On the Congruence Conjugate to a Surface.

By

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Introduction.

Darboux has proved the following theorem.[1]

When, passing through all the straight lines in a fixed plane, tangent planes to two surfaces (S), (S_1) are drawn, the straight lines joining the contact points form a congruence whose developables cut both surfaces (S) and (S_1) in a conjugate net.

Let us now suppose that a surface (S) with a conjugate net on it is given and consider the surface (S_1) which is in the following relation to (S).

1° There is a congruence whose developables cut (S) in given conjugate net and (S_1) also in a conjugate net.

2° Intersections of the tangent planes of (S) and (S_1) at the corresponding points describe a plane.

We shall say that such surfaces as (S_1) are in the relation (D) to (S).

We shall determine all the surfaces which are in the relation (D) to (S) and investigate their geometrical properties.

We assume the given surfaces is not developable surface and take as the parameter-curves the given conjugate net on (S).

Green has proved that in this case the homogeneous coordinates (x) of the points on (S) are a fundamental system of the solutions of a

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[1] Darboux, "Theorie générale des surfaces" II. Chap. X.

completely integrable system of partial differential equations of the second order.[1]

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial u^2} &= a\frac{\partial^2 x}{\partial v^2}+b\frac{\partial x}{\partial u}+c\frac{\partial x}{\partial v}+dx\\ \frac{\partial^2 x}{\partial u\partial v} &= \qquad\quad b_1\frac{\partial x}{\partial u}+c_1\frac{\partial x}{\partial v}+d_1 x\end{aligned}\right\}\qquad(\mathrm{I})$$

where a, b, c, d, b_1, c_2 and d_1 are functions of u, v.

We shall call, hereafter, (I) the equations of the given surface.

The higher derivatives of (x) can be expressed by linear combinations of $\frac{\partial^2 x}{\partial v^2}$, $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$ and x by (I)

In particular, we have

$$\frac{\partial^3 x}{\partial u\partial v^2}=a_2\frac{\partial^2 x}{\partial v^2}+b_2\frac{\partial x}{\partial u}+c_2\frac{\partial x}{\partial v}+d_2 x$$

$$\frac{\partial^3 x}{\partial v^3}=a_3\frac{\partial^2 x}{\partial v^2}+b_3\frac{\partial x}{\partial v}+c_3\frac{\partial x}{\partial v}+d_3 x$$

where

$$a_2=c_1,\quad b_2=b_1^2+\frac{\partial b_1}{\partial v},\quad c_2=b_1c_1+\frac{\partial c_1}{\partial v}+d_1$$

$$d_2=b_1d_1+\frac{\partial d_1}{\partial v},\quad a_3=\frac{1}{a}\left(ab_1-c-\frac{\partial a}{\partial v}\right),\quad b_3=\frac{1}{a}\left(b_1c_1+\frac{\partial b_1}{\partial u}-\frac{\partial b}{\partial v}+d_1\right)$$

$$c_3=\frac{1}{a}\left\{b_1c+c_1(c_1-b)+\frac{\partial c_1}{\partial u}-\frac{\partial c}{\partial v}-d\right\}$$

$$d_3=\frac{1}{a}\left\{b_1d+d_1(c_1-b)+\frac{\partial d_1}{\partial u}-\frac{\partial d}{\partial v}\right)$$

The functions $a, b, c, d, b_1, c_1, d_1$, are restricted by conditions of the integrability one of which is

$$\frac{\partial}{\partial v}(b+2c_1)=\frac{\partial}{\partial u}(a_3+b_1).$$

Hence we can find a function p such that

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[1] Amer. J. Math., **37**, 215–246 (1915).

$$b+2c_1=\frac{\partial p}{\partial u},\qquad a_1+b_1=\frac{\partial p}{\partial v}$$

As (x) are the fundamental system of solutions of (I), the general solution of (I) is

$$x=C_1x_1+C_2x_2+C_3x_3+C_4x_4$$

where C_1, C_2, C_3, C_4, are arbitarily constants.

Therefore we know that all the surfaces whose equation are (I), are the projective transforms of (S).

