Congruence of Circles in Non-Euclidean Space.

BY

Hidetoshi Kashiwagi.

(Received September 10, 1921.)

CHAPTER I.

CONGRUENCE OF CIRCLES.

§ 1. COORDINATES OF A POINT ON A CIRCLE.

Any circle in non-euclidean space may be given by the simultaneous equations

$$(1) \qquad (ax)^2 = \cos\frac{R}{k}(xx)(aa),$$

$$(bx) = 0,$$

where

$$(aa) = k^2, \ (bb) = k^2, \ (ab) = 0,$$

and (a) is the coordinates of the center of the main sphere,[1] (b) that of the plane of the circle, R the radius of the main sphere and $\frac{1}{k^2}$ the measure of curvature of the space. The center of the circle coincides with the center of the main sphere.

(1), See T. Nishiuchi, 'Oriented Circles in Non-Euclidean Space,' these Memoirs, Vol. iv, no. 6, p. 273...... .

When a_0, a_1, a_2, a_3; b_0, b_1, b_2, b_3; R are functions of two parameters u and v, we have a congruence of circles, by allowing unlimited variations to the parameters.

Let (a') be the coordinates of a point C' on the plane of the circle which is orthogonal to the center C of the circle and $(a'a')=k_t^2$, then

$$(aa')=0, \quad (a'b)=0,$$

and the coordinates of any point on the circle may be given by the equations

$$(2) \qquad \begin{aligned} x_0 &= a_0 \cos\frac{R}{k} + a_0' \sin\frac{R}{k}, \\ x_1 &= a_1 \cos\frac{R}{k} + a_1' \sin\frac{R}{k}, \\ x_2 &= a_2 \cos\frac{R}{k} + a_2' \sin\frac{R}{k}, \\ x_3 &= a_3 \cos\frac{R}{k} + a_3' \sin\frac{R}{k}. \end{aligned}$$

Next, consider an absolute polar triangle in the plane of a circle of which one of the vertices is the center C of the circle, and the remaining two vertices A and B, then the point C', A and B must lie on the plane axis[1] of the circle whose axial coordinates are

$$\pi_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, \quad (i, j=0, 1, 2, 3; \ i \neq j).$$

Let (ξ) and (η) be the coordinates of the vertices A and B respectively, and θ the angle between the two lines CA and CC', then

$$(3) \qquad a'_i = \xi_i \cos\theta + \eta_i \sin\theta, \quad (i=0, 1, 2, 3)$$

and

$$\lambda b_i = \frac{\partial}{\partial r_i} \begin{vmatrix} r_0 & r_1 & r_2 & r_3 \\ a_0 & a_1 & a_2 & a_3 \\ \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 \end{vmatrix},$$

(1) See T. Nishiuchi, 'Oriented circles in Non-Euclidean Space,' loc. cit.

$$(\xi\xi)=(\eta\eta)=k^2,$$

$$(a\xi)=(a\eta)=(\xi\eta)=0,$$

where λ is a proportional factor.

Then, the coordinates of a point on the circle given by (2) will be

$$(4)\quad \begin{aligned} x_0&=a_0\cos\frac{R}{k}+(\xi_0\cos\theta+\eta_0\sin\theta)\sin\frac{R}{k},\\ x_1&=a_1\cos\frac{R}{k}+(\xi_1\cos\theta+\eta_1\sin\theta)\sin\frac{R}{k},\\ x_2&=a_2\cos\frac{R}{k}+(\xi_2\cos\theta+\eta_2\sin\theta)\sin\frac{R}{k},\\ x_3&=a_3\cos\frac{R}{k}+(\xi_3\cos\theta+\eta_3\sin\theta)\sin\frac{R}{k}, \end{aligned}$$

where

$$(aa)=(\xi\xi)=(\eta\eta)=k^2,$$

$$(a\xi)=(a\eta)=(\xi\eta)=0.$$

§ 2. Focal Point.

Consider a congruence of circles defined by the equations

$$(5)\qquad x_i=a_i\cos\frac{R}{k}+(\xi_i\cos\theta+\eta_i\sin\theta)\sin\frac{R}{k},$$

$$(i=0,\ 1,\ 2,\ 3),$$

where a_i, ξ_i, η_i $(i=0, 1, 2, 3)$, R are analytic functions of the two parameters u and v, and θ is a variable along the circle.

The parameters u and v determine the circle and θ a point upon the circle.

If we establish a relation between u and v, say

$$v=f(u)$$

the circles of the congruence whose parameters satisfy this relation depend upon the single parameter u and consequently form a surface; the parametric equations of this surface are given by (5), when v has been replaced by $f(u)$. The surface will evidently be changed according as the function $f(u)$ changes, and are called the *surface of the cougruence* by Prof. Darboux.[1]

The equation of the tangent plane to the surface of the congruence is

$$(6)\quad \begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ x_0 & x_1 & x_2 & x_3 \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_3}{\partial \theta} \\ \frac{\partial x_0}{\partial u}+\frac{\partial x_0}{\partial v}f'(u) & \frac{\partial x_1}{\partial u}+\frac{\partial x_1}{\partial v}f'(u) & \frac{\partial x_2}{\partial u}+\frac{\partial x_2}{\partial v}f'(u) & \frac{\partial x_3}{\partial u}+\frac{\partial x_3}{\partial v}f'(u) \end{vmatrix} = 0,$$

where X_i ($i=0, 1, 2, 3$) are current coordinates and the accent denotes differentiation. And we easily have the following theorem of Darboux:[2]

Theorem. For any four surfaces of a congruence containing the same circle of the congruence, the anharmonic ratio of the tangent planes to these surfaces at a point is constant when the point moves along the circle.

From (6) it is seen that when (x) satisfies the condition

$$(7)\quad \frac{\begin{vmatrix} x_0 & x_1 & x_2 \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} x_0 & x_2 & x_3 \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_3}{\partial \theta} \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \end{vmatrix}}{\begin{vmatrix} x_0 & x_2 & x_3 \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_3}{\partial \theta} \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix}} = \ldots\ldots C \text{ (const.)},$$

the equation of the tangent plane is independent of the function $f(u)$ and consequently is the same for all surfaces of the congruence through

(1), (2) See Darboux, Leçons sur la théorie générale des surfaces, Vol. ii.

the corresponding point. These points are called the *focal point* and their locus the *focal surface* by Prof. Darboux.[1]

The equation (7) may be represented by

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_3}{\partial \theta} \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix} = 0,$$

from which we have the relation

$$\frac{\partial x_i}{\partial \theta} = \lambda x_i + \mu \frac{\partial x_i}{\partial u} + \nu \frac{\partial x_i}{\partial v}, \quad (i=0,\ 1,\ 2,\ 3),$$

where λ, μ and ν are a parameter respectively.

