Some Considerations on the Indeterminate Series

By

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(Received January 11, 1922.)

1. The present note is the continuation of the previous paper on some properties of the analytic elements on and outside their circles of convergence; but this paper is not necessarily related to the analytic continuations. In the previous paper I proved that a power-series (real) which is convergent for $0 \leq x < 1$ divides the interval $0 \leq x$, if possible, into three parts: 1°. It is convergent for $0 \leq x < 1$; 2°. infinite for $1 < x < x_0$ 3°. indeterminate for $x_0 < x$.[1] Now it is our problem whether x_0 may be unity, *i.e.* whether it gives a series which is convergent for $0 \leq x < 1$, and indeterminte for $1 < x$, but at $x = 1$, it is infinite. To verify this, we shall construct in the following a series which is really so.

2. Let the required series be

$$f(x) = a_0 + a_1 x + \ldots\ldots + a_n x^n + \ldots\ldots,$$

and put

$$S_n \equiv a_0 + a_1 + \ldots\ldots + a_n,$$
$$S'_n \equiv a_1 + 2a_2 + \ldots\ldots + na_n.$$

If S'_n be positive for any n, then since it is the differential quotient of the polynomial $a_0 + a_1 x + \ldots\ldots + a_n x^n$ at $x = 1$, this polynomial must be increasing at $x = 1$. It would be the same for $f(x)$

[1] Mem., Coll. Sci., Kyoto, **4**,**400**, (1921).

or $f(x)$ could not indeterminate in the neighbourhood of $x>1$. We have therefore to determine $a_0, a_1, \ldots\ldots, a_n, \ldots\ldots$ such that

$$\lim_{n\to\infty} S_n=\infty, \quad S'_n<0,$$

for certain increasing values of n. Since

$$S'_n=(S_1-S_0)+2(S_2-S_1)+\ldots\ldots+n(S_n-S_{n-1}),$$

we have

$$\frac{S'_n}{n+1}=S_n-\frac{S_0+S_1+\ldots\ldots+S_n}{n+1}. \tag{1}$$

Referring to this identity, take

$$\begin{array}{llll} S_0=0, & S_1=1, & S_2=2, & \\ S_{1+2}=1, & S_4=4, & S_5=5, & \\ S_{1+2+3}=2, & S_7=7, & S_8=8, & S_9=9, \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \end{array}$$

In general, if $n=1+2+3+\ldots\ldots+m=\frac{m(m+1)}{2}$,

$$S_n=m-1, \tag{2}$$

or else

$$S_n=n. \tag{3}$$

Then for $n=\frac{m(m+1)}{2}$, we have by easy calculations

$$S_0+S_1+\ldots\ldots+S_n=\frac{m^4}{8}+\frac{m^3}{12}+\frac{3m^2}{8}-\frac{7m}{12}+1. \tag{4}$$

Since for this value of n, $S_n=m-1$, (1) shows that S'_n tends to $-\infty$ when m tends to ∞. For the coefficients of the series we have

$$a_0=0,$$

$$\left.\begin{aligned} a_n=S_n-S_{n-1}&=-\frac{m(m-1)}{2}, \quad \text{for} \quad n=\frac{m(m+1)}{2}, \\ &=\frac{m^2-m+4}{2}; \quad \text{or} \quad n=\frac{m(m+1)}{2}+1. \end{aligned}\right\} \tag{5}$$

For the other cases, we have

$$a_n = 1. \tag{6}$$

Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = 1.$$

Now the series

$$f(x) = x + x^2 - x^3 + 3x^4 + x^5 - 3x^6 + 5x^7 + \quad \cdots \tag{7}$$

is what is required. For at first we have

$$\lim_{n\to\infty} S_n = f(1) = \infty.$$

Next we can easily verify that the series is indeterminate for any $x > 1$. To show that the series becomes positive by certain modes of summation, observe that for all $n = \frac{m(m+1)}{2}$, by (5),

$$a_n x^n + a_{n+1} x^{n+1} = \frac{1}{2} x^{\frac{m(m+1)}{2}} \{-m(m-1) + (m^2 - m + 4)x\}$$

$$= \frac{1}{2} x^{\frac{m(m+1)}{2}} \{m(m-1)(x-1) + 4x\} > 0$$

for any $x > 1$. The other terms of $f(x)$ are all positive. Therefore if we take in $f(x)$ the above sum as one term, then $f(x) \to \infty$, for any $x > 1$. To see that $f(x)$ becomes negative by certain modes of summation, consider the sums for all $n = \frac{m(m+1)}{2} + 1$, by (5) and (6),

$$T_n(x) \equiv a_n x^n + a_{n+1} x^{n+1} + \cdots\cdots + a_{\frac{(m+1)(m+2)}{2}} x^{\frac{(m+1)(m+2)}{2}}$$

$$= x^n \left\{ \frac{m^2 - m + 4}{2} + x + \cdots\cdots + x^{m-1} - \frac{m(m+1)}{2} x^m \right\}.$$