Denote by B and C the functions

$$B=b_1-\frac{1}{4}\frac{\partial p}{\partial v}-\frac{3}{8a}\frac{\partial a}{\partial v}$$

$$C=c_1-\frac{1}{4}\frac{\partial p}{\partial v}+\frac{1}{8a}\frac{\varrho a}{\partial u}$$

Then the functions a, B and C are invariant of the transformation

$$\bar{x}=\lambda(u_1v)x.$$

which do not alter, evidently, the surface nor the parameter-curves.

§ I. A congruence Γ whose developables correspond to given conjugate net.

Consider two points (ρ), (σ) such that

$$(\rho)=\left(\frac{\partial x}{\partial u}\right)+\mu(x)$$

$$(\sigma)=\left(\frac{\partial x}{\partial v}\right)+\lambda(x)$$

The points (ρ), (σ) evidently lie on the tangent lines at (x) of the curves C_u (v=const.), and C_v (u=const.) respectively which meet at (x).

When (x) moves on the given surface, the lines (ρ) (σ) form a congruence which is called by Green conguence Γ.

If the lines (ρ) (σ) form a developable surface when (x) moves along a curve C_u, the four point $\left(\frac{\partial \rho}{\partial u}\right)$, $\left(\frac{\partial \sigma}{\partial u}\right)$, (ρ) and (σ) must lie in a plane.

But we have

$$\left(\frac{\partial\rho}{\partial u}\right)=\left(\frac{\partial^2 x}{\partial u^2}\right)+u\left(\frac{\partial x}{\partial u}\right)+\frac{\partial\mu}{\partial u}(x),$$

$$\left(\frac{\partial\sigma}{\partial u}\right)= \qquad \left(b_1+\lambda\right)\left(\frac{\partial x}{\partial u}\right)+c_1\left(\frac{\partial x}{\partial v}\right)+\left(d_1+\frac{\partial\lambda}{\partial v}\right)(x).$$

Hence the points $\left(\frac{\partial\sigma}{\partial u}\right)$, (ρ), (σ) are in the tangent plane at (x), while the point $\left(\frac{\partial\rho}{\partial u}\right)$ is not on the tangent plane, for the point $\left(\frac{\partial^2 x}{\partial u^2}\right)$ does not lie in the tangent plane unless the surface (S) is developable.

Therefore if the points $\left(\frac{\partial\rho}{\partial u}\right)$, $\left(\frac{\partial\sigma}{\partial u}\right)$, (ρ), (σ) lie in a plane, the three points $\left(\frac{\partial\sigma}{\partial u}\right)$, (ρ), (σ) must lie in a straight line, that is, $\left(\frac{\partial\sigma}{\partial u}\right)$ must be of the form

$$\omega_1(\rho)+\omega_2(\sigma)$$

Eliminating ω_1 and ω_2 we have

$$\frac{\partial\lambda}{\partial u}-\lambda\mu-c_1\lambda-b_1\mu+d_1=o$$

Similarly if the lines (ρ) (σ) describe a developable surface when (x) moves along a curve C_v, we must have

$$\frac{\partial\mu}{\partial v}-\lambda\mu-c_1\lambda-b_1\mu+d_1=o$$

Hence we know that the developables of the congruence formed by lines (ρ) (σ) correspond to the given conjugate net, if, and only if

$$\frac{\partial\lambda}{\partial u}=\frac{\partial u}{\partial v}$$

$$\frac{\partial^2\theta}{\partial u\partial v}=b_1\frac{\partial\theta}{\partial u}+c_1\frac{\partial\theta}{\partial v}+d_1\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

where θ is a function such that

$$\lambda=-\frac{1}{\theta}\frac{\partial\theta}{\partial v}, \qquad \mu=-\frac{1}{\theta}\frac{\partial\theta}{\partial u}.$$

Hence the points (ρ), (σ) which satisfy the above conditions may be expressed in the form :

$$\left.\begin{aligned}(\rho)&=\theta\left(\frac{\partial x}{\partial u}\right)-\frac{\partial\theta}{\partial u}(x)\\(\sigma)&=\theta\left(\frac{\partial x}{\partial v}\right)-\frac{\partial\theta}{\partial v}(x)\end{aligned}\right\}\text{.........................(I)}$$

where θ is a solution of the partial differential equation of the second order (II).