Multiplying through by x_i and adding, we get

$$\lambda = 0,$$

and have the following relation

$$(8) \qquad \frac{\partial x_i}{\partial \theta} = \mu \frac{\partial x_i}{\partial u} + \nu \frac{\partial x_i}{\partial v}, \quad (i=0,\ 1,\ 2,\ 3).$$

Again, multiplying through by a_i and adding, we get

$$(9) \qquad \mu\left[\left(a\frac{\partial a'}{\partial u}\right) - k\frac{\partial R}{\partial u}\right] + \nu\left[\left(a\frac{\partial a'}{\partial v}\right) - k\frac{\partial R}{\partial v}\right] = 0,$$

and the relation (8) will be

$$(10) \qquad k^2 \sin\frac{R}{k}\,\frac{\partial a_i'}{\partial \theta} = \mu\left[k^2\left(\frac{\partial a_i}{\partial u}\cos\frac{R}{k} + \frac{\partial a_i'}{\partial u}\sin\frac{R}{k}\right)\right.$$

$$\left. - \left(a_i \sin\frac{R}{k} - a_i' \cos\frac{R}{k}\right)\left(a\frac{\partial a'}{\partial u}\right)\right] +$$

(1) See Darboux, Leçons sur la théorie générale des surfaces, Vol. ii.

$$+\nu\left[k^2\left(\frac{\partial a_i}{\partial v}\cos\frac{R}{k}+\frac{\partial a_i'}{\partial v}\sin\frac{R}{k}\right)-\left(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\right)\left(a\frac{\partial a'}{\partial v}\right)\right].$$

$$(i=0,\ 1,\ 2,\ 3).$$

Here we adopt the following notation in analogy to Kummer's classical method[1]

$$k^2(dada)-(a'da)=\begin{vmatrix} a_0' & a_1' & a_2' & a_3' \\ da_0 & da_1 & da_2 & da_3 \end{vmatrix}^2$$

$$=E_{11}du^2+(E_{12}+E_{21})dudv+E_{22}dv^2,\ (E_{12}=E_{21})$$

$$k^2(da'da')-(ada')^2=\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ da_0' & da_1' & da_2' & da_3' \end{Vmatrix}^2$$

$$(11)\qquad =E_{11}'du^2+E_{22}'dv^2+E_{33}'d\theta^2+(E_{23}'+E'_{32})dvd\theta$$

$$+(E_{31}'+E_{13}')d\theta du+(E_{12}'+E_{21}')dudv,\ (E_{ij}=E_{ji},\ i,j=1,2,3)$$

$$k^2(dada')=\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ da_0 & da_1 & da_2 & da_3 \end{Vmatrix}\cdot\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ da_0' & da_1' & da_2' & da_3' \end{Vmatrix}$$

$$-\begin{Vmatrix} a_0' & a_1' & a_2' & a_3' \\ da_0' & da_1' & da_2' & da_3' \end{Vmatrix}\cdot\begin{Vmatrix} a_0' & a_1' & a_2' & a_3' \\ da_0 & da_1 & da_2 & da_3 \end{Vmatrix}$$

$$=e_{11}du^2+e_{22}dv^2+e_{33}d\theta^2+(e_{23}+e_{32})dvd\theta$$

$$+(e_{31}+e_{13})d\theta du+(e_{12}+e_{21})dudv,$$

$$(e_{ij}\neq e_{ji},\ i,j=1,2,3,\ i\neq j)$$

$$(12)\qquad \begin{aligned} \left|E_{ij}\right| &= \left|a'\ a\ \frac{\partial a}{\partial u}\ \frac{\partial a}{\partial v}\right|^2\equiv\triangle^2, \\ \left|E'_{ij}\right| &= k^2\left|a\ a'\ \frac{\partial a'}{\partial u}\ \frac{\partial a'}{\partial v}\ \frac{\partial a'}{\partial \theta}\right|^2\equiv\triangle'^2, \end{aligned}$$

$$\triangle^2\triangle'^2=|e_{ij}|^2.$$

---

(1) See 'Allgemeine Theorie der geradlinigen Strahlensysteme,' Crelles Journal LVII, p. 189–230, 1860.

Multiplying through (10) by $\frac{\partial a_i'}{\partial u}$, $\frac{\partial a_i'}{dv}$ and $\frac{\partial a_i'}{\partial \theta}$ respectively and adding

$$(13)\quad \begin{aligned} \sin\frac{R}{k}E_{13}' &= \mu\left[e_{11}\cos\frac{R}{k}+E_{11}'\sin\frac{R}{k}\right]+\nu\left[e_{21}\cos\frac{R}{k}+E_{21}'\sin\frac{R}{k}\right],\\ \sin\frac{R}{k}E_{23}' &= \mu\left[e_{12}\cos\frac{R}{k}+E_{12}'\sin\frac{R}{k}\right]+\nu\left[e_{22}\cos\frac{R}{k}+E_{22}'\sin\frac{R}{k}\right],\\ \sin\frac{R}{k}E_{33}' &= \mu\left[e_{13}\cos\frac{R}{k}+E_{13}'\sin\frac{R}{k}\right]+\nu\left[e_{23}\cos\frac{R}{k}+E_{23}'\sin\frac{R}{k}\right]. \end{aligned}$$

Similarly, we have

$$(14)\quad \begin{aligned} \sin\frac{R}{k}e_{13} &= \mu\left[E_{11}\cos\frac{R}{k}+e_{11}\sin\frac{R}{k}\right]+\nu\left[E_{12}\cos\frac{R}{k}+e_{12}\sin\frac{R}{k}\right],\\ \sin\frac{R}{k}e_{23} &= \mu\left[E_{21}\cos\frac{R}{k}+e_{21}\sin\frac{R}{k}\right]+\nu\left[E_{22}\cos\frac{R}{k}+e_{22}\sin\frac{R}{k}\right]. \end{aligned}$$

From the above two equations, we can find the value of μ and ν. Eliminating μ and ν from (13), we get

$$(15)\quad \begin{vmatrix} e_{11}\cos\frac{R}{k}+E_{11}'\sin\frac{R}{k} & e_{21}\cos\frac{R}{k}+E_{21}'\sin\frac{R}{k} & E_{31}'\sin\frac{R}{k}\\ e_{12}\cos\frac{R}{k}+E_{12}'\sin\frac{R}{k} & e_{22}\cos\frac{R}{k}+E_{22}'\sin\frac{R}{k} & E_{32}'\sin\frac{R}{k}\\ e_{13}\cos\frac{R}{k}+E_{13}'\sin\frac{R}{k} & e_{23}\cos\frac{R}{k}+E_{23}'\sin\frac{R}{k} & E_{33}'\sin\frac{R}{k} \end{vmatrix}=0.$$

This equation can be looked upon as an equation in θ having its coefficients functions of u and v. When u and v are given particular values of θ satisfying this equation define the focal points upon the corresponding circle. When this equation is solved for θ the various solutions are substituted in (5), the latter will define the sheets of focal surface.

When the fixed point is common to all the planes of the circles of the congruence, the equation for the focal point (taking the center of the circles as the fixed point) will be

$$
(16)\qquad \begin{vmatrix} E_{11}' & E_{21}' & E_{31}' \\ E_{12}' & E_{22}' & E_{32}' \\ E_{13}' & E_{23}' & E_{33}' \end{vmatrix} = 0.
$$

Next, multiplying through the equation (10) by b_i and adding we get

$$
\mu\left[\left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k}\right]
+\nu\left[\left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k}\right]=0.
$$

Eliminating μ and ν from this equation and the equation (9) we get another form of equation for the focal points:

$$
(17)\qquad \begin{vmatrix} \left(a'\frac{\partial a}{\partial u}\right)+k\frac{\partial R}{\partial u} & \left(a'\frac{\partial a}{\partial v}\right)+k\frac{\partial R}{\partial v} \\ \left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k} & \left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k} \end{vmatrix}=0,
$$

or

$$
(18)\qquad \begin{vmatrix} \left(\xi\frac{\partial a}{\partial u}\right)\cos\theta+\left(\eta\frac{\partial a}{\partial u}\right)\sin\theta+k\frac{\partial R}{\partial u} & \left(\xi\frac{\partial a}{\partial v}\right)\cos\theta+\left(\eta\frac{\partial a}{\partial v}\right)\sin\theta+k\frac{\partial R}{\partial v} \\ \left(b\frac{\partial \xi}{\partial u}\right)\cos\theta+\left(b\frac{\partial \eta}{\partial u}\right)\sin\theta+\left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k} & \left(b\frac{\partial \xi}{\partial v}\right)\cos\theta+\left(b\frac{\partial \eta}{\partial v}\right)\sin\theta+\left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k} \end{vmatrix}=0.
$$

This equation leads to four values of θ in general, which settle the focal points of the circle, so that there are four focal points upon a circle in a general congruence.

