It contains no quotients of the first order. Hence all the quotients of second orders are zero. Therefore by the mathematical induction, we may conclude that all the quotients of higher orders are zero, hence we have

$$\eta_{M+1} = \mu x_{M+1}.$$

7. We consider next η_{M+t}, $(2 \leqq t \leqq n_m)$. Assume that

$$\eta_{M+t'} = \mu_1 x_{M+1} + \mu_2 x_{M+2} + \ldots + \mu_{t'} x_{M+t'}, \qquad 1 < t' < t. \tag{13}$$

For $t' = 1$, it is right. After the substitutions of the values of $\eta_{M+1}, \ldots, \eta_{M+t-1}$, in the equation

$$X\eta_{M+t} = \lambda_m \eta_{M+t} + \lambda_{M+t\ M+1} \eta_{M+1} + \ldots + \lambda_{M+t\ M+t-1} \eta_{M+t-1},$$

we have

$$X\eta_{M+t} = \lambda_m \eta_{M+t} + c_1 x_{M+1} + \ldots + c_{t-1} x_{M+t-1}, \tag{14}$$

where $c_1, \ldots, c_{t-1}$ are certain constants. Now we prove that η_{M+t} has the same form. First, it does not contain x_{A+r}, $(A < M)$. Since as before, after differentiations, we have

$$\lambda_a (x_{A+n_a}) = \lambda_m (x_{A+n_a}),$$

$$\lambda_a (x_{A+r}) + \ldots = \lambda_m (x_{A+r}), \qquad 1 \leqq r < n_a,$$

the dotted part being quotients of lower stages. Hence we are right. Secondly, since η_{M+t} has none of $x_{M+t+1}, \ldots, x_n$, in the first orders, it has except the terms of higher orders the same form with what is assumed. Thirdly, put

$$H \equiv \eta_{M+t} - \textit{terms of first orders of } \eta_{M+t}.$$

Then we have, by (14),

$$XH = \lambda_m H + c'_1 x_{M+1} + \ldots + c'_t x_{M+t},$$

where $c'_1, \ldots, c'$ are certain constants. Now all the quotients of the first order of H are zero. Therefore noticing that the quotients of the second order satisfy the relations

$$2\lambda_v (x_n^2) = \lambda_m (x_n^2), \qquad 1 \leqq r \leqq n_a,$$

$$(\lambda_a + \lambda_b)(x_{A+r} x_{B+s}) + \ldots = \lambda_m (x_{A+r} x_{B+s}), \qquad 1 \leqq s \leqq n_b,$$

the dotted part being quotients of lower orders, we way conclude that all the quotients of the second orders are zero. Continuing this consideration, by mathematical induction, we may prove that all the quotients of the higher orders are zero. Hence

$$H \equiv 0,$$

i.e., η_{M+t} has the same form as (13). Therefore, all the coefficients $\eta_{M+1}, \eta_{M+2}, \ldots, \eta_{M+n_m}$ have the same forms as $\xi_{M+1}, \xi_{M+2}, \ldots, \xi_{M+n_m}$ respectively.

8. Proceding further, consider the case of the multiple root λ_j which satisfies the relation (6) and the coefficients corresponding to it are given by (11), where

$$\begin{array}{ll} p_h' + \ldots + p_h'' = p_h, & o \leqq p_h' \leqq p_h, \\ & \ldots\ldots\ldots\ldots \\ & o \leqq p_h'' \leqq p_h, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ p_i' + \ldots + p_i'' = p_i, & o \leqq p_i' \leqq p_i, \\ & \ldots\ldots\ldots\ldots \\ & o \leqq p_i'' \leqq p_i. \end{array}$$