In this case the points (ρ), (σ) are focal points, for we have by differentiation

$$\left(\frac{\partial\rho}{\partial v}\right)=\left(b_1+\frac{1}{\theta}\frac{\partial\theta}{\partial v}\right)(\rho)+\left(c_1-\frac{1}{\theta}\frac{\partial\theta}{\partial u}\right)(\sigma)$$

$$\left(\frac{\partial\sigma}{\partial u}\right)=\left(b_1-\frac{1}{\theta}\frac{\partial\theta}{\partial v}\right)(\rho)+\left(c_1+\frac{1}{\theta}\frac{\partial\theta}{\partial u}\right)(\sigma)$$

We shall nextly find the condition that the line (ρ) (σ) given by (I) may describe a plane when (x) moves on the given surface (S).

If the lines (ρ) (σ) describe a plane the four points (σ), (ρ), $\left(\frac{\partial\sigma}{\partial v}\right)$, $\left(\frac{\partial\rho}{\partial u}\right)$ must necessarily lie in a plane.

But we have

$$\left(\frac{\partial\sigma}{\partial v}\right)=\theta\left(\frac{\partial^2 x}{\partial v^2}\right)-\frac{\partial^2\theta}{\partial v^2}(x)$$

$$\left(\frac{\partial\rho}{\partial u}\right)=a\theta\left(\frac{\partial^2 x}{\partial v^2}\right)+b\theta\left(\frac{\partial x}{\partial u}\right)+c\theta\frac{\partial x}{\partial v}+\left(d\theta-\frac{\partial^2\theta}{\partial u^2}\right)(x)$$

Therefore if the points (σ), (ρ), $\left(\frac{\partial\sigma}{\partial v}\right)$, $\left(\frac{\partial\rho}{\partial u}\right)$ lie in a plane we must have

$$\begin{vmatrix} 0 & \theta & 0 & -\dfrac{\partial\theta}{\partial u} \\ 0 & 0 & \theta & -\dfrac{\partial\theta}{\partial v} \\ \theta & 0 & 0 & -\dfrac{\partial^2\theta}{\partial v^2} \\ a\theta & b\theta & c\theta & d\theta-\dfrac{\partial^2\theta}{\partial u^2} \end{vmatrix}=0,$$

or

$$\frac{\partial^2\theta}{\partial u}=a\frac{\partial^2\theta}{\partial v^2}+b\frac{\partial\theta}{\partial u}+c\frac{\partial\theta}{\partial v}+d\theta,$$

that is, θ must be a solution of the system of simultaneous partial differential equations (I).

The condition is sufficient

For θ being a solution of (I), is of the form

$$\theta=C_1x_1+C_2x_2+C_3x_3+C_4x_4.$$

Substituting this value of θ in (1) we know that both (ρ) and (σ) satisfy the equation

$$C_1z_1+C_2z_2+C_3z_3+C_4z_4=o$$

that is, the line (ρ) (σ) describes a plane

§ 2. Congruences conjugate to the surface.

As the point $\left(\frac{\partial^2 x}{\partial v^2}\right)$ is out of the tangent plane at (x), any point can be expressen in the form

$$\omega_1\left(\frac{\partial^2 x}{\partial v^2}\right)+\omega_2\left(\frac{\partial x}{\partial u}\right)+\omega_3\left(\frac{\partial x}{\partial v}\right)+\omega_4(x).$$

Therefore any line passing through the point (x) and is not on the tangent plane may be regarded as the line which passes through the point (x) and through the point (z) which is of the form

$$(z)=\left(\frac{\partial^2 x}{\partial v^2}\right)+\lambda\left(\frac{\partial\lambda}{\partial u}\right)+\mu\left(\frac{\partial x}{\partial v}\right).$$

If λ and μ are given as the functions of u, v the lines (x) (z) form a congruence when the point (x) moves on the surface (S).

Now we shall determine λ and μ so that the developable of the congruence cut the surface in the curves C_u and C_v.