When the planes of the circles all pass through a fixed point, taking it as the center of the circles, then the equation for the focal points will be

$$
\left[\left(b\frac{\partial \xi}{\partial u}\right)\frac{\partial R}{\partial v}-\left(b\frac{\partial \xi}{\partial v}\right)\frac{\partial R}{\partial u}\right]\cos\theta
$$

$$+\left[\left(b\frac{\partial\eta}{\partial u}\right)\frac{\partial R}{\partial v}-\left(b\frac{\partial u}{\partial v}\right)\frac{\partial R}{\partial u}\right]\sin\theta=0.$$

And this equation leads to two values of θ determing two focal points upon each such a circle.

From the results above we get the following theorem:

Theorem. Upon a circle of a general congruence there are four focal points at which all the surfaces of the congruence through the circle admit the same tangent plane, whatever be the law according to which the circles have been assembled to generate the surfaces.

§ 3. Some Properties of the Focal Point.

We know that, in rectilinear congruences, certain selected consecutive rays intersect one another; the points of intersection of any ray with the different rays which meet it, are the focal points of the ray.[1] It is natural to enquire which circle (if any), consecutive to a given circle, do intersect. Now we shall find such circles.

Any circle

$$x_i=a_i\cos\frac{R}{k}+a_i'\sin\frac{R}{k},$$

$$a_i'=\xi_i\cos\theta+\eta_i\sin\theta,\ (i=0,\ 1,\ 2,\ 3)$$

intersects with a consecutive circle, if

$$a_i\cos\frac{R}{k}+a_i'\sin\frac{R}{k}=(a_i+da_i)\cos\frac{R+dR}{k}+(a_i'+da_i')\sin\frac{R+dR}{k},$$

$$(i=0,\ 1,\ 2,\ 3)$$

or

$$(19)\qquad da_i\cos\frac{R}{k}+da_i'\sin\frac{R}{k}-\frac{1}{k}\left(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\right)dR=0,$$

$$(i=0,\ 1,\ 2,\ 3).$$

Multiplying through by a_i, and adding

[1] See Fibbi, 'I sistemi doppiamente infiniti di raggi negli spazii di curvatura costante,' Annali della R. Scuola Normale Superiore, Pisa, 1891. Also, see Coolidge, Non-Euclidean Geometry.

(20) $$(ada')=kdR.$$

Rewriting the equation (19)

$$\Big[k^2\Big(\frac{\partial a_i}{\partial u}\cos\frac{R}{k}+\frac{\partial a_i'}{\partial u}\sin\frac{R}{k}\Big)$$

$$-\Big(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\Big)\Big(a\frac{\partial a'}{\partial u}\Big)\Big]du$$

(21) $$+\Big[k^2\Big(\frac{\partial a_i}{\partial v}\cos\frac{R}{k}+\frac{\partial a_i'}{\partial v}\sin\frac{R}{k}\Big)$$

$$-\Big(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\Big)\Big(a\frac{\partial a'}{\partial v}\Big)\Big]dv$$

$$+\Big[k^2\frac{\partial a_i'}{\partial\theta}\sin\frac{R}{k}-\Big(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\Big)\Big(a\frac{\partial a'}{\partial\theta}\Big)\Big]d\theta=0,$$

$$(i=0,\ 1,\ 2,\ 3).$$

Multiplying through by $\frac{\partial a_i'}{\partial u}$, $\frac{\partial a_i'}{\partial v}$, $\frac{\partial a_i'}{\partial\theta}$ and adding respectively, we get

$$\Big[e_{11}\cos\frac{R}{k}+E_{11}'\sin\frac{R}{k}\Big]du+\Big[e_{21}\cos\frac{R}{k}+E_{21}'\sin\frac{R}{k}\Big]dv$$

$$+E_{31}'\sin\frac{R}{k}d\theta=0,$$

(22) $$\Big[e_{12}\cos\frac{R}{k}+E_{12}'\sin\frac{R}{k}\Big]du+\Big[e_{22}\cos\frac{R}{k}+E_{22}'\sin\frac{R}{k}\Big]dv$$

$$+E_{32}'\sin\frac{R}{k}d\theta=0,$$

$$\Big[e_{13}\cos\frac{R}{k}+E_{13}'\sin\frac{R}{k}\Big]du+\Big[e_{23}\cos\frac{R}{k}+E_{23}'\sin\frac{R}{k}\Big]dv$$

$$+E_{33}'\sin\frac{R}{k}d\theta=0.$$

Eliminating $du:dv:d\theta$, we have

$$\begin{vmatrix} e_{11}\cos\frac{R}{k}+E_{11}'\sin\frac{R}{k} & e_{21}\cos\frac{R}{k}+E_{21}'\sin\frac{R}{k} & E_{31}'\sin\frac{R}{k} \\ e_{12}\cos\frac{R}{k}+E_{12}'\sin\frac{R}{k} & e_{22}\cos\frac{R}{k}+E_{22}'\sin\frac{R}{k} & E_{32}'\sin\frac{R}{k} \\ e_{13}\cos\frac{R}{k}+E_{13}'\sin\frac{R}{k} & e_{23}\cos\frac{R}{k}+E_{23}'\sin\frac{R}{k} & E_{33}'\sin\frac{R}{k} \end{vmatrix}=0,$$

which is the same equation as (15) giving the focal points of the circle. Hence we have the following theorem:

Theorem. The points of intersection of a circle with the consecutive circles are its focal points.

Again, multiplying through (21) by b_i and adding, we get

$$\left[\left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k}\right]du$$
$$+\left[\left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k}\right]dv=0,$$

and from (20)

$$\left[\left(a\frac{\partial a'}{\partial u}\right)-k\frac{\partial R}{\partial u}\right]du+\left[\left(a\frac{\partial a'}{\partial v}\right)-k\frac{\partial R}{\partial v}\right]dv=0.$$

Eliminating $du:dv$ from the above two equations, we get

$$\begin{vmatrix} \left(a\frac{\partial a'}{\partial u}\right)-k\frac{\partial R}{\partial u} & \left(a\frac{\partial a'}{\partial v}\right)-k\frac{\partial R}{\partial v} \\ \left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k} & \left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k} \end{vmatrix}=0,$$

and this is the same equation as (17) which gives the facal points on the circle.

Rewriting the above two equations, we have

$$\cos\frac{R}{k}\left[\left(b\frac{\partial a}{\partial u}\right)du+\left(b\frac{\partial a}{\partial v}\right)dv\right]+\sin\frac{R}{k}\left[\left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv\right]\cos\theta$$
$$+\sin\frac{R}{k}\left[\left(b\frac{\partial\eta}{\partial u}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv\right]\sin\theta=0,$$

(23)

$$k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)-\left[\left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv\right]\cos\theta$$

$$-\left[\left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv\right]\sin\theta=0.$$

Eliminating $\cos\theta$ and $\sin\theta$, we get

$$\sin^2\frac{R}{k}\begin{vmatrix}\left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv & \left(b\frac{\partial\eta}{\partial u}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv\\ \left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv & \left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv\end{vmatrix}^2$$

$$(24)\quad =\begin{vmatrix}\sin\frac{R}{k}\left[\left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv\right] & \cos\frac{R}{k}\left[\left(b\frac{\partial a}{\partial u}\right)du+\left(b\frac{\partial a}{\partial v}\right)dv\right]\\ \left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv & -k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{dv}dv\right)\end{vmatrix}^2$$

$$+\begin{vmatrix}\sin\frac{R}{k}\left[\left(b\frac{\partial\eta}{\partial u}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv\right] & \cos\frac{R}{k}\left[\left(b\frac{\partial a}{\partial u}\right)du+\left(b\frac{\partial a}{\partial v}\right)dv\right]\\ \left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{d\eta}{\partial v}\right)dv & -k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)\end{vmatrix}^2.$$

The circles, that are consecutive and intersect, are determined by quantities $u+du$, $v+dv$; and the common point with the consecutive circle is given by the value of $\theta+d\theta$ on that consecutive circle. Now the above relation is an equation for $du:dv$; its coefficient are functions of u and v only; and therefore it determines four consecutive values of u and v, which give consecutive values of a_i, b_i $(i=0, 1, 2, 3)$ and R, and therefore give fous consecutive intersecting circles,

Theorem. In a general congruence of circles, every circle meets four other consecutive circles; it intersects each of the circles at a single point, the four points being the focal points of the circle.