We assume as before $\lambda_h, \ldots, \lambda_i$ can not be represented by the others. $\eta_{J+1}, \eta_{J+2}, \ldots, \eta_{J+n_j}$ must satisfy the relations

$$\left.\begin{array}{l} X\eta_{J+1} = \lambda_j \eta_{J+1} + \sum', \\ X\eta_{J+2} = \lambda_j \eta_{J+2} + \lambda_{J+2\; J+1} \eta_{J+1} + \sum', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X\eta_{J+n_j} = \lambda_j \eta_{J+n_j} + \lambda_{J+n_j\; J+1} \eta_{J+1} + \ldots + \lambda_{J+n_j\; J+n_j-1} \eta_{J+n_j-1} + \sum'. \end{array}\right\} \quad (15)$$

The $\sum'$ in the first equation is

$$\begin{aligned} \sum' \equiv & \sum CY(x_{H+1}^{p_h'} \ldots x_{H+n_h}^{p_h''} \ldots x_{I+1}^{p_i'} \ldots x_{I+n_i}^{p_i''}) \\ = & \sum C \left\{ p_h' \eta_{H+1} x_{H+1}^{p_h'-1} \ldots x_{H+n_h}^{p_h''} \ldots x_{I+1}^{p_i'} \ldots x_{I+n_i}^{p_i''} \right. \\ & \left. + \ldots + p_i'' \eta_{I+n_i} x_{H+1}^{p_h'} \ldots x_{H+n_h}^{p_h''} \ldots x_{I+1}^{p_i'} \ldots x_{I+n_i}^{p_i''-1} \right\}, \end{aligned} \quad (16)$$

and the like expressions for the other $\sum'$. In the following, we shall use for products such as $x_r^{p_r} \ldots x_s^{p_s} \ldots x_t^{p_t}$ the same definiton of *stages*[1].

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[1] The first paper, *loc. cit.*, p. 270.

Then from the forms of $\eta_{H+1}, \ldots, \eta_{I+n_i}$, all the products under the sign $\sum'$ are respectively of same or higher stages than those under the sign $\sum$ of (11). All the products are constructed by the variables $x_{H+1}, \ldots, x_{I+n_i}$. For example, the first equation of (15) is the consequence of

$$X\eta_{J+1} \equiv Y\xi_{J+1}.$$

Since by the proof of the preceding sections, we have

$$\begin{aligned} \eta_{H+1} &= \mu\, x_{H+1}, \\ &\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots, \\ \eta_{I+n_i} &= \mu_1\, x_{I+1} + \ldots + \mu_{n_i}\, x_{I+n_i}, \end{aligned}$$

the calculation (16) of $\sum'$ teaches us that our statement is right.

9. After these considerations, we discuss the forms of $\eta_{J+1}, \ldots, \eta_{J+n_j}$ which satisfy the equations (15). Since the polynomials of orders higher than the first in the coefficients $\xi's$ of Xf can have effect only upon the terms of higher orders of the coefficients $\eta's$ of Yf, the terms of the first order in $\eta_{J+1}, \eta_{J+2}, \ldots, \eta_{J+n_j}$ have the same forms as (13), namely

$$\eta_{J+t} = \rho_1\, x_{J+1} + \rho_2\, x_{J+2} + \ldots + \rho_t\, x_{J+t} + \ldots, \qquad 1 \leq t \leq n_j,$$

$\rho_1\ \rho_2 \ldots, \rho_t$ are certain constants and the dotted part stands for terms of higher orders. Specially we have

$$\eta_{J+1} = \rho\, x_{J+1} + \ldots,$$

all the quotients of the first order, except (x_{J+1}), being zero. First consider the equation

$$X\eta_{J+1} = \lambda_j\, \eta_{J+1} + \sum{}'. \tag{17}$$

1°. When at least one of A and B is different from any of $H, \ldots, I$, then we have

$$(\lambda_a + \lambda_b)\,(x_{A+n_a}\, x_{B+n_b}) = \lambda_m\,(x_{A+n_a}\, x_{B+n_b}).$$