If the lines (x) (z) form a develpable when (x) moves along the curves C_u four points (x), (z), $\left(\frac{\partial x}{\partial u}\right)$, $\left(\frac{\partial z}{\partial u}\right)$ must lie in a plane, that is, $\left(\frac{\partial z}{\partial u}\right)$ must be of the form

$$\alpha(z)+\beta\left(\frac{\partial x}{\partial u}\right)+\gamma(x),$$

or of the form

$$\alpha\left(\frac{\partial^2 x}{\partial v^2}\right)+\beta'\left(\frac{\partial x}{\partial u}\right)+\alpha\mu\left(\frac{\partial x}{\partial v}\right)+\gamma(x).$$

But we have by differentiation

$$\left(\frac{\partial z}{\partial u}\right)=(a\lambda+a_2)\frac{\partial^2 x}{\partial v^2}+\left(b\lambda+b_1\mu+b_2+\frac{\partial\lambda}{\partial u}\right)\left(\frac{\partial x}{\partial u}\right)$$
$$+\left(c\lambda+c_1\mu+c_2+\frac{\partial\mu}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)$$
$$+\left(d\lambda+d_1\mu+d_2\qquad\right)(x).$$

Therefore we must have

$$(a\lambda+a_2)\mu=c\lambda+c_1\mu+c_2+\frac{\partial\mu}{\partial u}$$

or

$$\frac{\partial\mu}{\partial u}-a\lambda u-\lambda c+c_2=o.$$

Similarly, if the lines (x) (z) form a developable surface when (x) moves along a curve C_v, we must have

$$\frac{\partial\lambda}{\partial v}-\lambda u+\lambda(b_1-a_3)+b_3=o.$$

Therefore, if the developables of the congruence formed by the lines (x) (z) cut the surface (S) in curves C_u and C_v, we must have

$$\frac{\partial\mu}{\partial u}-a\lambda u+\lambda c+c_2=o$$

$$\frac{\partial\lambda}{\partial v}-\lambda u+\lambda(b_1-a_3)+b_3=o.$$

Multiplying the second by $-a$ and adding, we have

$$\frac{\partial}{\partial u}(u-b_1)=\frac{\partial}{\partial v}(a\lambda-b-c_1).$$

But there is the relation

$$\frac{\partial}{\partial v}(b+2c_1)=\frac{\partial}{\partial u}(a_3+b_1)$$

Hence we have

$$\frac{\partial}{\partial u}(u+a_3)=\frac{\partial}{\partial v}(a\lambda+c_1).$$

This shows that there exists a function θ of u, v such that

$$a\lambda+c_1=-\frac{1}{\theta}\frac{\partial\theta}{\partial u}$$

$$u+a_3=-\frac{1}{\theta}\frac{\partial\theta}{\partial v}.$$

Substituting these values of λ, u in any one of (2), we have

$$\frac{\partial^2\theta}{\partial u\partial v}=-\left(b_1-\frac{1}{a}\frac{\partial a}{\partial v}\right)\frac{\partial\theta}{\partial u}-c_1\frac{\partial\theta}{\partial v}$$

$$+\left(d_1+\frac{\partial b_1}{\partial u}+\frac{\partial c_1}{\partial v}-\frac{\partial^2 p}{\partial u\partial v}+\frac{c_1}{a}\frac{\partial a}{\partial v}\right)\theta. \quad \ldots\ldots\text{(III)}$$

Hence we know that any congruence whose developables cut the surfaces (S) in the system of the curves C_u and C_v is that which is formed by the lines which pass through the point (x) and through the point (z) such that

$$(z)=\theta\left(\frac{\partial^2 x}{\partial v^2}\right)-\frac{1}{a}\left(\frac{\partial\theta}{\partial u}+c_1\theta\right)\left(\frac{\partial x}{\partial u}\right)-\left(\frac{\partial\theta}{\partial v}+a_3\theta\right)\left(\frac{\partial x}{\partial v}\right)\ldots\ldots(3)$$

where θ is a solution of the partial differential equation of the second order (III).

§ 3. Focal surfaces of the congruences conjugate to the given surface.

By differentiation of (z) given by (3), we know in virtue of (III)

$$\left.\begin{aligned}\left(\frac{\partial z}{\partial u}\right)&=\xi\left(\frac{\partial x}{\partial u}\right)+\eta(x)\\ \left(\frac{\partial z}{\partial v}\right)&=\xi'\left(\frac{\partial x}{\partial v}\right)+\eta'(x).\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where

$$\xi=-\frac{1}{a}\frac{\partial^2\theta}{\partial u^2}-\frac{1}{a}\left(b+c_1-\frac{1}{a}\frac{\partial a}{\partial u}\right)\frac{\partial\theta}{\partial u}-b_1\frac{\partial\theta}{\partial v}$$