The terms within the blackets are equal to

$$\frac{m^2 - m + 2}{2} + \frac{x^m - 1}{x - 1} - \frac{m(m+1)}{2} x^m,$$

which is always negative for any $x > 1$, provided m is sufficiently great. Thus we can find N, such that $n \geqq N$ derives for each $x > 1$,

$$T_n(x) < 0.$$

By aid of this inequality, we sum up $f(x)$ in the following mode

$$a_0+a_1x+\ldots\ldots+a_{N-1}x^{N-1}+T_N(x)+T_{N+1}(x)+\ldots\ldots$$

Since $T_N(x)$, $T_{N+1}(x)$, ... are all negative, this series can not be ∞, moreover tends to $-\infty$. Thus *our series $f(x)$ is convergent for $0\leqq x<1$, ∞ at $x=1$, and indeterminate for $x>1$.*

3. We shall consider in the following sections some relations between $f(x)$ and $f'(x)$ at $x=1$, where as usual in the series

$$f(x)=a_0+a_1x+\ldots\ldots+a_nx^n+\ldots\ldots,$$

a_0, a_1, ..., a_n, ... are all real and $f(x)$ has the radius of convergence unity.

4. It is clear that when $f(1)$ is convergent, $f'(1)$ may converge, or be divergent. As a special case, we notice that Mr. Vivanti has given a series which is convergent with all its derivatives at $x=1$.[1] When $f(1)$ is ∞, $f'(1)$ may be ∞ or indeterminate. The former case is clear, while the latter is given by the series (7) found in the previous section. As we have said, for $n=\frac{m(m+1)}{2}$, $\lim\limits_{m\to\infty} S'_n=-\infty$. On the other hand by (1), (3), (4), we have

$$\frac{S'_{n+1}}{n+2}=\left\{\frac{m(m+1)}{2}+1\right\}-\frac{\frac{m^4}{8}+\ldots\ldots}{\frac{m(m+1)}{2}+2},$$

the dotted part stands for the lower degrees of m. Consequently

$$\lim_{m\to\infty} S'_{n+1}=\infty.$$

So that our series $f'(x)$ is indeterminate at $x=1$.

5. When $f(1)$ is indeterminate, $f'(1)$ is so also. We shall prove a more general proposition. Let us write

$$S\equiv a_0+a_1+\ldots\ldots+a_n+\ldots\ldots,$$

$$S_n\equiv a_0+a_1+\ldots\ldots+a_n,$$

$$S'\equiv\omega_0a_0+\omega_1a_1+\ldots\ldots+\omega_na_n+\ldots\ldots,$$

$$S'_n\equiv\omega_0a_0+\omega_1a_1+\ldots\ldots+\omega_na_n,$$

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1 Vivanti-Gutmer, Theorie der eindeutigen analytischen Funktionen, (1906), p. 413.

where $\omega_0, \omega_1, \ldots\ldots, \omega_n$ form a positive increasing sequence.

When S is indeterminate, S' must also be indeterminate. For, writing

$$\theta_i = \frac{1}{\omega_i}, \qquad i = o,\ 1,\ \ldots,\ n,\ \ldots,$$

we have

$$\theta_o \geqq \theta_1 \geqq \ldots\ldots \geqq \theta_n \geqq \ldots\ldots,$$

$$S_n = (\theta_o - \theta_1)S'_o + (\theta_1 - \theta_2)S'_1 + \ldots\ldots + (\theta_{n-1} - \theta_n)S'_{n-1} + \theta_n S'_n. \quad (8)$$

If $\lim_{n \to \infty} S'_n = \infty$, since for any integer $i \geqq o$, $\theta_i - \theta_{i+1} \geqq o$, $\theta_i > o$, the right-hand side of (8) must increase with n; hence S must be convergent or ∞. It is contrary to the assumption. If for any $n > N$, $|S'_n| < \varepsilon$, by Abel's inequality

$$| a_n + a_{n+1} + \ldots\ldots + a_{n+p} | \leqq 2\varepsilon\theta_n.$$

Therefore S becomes convergent. It is also absurd. *Q.E.D.*

Taking $\omega_i = i$,[1] *if a power-series $f(x) = a_o + a_1 x + \ldots\ldots a_n x^n + \ldots\ldots$ be indeterminate at $x = 1$, then $f'(1)$ must also be indeterminate and hence it can not be limited in both ways.*

As an example, since

$$\sin\alpha + \sin 2\alpha + \ldots\ldots + \sin n\alpha + \ldots\ldots$$

is indeterminate, the indeterminate series

$$\sin\alpha + 2\sin 2\alpha + \ldots\ldots + n\sin n\alpha + \ldots\ldots$$

can not be limited in both ways, since the power-series

$$x\sin\alpha + x^2\sin 2\alpha + \ldots\ldots + x^n\sin n\alpha + \ldots\ldots$$

has the convergent-radius unity.