For no coefficients except ξ_{A+n_a}, ξ_{B+n_b} have the variables x_{A+n_a}, x_{B+n_b}

in their first orders and moreover the coefficient ξ_{J+1} of $\frac{\partial f}{\partial x_{J+1}}$ lacks at least one of these variables. Hence the quotient $(x_{A+n_a}\, x_{B+n_b})$ is zero. 2°. Moreover we have

$$(\lambda_a+\lambda_b)\,(x_{A+r}\,x_{B+s})+\ldots=\lambda_j\,(x_{A+r}\,x_{B+s}),$$

the dotted part being quotients of lower stages. For in the expressions of the first order, x_{A+r} appears only in $\xi_{A+r}, \ldots, \xi_{A+n_a}$ and x_{B+s} only in $\xi_{B+s}, \ldots, \xi_{B+n_b}$, and the quotients in the dotted part are of the forms

$$(x_{A+r'}\,x_{B+s'}), \qquad \begin{array}{l} r<r'\leqq n_a, \\ s<s'\leqq n_b. \end{array}$$

Therefore by mathematical induction all the quotients of the second orders $(x_{A+r}\ x_{B+s})$ are zero where at least one of A and B is different from any one of $H, \ldots, I$. 3°. We may go further and conclude that all the quotients

$$(\overset{u}{x}_{A+n_a}\,\overset{v}{x}_{B+n_b}\ldots\overset{w}{x}_{C+n_c})$$

of order $u+v+\ldots+w$ are zero, provided at least one of $A, B, \ldots, C$ is different from any one of $H, \ldots, I$. Moreover we have

$$(u\lambda_a+v\lambda_b+\ldots+w\lambda_c)\,(\overset{u}{x}_{A+r}\,\overset{v}{x}_{B+s}\ldots\overset{w}{x}_{C+t})+\ldots$$
$$=\lambda_j\,(\overset{u}{x}_{A+r}\,\overset{v}{x}_{B+s}\ldots\overset{w}{x}_{C+t}),$$

the dotted part being quotients of lower stages. The considerations are quite the same, ... Thus applying the mathematical induction again and again, we conclude that all the quotients

$$(\overset{u}{x}_{A+r}\,\overset{v}{x}_{B+s}\ldots\overset{w}{x}_{C+t})=0,$$

where at least one of $A, B, \ldots, C$ is different from $H, \ldots, I$.

10. Let us put

$$H\equiv\eta_{J+1}-\rho x_{J+1}-\sum{}'',$$

where $\sum''$ is the polynomial in η_{J+1} constituting the terms of orders not higher than any of those of $\sum'$. Substituting this in (17), we have

$$XH \equiv \lambda_j H + \lambda_j \sum{}'' + \sum{}' - \rho \sum - X \sum{}''.$$

Since $\sum''$ contains only the variables $x_{H+1}, \ldots, x_{H+n_h}, \ldots, x_{I+1}, \ldots, x_{I+n}$ and by the properties of the coefficients of Xf, $X\sum''$ is of the same order as $\sum''$. Since the order of the terms of the lowest order of H and hence of XH is higher than any of $\sum$, $\sum'$, $\sum''$, we must have

$$XH \equiv \lambda_j H,$$

whence we may easily conclude that $H \equiv o$ and therefore we have

$$\eta_{J+1} = \rho x_{J+1} + \sum{}'',$$

$\sum''$ being of the same form as $\sum$ and $\sum'$.

11. Let us proceed to the discussion of η_{J+2}. Consider the second equation of (15)

$$X\eta_{J+2} = \lambda_j \eta_{J+2} + \lambda_{J+2\ J+1} \eta_{J+1} + \sum{}'.$$