$$+\left(b_2-\frac{bc_1}{a}-\frac{\partial}{\partial u}\left(\frac{c_1}{a}\right)-b_1a_3\right)\theta$$

$$\xi' = -\frac{\partial^2\theta}{\partial v^2} - \frac{c_1}{a}\frac{\partial\theta}{\partial u} - a_3\frac{\partial\theta}{\partial v} + \left(c_3 - \frac{c_1^2}{a} - \frac{\partial a_3}{\partial v}\right)\theta$$

$$\eta = -\frac{d}{a}\frac{\partial\theta}{\partial u} - d_1 + \frac{\partial\theta}{\partial v} + \left(d_2 - \frac{c_1 d}{a} - d_1 a_3\right)\theta$$

$$\eta' = -\frac{d_1}{a}\frac{\partial\theta}{\partial u} + \left(d_3 - \frac{d_1 c_1}{a}\right)\theta$$

Eliminating (z) from (4), we have

$$\left.\begin{aligned} -\frac{\partial\xi}{\partial v} + \eta' &= b_1(\xi - \xi') \\ +\frac{\partial\xi}{\partial u} - \eta &= c_1(\xi - \xi') \\ \frac{\partial\eta}{\partial v} - \frac{\partial\eta'}{\partial u} &= d_1(\xi - \xi') \end{aligned}\right\} \qquad \cdots\cdots(5)$$

If we put

$$\zeta = \xi - \xi'$$

We know from (5)

$$\frac{\partial^2\zeta}{\partial u \partial v} = -b_1\frac{\partial\zeta}{\partial u} - c_1\frac{\partial\zeta}{\partial v} + \left(d_1 - \frac{\partial b_1}{\partial u} - \frac{\partial c_1}{\partial v}\right)\zeta \qquad \cdots\cdots(6)$$

that is, ζ is a solution of the partial differential equation adjointed to (II).

Any point (ρ) on the line (x) (z) is given by the equation

$$(\rho) = (z) + \lambda(x).$$

From (4) we have

$$\left(\frac{\partial\rho}{\partial u}\right) = (\xi + \lambda)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial\lambda}{\partial u} + \eta\right)(x)$$

$$\left(\frac{\partial\rho}{\partial v}\right) = (\xi' + \lambda)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial\lambda}{\partial v} + \eta'\right)(x)$$

If the point (ρ) is a focal point on the line (x) (z), one of the $\left(\frac{\partial\rho}{\partial u}\right)$ and $\left(\frac{\partial\rho}{\partial v}\right)$ must be of the form

$$\alpha(x)+\beta(z)$$

or

$$\lambda=\xi \quad \text{or} \quad \lambda=\xi'.$$

Hence two focal points on the line (x) (z) are given by

$$(\omega)=(z)-\xi(x)$$
$$(\nu)=(z)-\xi'(x).$$

From (4) and (5) we have

$$\left.\begin{aligned}
&\left(\frac{\partial\omega}{\partial u}\right)=-\left(\frac{\partial\zeta}{\partial u}+c_1\zeta\right)(x)\\
&\left(\frac{\partial\omega}{\partial v}\right)=-\zeta\left(\frac{\partial x}{\partial v}\right)+b_1\zeta(x)\\
&\left(\frac{\partial\nu}{\partial\omega}\right)=\zeta\frac{\partial x}{\partial\omega}-c_1\zeta(x)\\
&\left(\frac{\partial\nu}{\partial v}\right)=\left(\frac{\partial\zeta}{\partial v}+b_1\zeta\right)(x).
\end{aligned}\right\}\quad\cdots\cdots\cdots(7)$$

The function ζ is a function of θ and its derivation to the second order with respect to u and also to v and satisfies a partial differential equation of the second order (6). Therefore we know that the equation $\zeta=o$ is satisfied by four particular solutions of (II).[1]

If we take one of those solitions, we have

$$\zeta=o\,; \quad \xi=\xi',$$

that is,

$$\omega=\nu$$

$$\frac{\partial\omega}{\partial u}=\frac{\partial\omega}{\partial v}=\frac{\partial\nu}{\partial u}=\frac{\partial\nu}{\partial v}=o.$$

Hence ω is independent of u and v, that is; the lines of the congruences corresponding to these solutions pass through a fixed point.