Two circles in space can not intersect at more than two points. So we can consider such a congruence of circles that two of the four focal points of a circle shall lie on one consecutive circle, and the other two lie on another consecutive circle. In this case, there will be two (and not four) values of $du:dv$ which give two consecutive intersectlng circles; for each of these two values of $du:dv$, there will be two values of θ, giving the two points upon the consecutive circle which are focal points of the first. In order that this may be possible, the two equations

$$\cos\frac{R}{k}(bda)+\sin\frac{R}{k}(bd\xi)\cos\theta+\sin\frac{R}{k}(bd\eta)\sin\theta=0,$$

$$kdR-(ad\xi)\cos\theta-(ad\eta)\sin\theta=0,$$

can not be independent; for if not, they would determine $\cos\theta$ and $\sin\theta$ uniquely for an assigned value of $du:dv$, and so there would be only a single focal point on the consecutive circle. Hence, all the determinants of the matrix

$$(25)\qquad \left\|\begin{matrix}(bda) & \tan\frac{R}{k}(bd\xi) & \tan\frac{R}{k}(bd\eta)\\ -kdR & (ad\xi) & (ad\eta)\end{matrix}\right\|$$

must be zero, and these equations are to determine two values of θ.

But, since, if two circles (consecutive) intersect at two points they are cospherical and the conditions for that is

$$\begin{vmatrix}a_0 & a_1 & a_2 & a_3\\ b_0 & b_1 & b_2 & b_3\\ da_0 & da_1 & da_2 & da_3\\ db_0 & db_1 & db_2 & db_3\end{vmatrix}=0.$$

and all the determinants of the matrix

$$\left\|\begin{matrix}a_0\sin\frac{R}{k}\frac{dR}{k}+da_0\cos\frac{R}{k} & a_1\sin\frac{R}{k}\frac{dR}{k}+da_1\cos\frac{R}{k} & a_2\sin\frac{R}{k}\frac{dR}{k}+da_2\cos\frac{R}{k} & a_3\sin\frac{R}{k}\frac{dR}{k}+da_3\cos\frac{R}{k}\\ b_0 & b_1 & b_2 & b_3\\ db_0 & db_1 & db_2 & db_3\end{matrix}\right\|$$

are equal to zero.[1] From these relations we can also find the conditions (25) easily.

---

[1] See, T. Nishiuchi, 'Oriented Circles in Non-Euclidean Space,' loc. cit.

Next, from the conditions (25) we can find two quantities λ and μ such that

$$\tan\frac{R}{k}\left[\lambda\left(b\frac{\partial\xi}{\partial u}\right)+\mu\left(b\frac{\partial\eta}{\partial u}\right)\right]=\left(b\frac{\partial a}{\partial u}\right),$$

$$\tan\frac{R}{k}\left[\lambda\left(b\frac{\partial\xi}{\partial v}\right)+\mu\left(b\frac{\partial\eta}{\partial v}\right)\right]=\left(b\frac{\partial a}{\partial v}\right),$$

$$\lambda\left(a\frac{\partial\xi}{\partial u}\right)+\mu\left(a\frac{\partial\eta}{\partial u}\right)=k\frac{\partial R}{\partial u},$$

$$\lambda\left(a\frac{\partial\xi}{\partial v}\right)+\mu\left(a\frac{\partial\eta}{\partial v}\right)=k\frac{\partial R}{\partial v}.$$

Therefore, the two conditions represented by the following equations

$$\left\|\begin{matrix} \left(a\frac{\partial\xi}{\partial u}\right) & \left(a\frac{\partial\xi}{\partial v}\right) & \left(b\frac{\partial\xi}{\partial u}\right) & \left(b\frac{\partial\xi}{\partial v}\right) \\ \left(a\frac{\partial\eta}{\partial u}\right) & \left(a\frac{\partial\eta}{\partial v}\right) & \left(b\frac{\partial\eta}{\partial u}\right) & \left(b\frac{\partial\eta}{\partial v}\right) \\ k\frac{\partial R}{\partial u} & k\frac{\partial R}{\partial v} & \left(b\frac{\partial a}{\partial u}\right) & \left(b\frac{\partial a}{\partial v}\right) \end{matrix}\right\|=0,$$

must be satisfied by the magnitudes that occur in the expression of the congruence. The two values of $du:dv$ are the roots of the quadratic

$$\begin{vmatrix} \left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv & \left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv \\ \left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv & \left(b\frac{\partial\eta}{du}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv \end{vmatrix}=0.$$

And the two values of θ, which correspond to one of the values of $du:dv$ are the roots of the equation

$$k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)+\left[\left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv\right]\cos\theta$$
$$+\left[\left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv\right]\sin\theta=0.$$

CHAPTER. II.

CYCLICAL SYSTEMS OF CONGRUENCES OF CIRCLES.

§ 4. CONDITION THAT A CONGRUENCE OF CIRCLES HAS AN ORTHOGONAL SURFACE.

The Plücker line-coordinates of the tangent to any circle of a congruence represented by

$$(1) \qquad x_i = a_i \cos\frac{R}{k} + (\xi_i \cos\theta + \eta_i \sin\theta)\sin\frac{R}{k},$$
$$(i=0.\ 1,\ 2,\ 3)$$

at a point (x) are given by

$$\begin{vmatrix} x_i & x_j \\ \dfrac{\partial x_i}{\partial\theta} & \dfrac{\partial x_j}{\partial\theta} \end{vmatrix}, \quad (i, j=0,\ 1,\ 2,\ 3\ ;\ i \neq j).$$

Hence if every line at that point whose Plücker line coordinates are

$$\begin{vmatrix} x_i & x_j \\ dx_i & dx_j \end{vmatrix}, \quad (i, j=0,\ 1,\ 2,\ 3\ ;\ i \neq j)$$

perpendicular to the tangent to the circle, must satisfy the relation

$$(2) \qquad \left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ \dfrac{\partial x_0}{\partial\theta} & \dfrac{\partial x_1}{\partial\theta} & \dfrac{\partial x_2}{\partial\theta} & \dfrac{\partial x_3}{\partial\theta} \end{matrix} \right\| \cdot \left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ dx_0 & dx_1 & dx_2 & dx_3 \end{matrix} \right\| = 0,$$

i. e.

$$(3) \qquad \left(\frac{\partial x}{\partial\theta} dx\right) = 0.$$

Now, we put

$$U = \frac{1}{\sin\frac{R}{k}} \left(\frac{\partial x}{\partial\theta}\ \frac{\partial x}{\partial u}\right)$$
$$= -\left(\xi \frac{\partial a}{\partial u}\right)\cos\frac{R}{k}\sin\theta + \left(\eta\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}\cos\theta + \left(\eta\frac{\partial \xi}{\partial u}\right)\sin\frac{R}{k},$$