Putting the value of η_{J+1} we have

$$X\eta_{J+2} = \lambda_j \eta_{J+2} + \rho \lambda_{J+2\ J+1} x_{J+1} + \sum{}''',$$

where $\sum'''$ is a similar polynomial as $\sum'$, $\sum''$. As we have said

$$\eta_{J+2} = \rho_1 x_{J+1} + \rho_2 x_{J+2} + \ldots,$$

the dotted part being terms of higher orders. To treat in the same way, put

$$\eta'_{J+2} \equiv \eta_{J+2} - \rho_1 x_{J+1}, - \rho_2 x_{J+2}$$

then we have

$$X\eta'_{J+2} = \lambda_j \eta'_{J+2} + \sum{}_1''',$$

where by the forms of ξ_{J+1}, ξ_{J+2}, $\sum_1'''$ has the same meaning as $\sum'''$. Now we have only to repeat the processes carried out in the preceding sections and the same result must follow. Continuing these considerations, we conclude that

$$\eta_{J+1}=\rho\, x_{J+1}+\sum\nolimits_1,$$

$$\eta_{J+2}=\rho_1\, x_{J+1}+\rho_2\, x_{J+2}+\sum\nolimits_2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\eta_{J+n_j}=\rho_1'\, x_{J+1}+\rho_2'\, x_{J+2}+\dots+\rho_{n_j}\, x_{J+n_j}+\sum\nolimits_{n_j},$$

where $\rho,\ \rho_1, \dots, \rho_{n_j}$ are certain constants and all the $\sum$ indicate certain polynomials of the variables $x_{H+1}, \dots, x_{H+n_h}, \dots, x_{I+1}, \dots, x_{I+n_i}$ and of the same order as $\sum$.

12. Proceeding further, consider the case for a multiple root λ_l which is expressible by the other, some number of which are already expressible by the other. Suppose for simplicity we have such relations as

$$p_g\, \lambda_g+\dots+p_j\, \lambda_j+\dots+p_k\, \lambda_k=\lambda_l,$$

where $\lambda_g, \dots, \lambda_k$ except λ_j are not expressible by the other. For λ_j we have as before the relations such as

$$p_h\, \lambda_h+\dots+p_i\, \lambda_i=\lambda_j.$$

By these relations we see easily that in the coefficient ξ_{L+1} there are such products as

$$x_{G+1}^{p_g'}\cdots x_{G+n_g}^{p_g''}\cdots\{x_{H+1}^{p_h'}\cdots x_{H+n_h}^{p_h''}\cdots x_{I+1}^{p_i'}\cdots x_{I+n_i}^{p_i''}\}^{p_j}\cdots x_{K+1}^{p_k'}\cdots x_{K+n_k}^{p_k''}, \quad (18)$$

$$x_{G+1}^{p_g'}\cdots x_{G+n_g}^{p_g''}\cdots\{x_{H+1}^{p_h'}\cdots x_{H+n_h}^{p_h''}\cdots x_{I+1}^{p_i'}\cdots x_{I+n_i}^{p_i''}\}^{p_{j-m}}\ x_{J+1}^{m}\cdots x_{K+1}^{p_k'}\cdots x_{K+n_k}^{p_k''}, \quad (19)$$

$$p_t'+\dots+p_t''=p_t,$$

where m is a positive integer. If $m=o$ (19) becomes (18). Therefore if we calculate the relation

$$X\eta_{L+1}=Y\xi_{L+1},$$

we have

$$X\eta_{L+1}=\lambda_l\, \eta_{L+1}+S_1+S_2,$$

where S_1 is obtained by putting such products as (18) in Yf and S_2, by putting such products as (19). If we put the product (18) in Yf, the result is a polynomial of the same order as (18). For all the coefficients which are introduced into the polynomial are of the first order. But if we put the product (19) in Yf, then, since the coefficients $\eta_{J+1}, \ldots, \eta_{J+n_j}$, as we have proved, contain such products as

$$x_{H+1}^{p_h'} \ldots x_{H+n_h}^{p_h''} \ldots x_{I+1}^{p_i'} \ldots x_{I+n_i}^{p_i''},$$

the resulting polynomial is at most of order

$$p_g + \ldots + (p_h + \ldots + p_i)(p_j - m) + (m-1) + (p_h + \ldots + p_i) + \ldots + p_k$$
$$= p_g + \ldots + (p_h + \ldots + p_i)(p_j - m + 1) + (m-1) + \ldots + p_k.$$

Thus the polynomial $S_1 + S_2$ is of the same form as the polynomial in the coefficient ξ_{L+1}; and the like for the others.