Returning to the general case eliminate (x) from the last two equations of (7) then we know ν is a solution of the partial differential equation of the second order

[1] Darboux, loc cit. II, p. 174.

$$\frac{\partial^2\Theta}{\partial u\partial v}=\left(b_1+\frac{1}{\zeta}\frac{\partial\zeta}{\partial v}\right)\frac{\partial\Theta}{\partial u}$$

$$+\left\{c_1+\frac{1}{b_1\zeta+\frac{\partial\zeta}{\partial v}}\frac{\partial}{\partial u}\left(b_1\zeta+\frac{\partial\zeta}{\partial v}\right)\right\}\frac{\partial\Theta}{\partial v}\quad\text{.........(IV)}$$

Reciprocally, if we take any solution Θ of (IV), the function θ given by

$$\theta=\frac{1}{b_1\zeta+\frac{\partial\zeta}{\partial v}}\frac{\partial\Theta}{\partial v}\quad\text{................................(V)}$$

is a solution of (II).

For differentiating the equation

$$\frac{\partial\Theta}{\partial v}=\theta\left(b_1\zeta+\frac{\partial\zeta}{\partial v}\right)$$

by u, we have by (IV).

$$\frac{\partial\theta}{\partial u}=\frac{1}{\zeta}\frac{\partial\Theta}{\partial u}+c_1\theta\quad\text{..................................(8)}$$

Nextly differntiating this equation by v, we have by (V), (6) and (8)

$$\frac{\partial^2\theta}{\partial uuv}=b_1\frac{\partial\theta}{\partial u}+c_1\frac{\partial\theta}{\partial v}+d_1\theta.$$

From (V) and (9) we know that there are between θ and Θ the relation

$$\left.\begin{aligned}\frac{\partial\Theta}{\partial u}&=\zeta\left(\frac{\partial\theta}{\partial u}-c_1\theta\right)\\ \frac{\partial\Theta}{\partial v}&=\theta\left(\frac{\partial\zeta}{\partial v}+b_1\zeta\right)\end{aligned}\right\}\quad\text{......................(VI)}$$

In the above discussion we have assumed

$$\frac{\partial\zeta}{\partial v}+b_1\zeta\neq o$$

But, if

$$\frac{\partial\zeta}{\partial v}+b\zeta=o$$

combining this with

$$o=\frac{\partial}{\partial u}\left(\frac{\partial \zeta}{\partial v}+b_1\zeta\right)=-c_1\frac{\partial \zeta}{\partial v}+\left(d_1-\frac{\partial c_1}{\partial v}\right)\zeta.$$

we have

$$d_1+b_1c_1-\frac{\partial c_1}{\partial v}=o$$

In this case Darboux-Laplace transform of (S) must be reduced to a curve or a developable.

§ 4. Surfaces which are in the relation (D) to the given surface.

Any point (ρ) on the line (x) (z), being also a point on the line (x) (ν), can be expressed in the form

$$(\rho)=(\nu)+\lambda(x)=(\nu)+\frac{\lambda}{b_1\zeta+\frac{\partial \zeta}{\partial v}}\left(\frac{\partial \nu}{\partial \zeta}\right)$$

But the surface described by (ν) is one of the focal surfaces of the congruence. When the developables of this congruence cut the surface which is described by (ρ) in a conjugate net, (ρ) must satisfy the partial differential equation of the second order of the form.

$$\frac{\partial^2 \rho}{\partial u \partial v}=p\frac{\partial \rho}{\partial u}+q\frac{\partial \rho}{\partial v}+r\rho$$

From this follows that λ must satisfy the equation

$$\frac{\lambda}{b_1\zeta+\frac{\partial \zeta}{\partial v}}=-\frac{\theta}{\frac{\partial \theta}{\partial v}} \quad \text{or} \quad \lambda=-\theta\frac{b_1\zeta+\frac{\partial \zeta}{\partial v}}{\frac{\partial \theta}{\partial v}}$$

where θ is a solution of (IV).[1]

But we have proved that $\dfrac{\frac{\partial \theta}{\partial v}}{b_1\zeta+\frac{\partial \zeta}{\partial v}}$ is a solution of (II).

Hence we know that all the surfaces such that developables of the same congruence cut them in a conjugate net and the given surface in the given conjugate net are given by

$$(\rho)=(\nu)-\frac{\theta}{\theta}(x)$$

1 Darboux, loc. cit. Vol. II. p. 222.

where Θ is a solution of (IV) and θ is a solution of (II) and Θ and θ are connected by (V).