(4) $$V=\frac{1}{\sin\frac{R}{k}}\left(\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial v}\right)$$

$$=-\left(\xi\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}\sin\theta+\left(\eta\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}\cos\theta+\left(\eta\frac{\partial\xi}{\partial v}\right)\sin\frac{R}{k},$$

$$\Theta=\frac{1}{\sin\frac{R}{k}}\left(\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\theta}\right)=k^2\sin\frac{R}{k},$$

then the equation (3) will be

(5) $$Udu+Vdv+\Theta d\theta=0.$$

And the necessary and sufficient condition that there is a system of surface which cuts the circle of the congruence orthogonally, is that the differential equation (5) shall be integrable, i. e. for all values of u, v and θ, the condition of integrability

(6) $$\Theta\left[\frac{\partial U}{\partial v}-\frac{\partial V}{\partial u}\right]+U\left[\frac{\partial V}{\partial\theta}-\frac{\partial\Theta}{\partial v}\right]+V\left[\frac{\partial\Theta}{\partial u}-\frac{\partial U}{\partial\theta}\right]=0,$$

must satisfy. When the values of U, V and Θ are substituted, the condition will be

(7) $$A+B\sin\theta+C\cos\theta=0.$$

where

$$A=\cos^2\frac{R}{k}\left[\left(\xi\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial a}{\partial v}\right)-\left(\eta\frac{\partial a}{\partial u}\right)\left(\xi\frac{\partial a}{\partial v}\right)\right]$$

$$+k^2\sin^2\frac{R}{k}\left[\frac{\partial}{\partial v}\left(\eta\frac{\partial\xi}{\partial u}\right)-\frac{\partial}{\partial u}\left(\eta\frac{\partial\xi}{\partial v}\right)\right],$$

(8) $$B=\sin\frac{R}{k}\cos\frac{R}{k}\left[\left(\eta\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial\xi}{\partial v}\right)-\left(\eta\frac{\partial a}{\partial v}\right)\left(\eta\frac{\partial\xi}{\partial u}\right)\right.$$

$$\left.+k^2\left\{\frac{\partial}{\partial u}\left(\xi\frac{\partial a}{\partial v}\right)-\frac{\partial}{\partial v}\left(\xi\frac{\partial a}{\partial u}\right)\right\}\right]+k\left[\left(\xi\frac{\partial a}{\partial u}\right)\frac{\partial R}{\partial v}-\left(\xi\frac{\partial a}{\partial v}\right)\frac{\partial R}{\partial u}\right],$$

$$C=\sin\frac{R}{k}\cos\frac{R}{k}\left[\left(\xi\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial\xi}{\partial v}\right)-\left(\xi\frac{\partial a}{\partial v}\right)\left(\eta\frac{\partial\xi}{\partial u}\right)\right.$$

$$\left.+k^2\left\{\frac{\partial}{\partial v}\left(\eta\frac{\partial a}{\partial u}\right)-\frac{\partial}{\partial u}\left(\eta\frac{\partial a}{\partial v}\right)\right\}\right]+k\left[\left(\eta\frac{\partial a}{\partial v}\right)\frac{\partial R}{\partial u}-\left(\eta\frac{\partial a}{\partial u}\right)\frac{\partial R}{\partial v}\right],$$

and A, B and C are manifestly independent of θ.

If the condition be satisfied identically, then

$$A=0, \quad B=0, \quad C=0.$$

In this case, let

$$\Phi=\text{const.}$$

be the integral solution of the differential equation; it is the equation of the family of surfaces cutting the circles of the congruence orthogonally. If the condition is not satisfied identically, it is possible that the two solutions of the equation (7) will satisfy (5), and thus determine two surfaces orthogonal to the circles of the congruence. Hence we have the theorem of Ribaucour[1]:

Theorem. If the circle of a congruence are normal to more than two surfaces, they form a cyclical system.

§ 5. A Special Congruence of Circles Which From a Cyclical System.

Consider a congruence of circles, which lie in the tangent planes to a surface and have their centers at the point of contact of the tangent plane with the surface.

Let the surface be referred to the lines of curvature as the parametric curves; then we can take the points (ξ) and (η) on the tangents to the parametric curves $v=$const, $u=$const. respectively, so that

$$\xi_i=\lambda a_i+\mu\frac{\partial a_i}{\partial u}du,$$

$$\eta_i=\lambda' a_i+\mu'\frac{\partial a_i}{\partial v}dv, \qquad (i=0, 1, 2, 3)$$

where λ, μ and λ', μ' are parameters.

But since

$$(a\xi)=0, \quad (a\eta)=0,$$

$$(\xi\xi)=(\eta\eta)=k^2,$$

---

1 See Ribaucour, 'Mèmoire sur la théorie générale des surfaces courbées,' Journal des Mathématiques, Ser. 4, Vol. vii, 1891.

we have

$$\xi_i = \frac{k}{\sqrt{\left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial u}\right)}}\frac{\partial a_i}{\partial u} = \frac{k}{\sqrt{E}}\frac{\partial x_i}{\partial u},$$

$$\eta_i = \frac{k}{\sqrt{\left(\frac{\partial a}{\partial v}\frac{\partial a}{\partial v}\right)}}\frac{\partial a_i}{\partial v} = \frac{k}{\sqrt{G}}\frac{\partial x_i}{\partial v}, \quad (i=0,\ 1,\ 2,\ 3)$$

and

$$\left(\xi\frac{\partial a}{\partial u}\right) = k\sqrt{E}, \quad \left(\xi\frac{\partial a}{\partial v}\right) = 0,$$

$$\left(\eta\frac{\partial a}{\partial u}\right) = 0, \quad \left(\eta\frac{\partial a}{\partial v}\right) = k\sqrt{G},$$

$$\left(\eta\frac{\partial \xi}{\partial u}\right) = -\frac{k^2}{2\sqrt{EG}}\frac{\partial E}{\partial v},$$

$$\left(\eta\frac{\partial \xi}{\partial v}\right) = \frac{k^2}{2\sqrt{EG}}\frac{\partial G}{\partial v}.$$ [1]

Substituting these relations above to (8), we have

$$A = k^2\sqrt{EG} - k^2\sqrt{EG}\sin^2\frac{R}{k}\Big[1 + \frac{k^2}{2EG}\Big\{\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 E}{\partial v^2}$$

$$-\frac{1}{2EG}\frac{\partial G}{\partial u}\left(G\frac{\partial E}{\partial u} + E\frac{\partial G}{\partial u}\right) - \frac{1}{2EG}\frac{\partial E}{\partial v}\left(G\frac{\partial E}{\partial v} + E\frac{\partial G}{\partial v}\right)\Big\}\Big],$$

$$B = k^2\sqrt{E}\frac{\partial R}{\partial v},$$

$$C = k^2\sqrt{G}\frac{\partial R}{\partial u}.$$

Let K be the total relative curvature of the surface, then

---

[1] E, F, G are fundamental coefficients of the first order, and

$$E = \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial u}\right), \quad F = \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right), \quad G = \left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial v}\right).$$