Moreover, to calculate the coefficients η_{L+t}, we may, as we have done many times, conclude that all of their quotients which imply some variables other than $x_{L+1}, \ldots, x_{L+n_l}$ are zero. Thus we may conclude that all the coefficients $\eta_{L+1}, \ldots, \eta_{L+n_l}$, have the same forms as the cocfficients $\xi_{L+1}, \ldots, \xi_{L+n_l}$, respectively.

13. We remark that for any suffix, all the coefficients corresponding to $\xi_{M+1}, \xi_{M+2}, \ldots, \xi_{M+n_m}$, with respect to their terms of the first order, have the forms:

$$\eta_{M+1} = \mu_m x_{M+1} + \ldots,$$
$$\eta_{M+2} = \mu_{M+2\ M+1} x_{M+1} + \mu_m x_{M+2} + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\eta_{M+n_m} = \mu_{M+n_m\ M+1} x_{M+1} + \ldots + \mu_m x_{M+n_m} + \ldots$$

The multiple roots $\mu_1, \ldots, \mu_\nu$ are different with the exception of some special cases. Concise knowledge about these can not be obtained from the idea of the permutability of two matrices. We may conclude these results above obtained as follows:—*When the infinitesimal transformations Xf and Yf are permutable with each other, where their coefficients commence by terms of the first order and the characteristic equation made by the coefficients of the first order in those of Xf has multiple roots which satisfy the second of Poincaré's conditions, then the other iufinitesimal transformation Yf has exactly the same form as the infinitesimal transformation Xf.* The special case of this was proved in my first paper. We shall not here discuss the forms of the common solutions of the simultaneous partial differential equations, for, after the reductions have been made the next task is easy. In the following, a simple example, to explain §12 is given.

12. Consider the transformations

$$Xf \equiv x\frac{\partial f}{\partial x} + (2y + x^2)\frac{\partial f}{\partial y} + (3z + 2xy + x^3)\frac{\partial f}{\partial z},$$

$$Yf \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z},$$

where $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, so that $2\lambda_1 = \lambda_2$, $3\lambda_1 = \lambda_3$, $\lambda_1 + \lambda_2 = \lambda_3$. For the permutability we must have

1°. $X\xi = \xi,$

2°. $X\eta = 2\eta + 2x\xi,$

3°. $X\zeta = 3\zeta + 2(\xi y + \eta x) + 3x^2\xi$

From 1°. we obtain

$$\xi = \lambda x,$$

where λ is arbitrary. Substituting in 2°. we have from the equation

$$X\eta = 2\eta + 2\lambda x^2,$$

that

$$\eta = 2\lambda y + \nu x^2,$$

where ν is arbitrary. Substituting in the third equation, we have

$$X\zeta = 3\zeta + 6\lambda xy + (2\nu + 3\lambda)x^3,$$

from which we have

$$\zeta = 3\lambda z + 2\lambda xy + \omega x^3,$$

where ω is arbitrary.

Therefore the required expression for Yf is

$$Yf = \lambda x\frac{\partial f}{\partial x} + (2\lambda y + \nu x^2)\frac{\partial f}{\partial y} + (3\lambda z + 2\nu xy + \omega x^3)\frac{\partial f}{\partial z},$$

λ, ν, ω being arbitrary.

Thus we notice the coefficients have the same forms as those of Xf.

The solutions of the equation $Xf=o$ are

$$f_2 \equiv yx^{-2} - \log x, \quad f_3 \equiv zx^{-3} - (2yx^{-2}+1)\log x + (\log x)^2.$$

The common solution of the simultaneous equations is the arbitrary function of $f_3 - f_2^2 - \frac{\omega-\lambda}{\nu-\lambda} f_2$, provided $\nu \neq \lambda$.

Two permutable linear homogeneous transformations have at least a common pole. What is the case for Lie's group? This is the motive of my study. About ordinary points, the study is clear. So we have to study the singular points. By these successive papers, I think, the properties became a little clearer.