But when $\zeta=o$, as noticed in § 3 $\nu=$const., that is, the surface described by (ρ) is reduced to a point.

In this case we take as the form of (ρ)

$$(\rho)=\lambda(\nu)+(x) \quad\cdots\cdots\cdots\cdots (10)$$

from which we have

$$\frac{\partial^2\rho}{\partial u\partial v}=b_1\frac{\partial\rho}{\partial u}+c_1\frac{\partial\rho}{\partial v}+d_1\rho+\left(\frac{\partial^2\lambda}{\partial u\partial v}-b_1\frac{\partial\lambda}{\partial u}-c_1\frac{\partial\lambda}{\partial v}-d_1\lambda\right)\nu$$

When the surface described by (ρ) satisfies our condition, we must have

$$\frac{\partial^2\lambda}{\partial u\partial v}=b_1\frac{\partial\lambda}{\partial u}+c_1\frac{\partial\lambda}{\partial v}+d_1\lambda.$$

Therefore also in this case, dividing (10) by λ, (ρ) can be expressed in the form

$$(\rho)=(\nu)+\frac{(x)}{\theta}$$

where θ is a solution of (II).

By differentiation from (9), we have

$$\left.\begin{aligned}\left(\frac{\partial\rho}{\partial u}\right)&=(\zeta\theta-\Theta)\frac{\partial}{\partial u}\left\{\frac{(x)}{\theta}\right\}\\ \left(\frac{\partial\rho}{\partial v}\right)&=-\Theta\frac{\partial}{\partial v}\left\{\frac{(y)}{\theta}\right\}.\end{aligned}\right\}\quad\cdots\cdots\cdots (11)$$

Those formulas are given by Darboux by another method.

The points $\left(\frac{\partial\rho}{\partial u}\right)$, $\left(\frac{\partial\rho}{\partial v}\right)$ are on the tangent plane at (ρ) to its surface.

The points $\frac{\partial}{\partial u}\left\{\frac{(x)}{\theta}\right\}$, $\frac{\partial}{\partial v}\left\{\frac{(x)}{\theta}\right\}$ are on the tangent plane at (x) to the given surface.

But the equation (11) shows that the points $\left(\frac{\partial\rho}{\partial u}\right)$, $\left(\frac{\partial\rho}{\partial v}\right)$ coincide respectively with the points $\frac{\partial}{\partial u}\left\{\frac{(x)}{\theta}\right\}$, $\frac{\partial}{\partial v}\left\{\frac{(x)}{\theta}\right\}$.

Hence the line joining the points $\frac{\partial}{\partial u}\left\{\frac{(x)}{\theta}\right\}$, $\frac{\partial}{\partial v}\left\{\frac{(x)}{\theta}\right\}$ is the intersection of the corresponding tangent planes to the given surface and to a surface given by (9).

But we have shown in § 1 that when the line joining the points $\frac{\partial}{\partial u}\left\{\frac{(x)}{\theta}\right\}$, $\frac{\partial}{\partial v}\left\{\frac{(x)}{\theta}\right\}$ describes a plane as the point (x) moves on the given surfaces (S), θ must be a solution of the simultaneous partial differential equations (I).

Hence we know that surfaces which are in the relation (D) to the given surface are given by

$$(\rho)=\nu-\frac{\theta}{\theta}(x).$$

where θ is a solution of the simultaneous partial differential equation (I).

Put

$$x=\theta y \qquad \text{...(12)}$$

Then y satisfies also a system of the simultaneous partial differential equations of the form (I). These equations, being satisfied by $y=1$ (which corresponds to the solution $x=\theta$ of the original equations), must be of the form

$$\frac{\partial^2 y}{\partial u^2}=a\frac{\partial^2 y}{\partial v^2}+\beta\frac{\partial y}{\partial u}+\gamma\frac{\partial y}{\partial v}$$

$$\frac{\partial^2 y}{\partial u \partial v}=\beta_1\frac{\partial y}{\partial u}+c_1\frac{\partial y}{\partial v}.$$

As we have noticed in the introduction, the surface and also the parameter-curves are not altered by this transformation (12) and also the functions a, B, and C.