But in this case, from the hypothesis, $(x)=(a)$ and

$$E = \left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial u}\right), \quad F = \left(\frac{\partial a}{\partial u}\frac{\partial a}{\partial v}\right) = 0, \quad G = \left(\frac{\partial a}{\partial v}\frac{\partial a}{\partial v}\right).$$

$$K=\frac{DD''}{k^2EG}=\frac{1}{k^2E^2G^2}\begin{vmatrix} k^2 & 0 & 0 & -G \\ 0 & E & 0 & -\frac{1}{2}\frac{\partial G}{\partial u} \\ 0 & 0 & G & \frac{1}{2}\frac{\partial G}{\partial v} \\ -E & \frac{1}{2}\frac{\partial E}{\partial u} & -\frac{1}{2}\frac{\partial E}{\partial v} & \left(\frac{\partial^2 x}{\partial u^2}\frac{\partial^2 x}{\partial v^2}\right) \end{vmatrix},$$ [1]

but

$$\left(\frac{\partial^2 x}{\partial u^2}\frac{\partial^2 x}{\partial v^2}\right)=\frac{1}{4EG}\left\{G\left(\frac{\partial E}{\partial v}\right)^2+E\left(\frac{\partial G}{\partial u}\right)^2\right\}-\frac{1}{2}\left\{\frac{\partial^2 E}{\partial v^2}+\frac{\partial^2 G}{\partial u^2}\right\},$$

we have

$$K=-\frac{1}{k^2}\left[1+\frac{k^2}{2EG}\left\{\frac{\partial^2 G}{\partial u^2}+\frac{\partial^2 E}{\partial v^2}-\frac{1}{2EG}\frac{\partial G}{\partial u}\left(G\frac{\partial E}{\partial u}+E\frac{\partial G}{\partial u}\right)\right.\right.$$
$$\left.\left.-\frac{1}{2EG}\frac{\partial E}{\partial v}\left(G\frac{\partial E}{\partial v}+E\frac{\partial G}{\partial v}\right)\right\}\right]$$

therefore

$$A=k^2\sqrt{EG}\left(1+k^2\sin^2\frac{R}{k}K\right).$$

In order that the congruence of the circles may form a cyclical system, we are to have

$$A=0,\quad B=0,\quad C=0.$$

Hence, we must have

$$R=\text{constant},$$

$$K=-\frac{1}{k^2\sin^2\frac{R}{k}}\ \text{(constant)},$$

---

[1] D, D', D'' are fundamental coefficients of the second order, and

$$D=\frac{\begin{vmatrix} x\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial^2 x}{\partial u^2}\end{vmatrix}}{\sqrt{EG-F^2}},\quad D'=\frac{\begin{vmatrix} x\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial^2 x}{\partial u\partial v}\end{vmatrix}}{\sqrt{EG-F^2}},\quad D''=\frac{\begin{vmatrix} x\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial^2 x}{\partial v^2}\end{vmatrix}}{\sqrt{EG-F^2}}.$$

and the total Gaussian curvature[1] of the surface will be constant and equal to $-\dfrac{1}{k^2\tan^2\frac{R}{k}}$.

Therefore, we get the following result:

A congruence of circle of constant radius which lie in the tangent planes to a surface, whose total Gaussian curvature is constant and its absolute valve is equal to the square of the curvature of the circles, and have their centers at the point of contact of the tangent plane with the surface, form a cyclical system.

§ 6. Formula for a Cyclical System.

Let S be an orthogonal surface of a cyclical system and take lines of curvature on S as parametric curves. Then through every point (x) on S, a circle (u, v), which is orthogonal to the surface S, passes.

Take points (X) and (Y) on the tangents to the parametric curves $v=$const. and $u=$const. respectively, such that

$$(Xx)=0,\ (Yx)=0,\ (XX)=k^2,\ (YY)=k^2.$$

Then as the lines of curvature are parametric,

$$(XY)=0,$$

and (X), (Y), (x) will be the vertices of a moving self-conjugate triangle respectively.

Again, let $\frac{\pi}{2}-\omega$ be the angle between the parametric curve $v=$ const. and the plane of the circle, and (z) the coordinates of the point of intersection of the line joining the points (X) and (Y) with the plane of the circle, then

$$(9)\qquad z_i=X_i\cos\omega+Y_i\sin\omega,\ (zz)=k^2.$$

$$(i=0,\ 1,\ 2,\ 3).$$

But the point (z) will be the absolute pole of the tangent to the

[1] See Coolidge, Non-Euclidean Geometry, p. 204.

circle at the point (x), the coordinates of the center of the circle will be given by

$$(10) \qquad a_i = x_i \cos\frac{R}{k} + (X_i \cos\omega + Y_i \sin\omega)\sin\frac{R}{k},$$

$$(i=0, 1, 2, 3)$$

Next, let (Z) be the absolute pole of the tangent plane to the surface S at the point (x), the point (Z) will lie on the plane of the circle and the line joining the points (x) and (Z) will be the tangent to the circle at the point (x), and

$$(xZ)=0.$$

Hence, we can put

$$(11) \qquad \xi_i = (X_i \cos\omega + Y_i \sin\omega)\cos\frac{R}{k} - x_i \sin\frac{R}{k},$$

$$\eta_i = Z_i,$$

$$(i=0, 1, 2, 3).$$

But

$$(12) \begin{cases} X_i = \dfrac{k}{\sqrt{E}}\dfrac{\partial x_i}{\partial u}, \quad Y_i = \dfrac{k}{\sqrt{G}}\dfrac{\partial x_i}{\partial v}, \\[2ex] \dfrac{\partial Z_i}{\partial u} = \dfrac{1}{\tan\dfrac{r_2}{k}}\dfrac{\partial x_i}{\partial u} = \dfrac{\sqrt{E}}{k\tan\dfrac{r_2}{k}}X_i, \\[2ex] \dfrac{\partial Z_i}{\partial v} = \dfrac{\sqrt{G}}{k\tan\dfrac{r_1}{k}}Y_i, \qquad (i=0, 1, 2, 3) \end{cases}$$

where $\dfrac{1}{k\tan\dfrac{r_1}{k}}$ and $\dfrac{1}{k\tan\dfrac{r_2}{k}}$ are the curvatures of the normal sections through the tangents to the lines of curvature $u=$const. and $v=$const. respectively, and

$$DZ_i=\frac{1}{EG}\begin{vmatrix} x_i & k^2 & 0 & 0 \\ \frac{\partial x_i}{\partial u} & 0 & E & 0 \\ \frac{\partial x_i}{\partial v} & 0 & 0 & G \\ \frac{\partial^2 x_i}{\partial u^2} & -E & \frac{1}{2}\frac{\partial E}{\partial u} & -\frac{1}{2}\frac{\partial E}{\partial v} \end{vmatrix}, \quad (i=0,\ 1,\ 2,\ 3)$$

we have

$$\frac{\partial^2 x_i}{\partial u^2}=\frac{1}{2E}\frac{\partial E}{\partial u}\frac{\partial x_i}{\partial u}-\frac{1}{2G}\frac{\partial E}{\partial v}\frac{\partial x_i}{\partial v}-\frac{E}{k^2}x_i-\frac{DZ_i}{k^2},$$

$$(i=0,\ 1,\ 2,\ 3).$$

and similarly

$$(13)\qquad \frac{\partial^2 x_i}{\partial u \partial v}=\frac{1}{2E}\frac{\partial E}{\partial v}\frac{\partial x_i}{\partial u}+\frac{1}{2G}\frac{\partial G}{\partial u}\frac{\partial x_i}{\partial v},$$

$$\frac{\partial^2 x_i}{\partial v^2}=-\frac{1}{2E}\frac{\partial G}{\partial u}\frac{\partial x_i}{\partial u}+\frac{1}{2G}\frac{\partial G}{\partial v}\frac{\partial x_i}{\partial v}-\frac{G}{k^2}x_i-\frac{D''Z_i}{k^2},$$

$$(i=0,\ 1,\ 2,\ 3).$$

Hence we have

$$\frac{\partial X_i}{\partial u}=-\frac{k}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}Y_i-\frac{\sqrt{E}}{k}x_i-\frac{\sqrt{E}}{k\tan\frac{r_2}{k}}Z_i,$$

$$(14)\qquad \frac{\partial Y_i}{\partial u}=\frac{k}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}X_i,$$

$$\frac{\partial X_i}{\partial v}=\frac{k}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}Y_i,$$

$$\frac{\partial Y_i}{\partial v}=-\frac{k}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}X_i-\frac{\sqrt{G}}{k}x_i-\frac{\sqrt{G}}{k\tan\frac{r_1}{k}}Z_i,$$

$$(i=0,\ 1,\ 2,\ 3).$$

From the results above we have

$$\left(\xi \frac{\partial a}{\partial u}\right)=k\left\{\sqrt{E}\cos\omega+\frac{\partial R}{\partial u}\right\},$$

$$\left(\xi \frac{\partial a}{\partial v}\right)=k\left\{\sqrt{G}\sin\omega+\frac{\partial R}{\partial v}\right\},$$