6. The relations of the oscillation-abscissa of S' with respect to that of S is very important. But it is in general extremely difficult to find. We shall merely prove that if $\frac{S'_n}{n}$ is limited in a positive way (*i.e.*, it is less than a number G for any n), then $\frac{S_n}{\log n}$ is also limited in a positive way, where

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[1] When $\omega_0 = o$, we consider $S - a_0$ instead of S.

$$S^n = a_0 + a_1 + \ldots\ldots a_n,$$
$$S'_n = \quad a_1 + \ldots\ldots + na_n.$$

For this, we use the identity (8) for $\omega_i = i$:

$$S_n - a_0 = \left(1 - \frac{1}{2}\right)S'_1 + \ldots\ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)S'_{n-1} + \frac{1}{n}S'_n$$

Since $\frac{S'_i}{i} < G$, $i = 1, 2, \ldots, n, \ldots$, we have

$$S_n - a_0 < \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right)G.$$

We know that

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}}{\log n} = 1.$$

Therefore we can find a number G' such that for all n

$$\frac{S_n}{\log_n} < G'. \qquad Q.E.D.$$

To obtain deeper relations, we have probably to use more transcendental analytical arms or add another condition. Tauber's problem is an allied one. He proved that if $S'_n = o(n)$, $\lim_{x\to 1} f(x) = o$, then $S = o$. Messrs. Hardy and Littlewood extended this theorem.[1] They also proved another, in which instead of $f(x)$, Borel's generalised sum of S was used. It was extended by Mr. Fujiwara.[2]

7. When $f(x) = a_0 + a_1x + \ldots\ldots + a_nx^n + \ldots\ldots$ may be continued over $x = 1$, may $f_\omega(x) \equiv \omega_0 a_0 + \omega_1 a_1 x + \ldots\ldots + \omega_n a_n x^n + \ldots\ldots$ also be continued over $x = 1$? $\omega_0, \omega_1, \ldots, \omega_n, \ldots$ are the same quantities defined above. In the special case where $\omega_0 = o$, $\omega_n = n$, *i.e.*, in the case of analytic functions, the answer is clearly affirmative. But in general it is not the case and the problem is related to Hadamard's theorem. For if $x = -1$ be a singular point of $f(x)$ and of the function defined by

$$\Omega(x) \equiv \omega_0 + \omega_1 x + \ldots\ldots + \omega_n x^n + \ldots\ldots,$$

1 Landau, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, (1916; J. Springer).

2 Tôhoku Math. J., **17**, Nos. 3, 4, (1920).

and even when $f(x)$ may be continued over $x=1$, the point $x=1$ is a singular point of the analytic function defined by $f_\omega(x)$. Therefore given a series $f(x)$ which may be continued over $x=1$, we may divide all the positive increasing sequences into two parts in one of which all the sequences for which $f_\omega(x)$ may be contined over $x=1$, are classified; while in the other, all the sequences for which $f_\omega(x)$ may not be continued. Then if $f_\omega(x)$ and $f_{\omega'}(x)$, corresponding to the sequences

$$\omega_0, \omega_1, \ldots\ldots, \omega_n, \ldots\ldots,$$

$$\omega'_0, \omega'_1, \ldots\ldots, \omega'_n, \ldots\ldots$$

respectively may be continued, the series $f_{\omega+\omega'}(x)$ corresponding to the sequence $\omega_0+\omega_0, \omega_1+\omega'_1, \ldots\ldots, \omega_n+\omega'_n, \ldots\ldots$ may also be continued. On the other hand consider a sequence $\varepsilon_0, \varepsilon_1, \ldots\ldots, \varepsilon_n, \ldots\ldots,$ tending to zero. Even when $f_\omega(x)$ may be continued, yet it may occur that $f_{\omega+\varepsilon}(x)$ may not be continued. For if otherwise, $f_\varepsilon(x)$ must be continued over $x=1$. This is in general untrue.

We will not go further since probably such considerations can not throw light on the indeterminate series, though these considerations themselves are interesting. Here the researches of Borel and of Pincherle should be referred to.[1]

8. Given a limited function $F(x)$ defined in an interval, Darboux established the so-called *intégrales par excès* and *par défaut*. To these integrals, it seems to correspond the superior and inferior limits of the indeterminate series which are limited.

[1] See *e.g.*, Montel, Leçons sur les séries de polynomes, (1910), p. 37.