From (12) and (11) we have

$$\left(\frac{\partial\rho}{\partial u}\right)=\left(\zeta\theta-\theta\right)\left(\frac{\partial y}{\partial u}\right)$$

$$\left(\frac{\partial\rho}{\partial v}\right)=-\theta\left(\frac{\partial y}{\partial v}\right).$$

By differentiation of the above equation, we know that (ρ) satisfies the following system of the simultaneous partial differential equations of the second order which are completely integrable.

$$\frac{\partial^2\rho}{\partial u^2}=a\left(\frac{\theta-\zeta\theta}{\theta}\right)\frac{\partial^2\rho}{\partial v^2}+\left(\beta+\frac{\frac{\partial}{\partial u}(\theta-\zeta\theta)}{\theta-\zeta\theta}\right)\frac{\partial\rho}{\partial u}$$

$$+\frac{\theta-\zeta\theta}{\theta}\left(\gamma-\frac{a}{\theta}\frac{\partial\theta}{\partial v}\right)\frac{\partial\rho}{\partial v}\Bigg\}\text{..........(VII)}$$

$$\frac{\partial^2\rho}{\partial u\partial v}=\left\{\beta_1+\frac{\frac{\partial}{\partial v}(\theta-\zeta\theta)}{\theta-\zeta\theta}\right\}\frac{\partial\rho}{\partial u}+\left(\gamma_1+\frac{1}{\theta}\frac{\partial\theta}{\partial u}\right)\frac{\partial\rho}{\partial v}$$

If the asymptotic lines of the surface described by (ρ) to those of the given surface (ζ), we must have

$$a\frac{\theta-\zeta\theta}{\theta}=a$$

or

$$\zeta\theta=o$$

In this case the equations (VII) coincide with (I). This shows that the surface described by (ρ) is a projective transform of the given surface.

In fact, $\zeta=o$ is the case we have noticed in § 3. When $\theta=o$, the point (ρ) coincides with (x).

Hence we have the following theorem.

The asymptotic lines of a surface correspond to those of the surface which is in the relation (D) to it when and only when the latter is a projective transform of the former.

Let $\bar{a}$, $\bar{B}$ and $\bar{C}$ be the same functions as a, B and C respectively but formed with the coefficient of (VII).

Then we know that

$$\bar{a}=a\frac{\theta-\zeta\theta}{\theta}$$

$$\bar{B}=B+\frac{3}{8}\frac{\partial}{\partial v}\log(\theta-\zeta\theta)-\frac{1}{8}\frac{\partial}{\partial v}\log\theta$$

$$\bar{c}=c-\frac{1}{8}\frac{\partial}{\partial u}\log(\theta-\zeta\theta)+\frac{3}{8}\frac{\partial}{\partial u}\log\theta$$

from which follows

$$2\frac{\partial\bar{B}}{\partial u}-2\frac{\partial\bar{C}}{\partial v}-\frac{\partial^2}{\partial u\partial v}\log\bar{a}=2\frac{\partial B}{\partial u}-2\frac{\partial C}{\partial v}-\frac{\partial^2}{\partial u\partial v}\log a$$

We know easily

$$2\frac{\partial B}{\partial u}-2\frac{\partial c}{\partial v}-\frac{\partial^2}{\partial u\partial v}log\ a=-2\Big(H-K+\frac{\partial^2}{\partial u\partial v}log\ a\Big).$$

where H and K are Laplace-Darboux invariants, that is,

$$H=d_1+b_1c_1-\frac{\partial b_1}{\partial u}$$

$$K=d_1+b_1c_1-\frac{\partial c_1}{\partial v}$$

Wilczynski has called the line of the intersection of the osculating planes at (x) of the curves of the conjugate net on a surface (Σ) which meet at (x) the axis of the point (x) with respect to this conjugate net.

When the point (x) moves on the surface (Σ), the axes form a congruence which is called by him the axis congruence. The curves in which the developables of the axis congruence cut the surface (Σ) are called the axis curve.

Green[1] has proved that the condition that the axis curve with respect to the given conjugate net may also form a conjugate net is[1]

$$H-K+\frac{\partial^2}{\partial u\partial v}log\ a=o$$

Hence we have the following theorem.

If the axis curves with respect to the given conjugate net form a conjugate net, the same is the case for all the snrfaces which are in the relation (D) to it.

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1 Amer. J. Math., 38, (1916).