(15)
$$\left(\eta \frac{\partial a}{\partial u}\right)=-k\sin\frac{R}{k}\,\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}},$$

$$\left(\eta \frac{\partial a}{\partial v}\right)=-k\sin\frac{R}{k}\,\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}},$$

$$\left(\eta \frac{\partial \xi}{\partial u}\right)=-k\cos\frac{R}{k}\,\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}},$$

$$\left(\eta \frac{\partial \xi}{\partial v}\right)=-k\cos\frac{R}{k}\,\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}},$$

Hence, the differential equation (5) will be

(16)
$$\begin{aligned} kd\theta=&\left[\left\{\frac{\sqrt{E}\cos\omega}{R'}+\frac{1}{R'(1+R'^2)}\frac{\partial R}{\partial u}\right\}\sin\theta\right.\\ &\left.+\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}}\frac{1}{\sqrt{1+R'^2}}(1+\cos\theta)\right]du\\ &+\left[\left\{\frac{\sqrt{G}\sin\omega}{R'}+\frac{1}{R'(1+R'^2)}\frac{\partial R}{\partial v}\right\}\sin\theta\right.\\ &\left.+\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}}\frac{1}{\sqrt{1+R'^2}}(1+\cos\theta)\right]dv,\end{aligned}$$

where

$$R'=\tan\frac{R}{k}.$$

Or

$$k\tan\frac{R}{k}d\theta=\Bigg[\left\{\sqrt{E}\cos\omega+\frac{\partial R}{\partial u}\right\}\sin\theta$$

$$(17)\qquad +\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}}\sin\frac{R}{k}(1+\cos\theta)\Bigg]du$$

$$+\Bigg[\left\{\sqrt{G}\sin\omega+\frac{\partial R}{\partial v}\right\}\sin\theta+\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}}\sin\frac{R}{k}(1+\cos\theta)\Bigg]dv.$$

Therefore, as remarked before, if there are three surfaces orthogonal to a congruence of circles, the congruence form a cyclical system.

The conditions of integrability will be reduced to the following two equations:

$$\frac{\partial}{\partial u}\left(\frac{\sqrt{G}\sin\omega}{\tan\frac{R}{k}}\right)-\frac{\partial}{\partial v}\left(\frac{\sqrt{E}\cos\omega}{\tan\frac{R}{k}}\right)=0,$$

$$(18)\qquad \frac{\partial}{\partial u}\left(\frac{1}{\tan\frac{r_1}{k}}\frac{\sqrt{G}\sin\omega}{\tan\frac{R}{k}}\right)-\frac{\partial}{\partial v}\left(\frac{1}{\tan\frac{r_2}{k}}\frac{\sqrt{E}\cos\omega}{\tan\frac{R}{k}}\right)$$

$$+\frac{\sqrt{EG}\sin\omega\cos\omega}{k\tan^2\frac{R}{k}}\left(\frac{1}{\tan\frac{r_2}{k}}-\frac{1}{\tan\frac{r_1}{k}}\right)=0.$$

Hence, if the surface S be a sphere or a plane, the second equation of the conditions will be reduced to the first. And we have incidentally the following theorem:

Theorem. A congrnence of circles orthogonal to a sphere (or a plane) and to any other surface constitutes a cyclical system.

Next, we assume that the surface S is not the sphere, then from the relation

$$(19)\qquad \begin{aligned}\frac{\partial}{\partial u}\left(\frac{\sqrt{G}}{\tan\frac{r_1}{k}}\right)&=\frac{1}{\tan\frac{r_2}{k}}\frac{\partial\sqrt{G}}{\partial u},\\ \frac{\partial}{\partial v}\left(\frac{\sqrt{E}}{\tan\frac{r_2}{k}}\right)&=\frac{1}{\tan\frac{r_1}{k}}\frac{\partial\sqrt{E}}{\partial v},\end{aligned}$$ [1]

[1] These relations can be obtained from the equation of Gauss (13) easily.

and the conditions (18), we have

$$k\frac{\partial \tan \frac{R}{k}}{\partial u}=k \tan \frac{R}{k} \cot \omega\left(\frac{\partial \omega}{\partial u}-\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v}\right)-\sqrt{E} \cos \omega,$$

(20)

$$k\frac{\partial \tan \frac{R}{k}}{\partial v}=-k \tan \frac{R}{k} \tan \omega\left(\frac{\partial \omega}{\partial v}+\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u}\right)-\sqrt{G} \sin \omega.$$

From the first condition of (18) the expression

$$(21) \qquad \frac{\sqrt{E} \cos \omega}{\tan \frac{R}{k}} du+\frac{\sqrt{G} \sin \omega}{\tan \frac{R}{k}} du$$

is the total differential. If we take the unknown complementary function ψ, then we can put

$$(22) \qquad \frac{\sqrt{E} \cos \omega}{\tan \frac{R}{k}}=-\frac{k}{\psi} \frac{\partial \psi}{\partial u}, \quad \frac{\sqrt{G} \sin \omega}{\tan \frac{R}{k}}=-\frac{k}{\psi} \frac{\partial \psi}{\partial v},$$

and the second condition will be reduced to the following Laplace equation by the aid of the relation (19):

$$(23) \qquad \frac{\partial^2 \psi}{\partial u \partial v}=\frac{\partial \log \sqrt{E}}{\partial v} \frac{\partial \psi}{\partial u}+\frac{\partial \log \sqrt{G}}{\partial u} \frac{\partial \psi}{\partial v}.$$

Conversely, if ψ be a solution of this differential equation, then the corresponding cyclical system of circles can be given by the following equations:

$$\frac{1}{\tan^2 \frac{R}{k}}=\frac{k^2}{E}\left(\frac{\partial \log \psi}{\partial u}\right)^2+\frac{k^2}{G}\left(\frac{\partial \log \psi}{\partial v}\right)^2,$$

(24)

$$\cos \omega=-\frac{k \tan \frac{R}{k}}{\sqrt{E}} \frac{\partial \log \psi}{\partial u},$$

$$\sin\omega=-\frac{k\tan\dfrac{R}{k}}{\sqrt{G}}\,\frac{\partial\log\psi}{\partial v}.$$

Let

$$\psi=(ax)+C$$

where (a) is the coordinate of a fixed point and C constant, then

$$\frac{\partial^2\psi}{\partial u\partial v}=\left(a\frac{\partial^2 x}{\partial u\partial v}\right)$$

$$=\frac{\partial\log\sqrt{E}}{\partial v}\left(a\frac{\partial x}{\partial u}\right)+\frac{\partial\log\sqrt{G}}{\partial u}\left(a\frac{\partial x}{\partial v}\right).$$

Hence

$$\psi=(ax)+C$$

is a particular solution of the Laplace equation (23) and the corresponding cyclical system of circles is orthogonal to the surface S and a sphere whose equation is

$$(ax)^2=C'.$$

This is the fact which we discussed before. When $C'=0$, the sphere will be a plane.

§ 7. The Theorem of Ribaucour.

Now, we consider a family of surfaces of a triply orthogonal system

$$x_i=f_i(u,\ v),$$

$$(25)\qquad x_i=\phi_i(v,\ w),$$

$$x_i=\psi_i(w,\ u),\quad (i=0,\ 1,\ 2,\ 3),$$

whose lines of curvature are parametrie curves, and the system of the osculating circles of the parametric curves w of an orthogonal system at their point of intersection with a surface $w=$const. $(x_i=f_i(u,\ v))$.

Then the geodesic curvature $\dfrac{1}{\rho_g}$ of the osculating circles on the surface $w=$const. is given by

$$\frac{1}{\rho_g}=\frac{\sin\omega}{k\tan\dfrac{R}{k}}$$ [1]

(26)

$$=-\frac{1}{\sqrt{GH}}\frac{\partial\sqrt{H}}{\partial v},$$

and

(27) $$\frac{\cos\omega}{k\tan\dfrac{R}{k}}=-\frac{1}{\sqrt{EH}}\frac{\partial\sqrt{H}}{\partial u},$$

where

$$E=\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial u}\right),\quad G=\left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial v}\right),\quad H=\left(\frac{\partial x}{\partial w}\frac{\partial x}{\partial w}\right).$$

The coordinates of the pole of the tangent plane to the surface $w=$const. at that point are

$$y_i=\frac{1}{\sqrt{EG}}\frac{\partial}{\partial r_i}\left|r\ x\ \frac{\partial x}{\partial u}\ \frac{\partial x}{\partial v}\right|,\quad (i=0,\ 1,\ 2,\ 3),$$

$$(yy)=k^2.$$

But since

$$\left(x\frac{\partial x}{\partial w}\right)=\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial w}\right)=\left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial w}\right)=0,$$

we get

$$y_i=\lambda\frac{\partial x_i}{\partial w},\quad (i=0,\ 1,\ 2,\ 3)$$

where λ is a parameter.

And therefore, we have

(28) $$y_i=\frac{k}{\sqrt{H}}\frac{\partial x_i}{\partial w},\quad (i=0,\ 1,\ 2,\ 3)$$

and

$$D=\left(y\frac{\partial^2 x}{\partial u^2}\right)=-k\frac{\sqrt{E}}{\sqrt{H}}\frac{\partial\sqrt{E}}{\partial w},$$

---

1 See Coolidge, Non-Euclidean Geometry, p. 188, 208.

$$(29)\qquad D' = \left(y\frac{\partial x}{\partial u \partial v}\right) = 0,$$

$$D'' = \left(y\frac{\partial^2 x}{\partial v^2}\right) = -k\frac{\sqrt{G}}{\sqrt{H}}\frac{\partial\sqrt{G}}{\partial w}.$$

Hence

$$(30)\qquad \begin{aligned} \frac{1}{k\tan\frac{r_1}{k}} &= -\frac{1}{\sqrt{GH}}\frac{\partial\sqrt{G}}{\partial w}, \\ \frac{1}{k\tan\frac{r_2}{k}} &= -\frac{1}{\sqrt{EH}}\frac{\partial\sqrt{E}}{\partial w}. \end{aligned}$$

And the equations of Lamé[1] will be

$$(31)\qquad \begin{aligned} \frac{\partial^2\sqrt{E}}{\partial v\partial w} &= \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial w}\frac{\partial\sqrt{E}}{\partial v} + \frac{1}{\sqrt{H}}\frac{\partial\sqrt{H}}{\partial v}\frac{\partial\sqrt{E}}{\partial w}, \\ \frac{\partial^2\sqrt{G}}{\partial w\partial u} &= \frac{1}{\sqrt{H}}\frac{\partial\sqrt{H}}{\partial u}\frac{\partial\sqrt{G}}{\partial w} + \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial w}\frac{\partial\sqrt{H}}{\partial u}, \\ \frac{\partial^2\sqrt{H}}{\partial u\partial v} &= \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\frac{\partial\sqrt{H}}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}\frac{\partial\sqrt{H}}{\partial v}. \end{aligned}$$

$$(32)\qquad \begin{aligned} &\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) + \frac{1}{H}\frac{\partial\sqrt{E}}{\partial w}\frac{\partial\sqrt{G}}{\partial w} = 0, \\ &\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{H}}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{1}{\sqrt{H}}\frac{\partial\sqrt{G}}{\partial w}\right) + \frac{1}{E}\frac{\partial\sqrt{H}}{\partial u}\frac{\partial\sqrt{G}}{\partial u} = 0, \\ &\frac{\partial}{\partial w}\left(\frac{1}{\sqrt{H}}\frac{\partial\sqrt{E}}{\partial w}\right) + \frac{1}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{H}}{\partial u}\right) + \frac{1}{G}\frac{\partial\sqrt{H}}{\partial v}\frac{\partial\sqrt{E}}{\partial v} = 0. \end{aligned}$$

When these values which can be got from (26), (27) and (30) are substituted in the equation (18), the first vanishes identically, likewise, the second, in consequence of the equations (31) and (32). Hence we get the following theorem of Ribaucour:[2]

1 See Lamé, Leçons sur les coordonnées curvilignes et leurs diverses applications, p. 73–79, 1859.

2 See Ribaucour, Comptes Rendus, Vol. LXX, p. 330–333, 1870.

Theorem. Given a family of surfaces of a triply orthogonal system and their orthogonal trajectories; the osculating circles to the latter at their point of meeting with any surface of the family form a cyclical system.

§ 8. A Sphere and a Cyclical System.

Let a sphere

$$(ax)^2=(aa)(xx)\cos^2\frac{R'}{k}$$

be orthogonal to a cyclical system of circles, and take the lines of curvature on the sphere (i. e. a system of great circles which are perpendicular to each other) as the parametric curves. Then the coordinates of the center of the circles of the cyclical system are given by (see § 6, (10))

$$a_i=x_i\cos\frac{R}{k}+(X_i\cos\omega+Y_i\sin\omega)\sin\frac{R}{k}$$

$$(i=0,\ 1,\ 2,\ 3).$$

As a tangent plane to a sphere is perpendicular to the diameter through the point of contact, the coordinates of the center of the sphere will be given by

$$a_i=x_i\cos\frac{R'}{k}+Z_i\sin\frac{R'}{k},\quad (i=0,\ 1,\ 2,\ 3)$$

where (x) is the coordinates of the point of contact, and (Z) that of the absolute pole of the tangent plane.

Hence, we get

$$(aa)=k^2\cos\frac{R}{k}\cos\frac{R'}{k},$$

and since the point (Z) i. e. the pole of the tangent plane to the sphere, lies on the plane of the circle, the center of the sphere (a) must lie on the planes of the circles.

From these results, we derive the following theorem:

Theorem. If a sphere be orthogonal to a cyclical system, then the planes of the circles of the system pass through the center of the sphere *and the sphere is orthogonal to every main sphere of the circles of the system.*

Conversely, we can prove the following theorem easily:

Theorem. If the planes of the circles of a cyclical system pass through a fixed point, then the circles are cut orthogonally be a sphere with its center at that point.

These properties of the circles and the spheres have been discussed already by Prof. T. Nishiuchi.[1]

In conclusion the author wishes to express his sincere thanks to Prof. T. Nishiuchi for his kind guidance and encouraging remarks.

---

1 See T. Nishiuchi, 'Oriented Circles in Non-Euclidean Space,' loc. cit.