Oriented Circles in Non-Euclidean Space II.

By

Hidetoshi Kashiwagi.

(Received March 17, 1922.)

Introduction.

In the present paper we want to discuss some properties about the circles in non-euclidean space, other than those which were not touched in a previous memoir.[1] In that paper Study's coordinates[2] were used. To study the incidence and the orthogonality of the circles, the coordinates of circles in analogy to those of Plückers[3] in euclidean space may be used. Since the metric properties of the circles must be expressed by the relation between the radii of the circles, the coordinates of the centers and those .of the planse of the circles, in the present study, I think it convenient to apply Study's coordinates of the circles in non-euclidean space.

I.

First, we set up our coordinates system as follows:

$$x_0 = k\cos\frac{\rho}{k},$$

$$x_1 = k\sin\frac{\rho}{k}\cos\alpha,$$

1, 2 T. Nishiuchi and H. Kashiwagi, Oriented circles in Non-Euclidean Space, Mem. Coll. Sci., Kyoto, **4**, No. 6.

3 The first writer to use these seems to have been Stephanos, 'Sur une configuration remarquable de cercles dans l'espace,' C. R., **93**, 1881. See also Coolidge, 'Study of the circle cross,' Transac. Amer. Math. Soc., **14**.

$$x_2 = k \sin\frac{\rho}{k}\cos\beta,$$

$$x_3 = k \sin\frac{\rho}{k}\cos\gamma,$$

$$(xx) = x_0^2 + x_1^2 + x_2^2 + x_3^2 = k^2$$

with usual notations.[1]

A sphere in non-euclidean space may be given by the equations

$$(ax)^2 = k^4 \cos^2\frac{r}{k},$$

$$(aa) = (xx) = k^2$$

where (a) is the coordinates of the center of the sphere, r the radius of the sphere and $\frac{1}{k^2}$ the measure of curvature of the space.

According to the idea by which Sommerville[2] has treated the circle in non-euclidean plane, it is natural, for the equation of the sphere, to adopt the following linear one :

$$(ax) = k^2 \cos\frac{r}{k}.$$

Moreover I think it evident to take

$$\frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\cos\frac{r}{k}$$

as the coordinates of the sphere, where we take $\frac{a_0}{k} > o$ but $\frac{a_1}{k} > o$ provided $a_0 = o$ and so on. Let us adopt the following notation :

$$\frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\cos\frac{r}{k}$$

$$\equiv x_0,\ x_1,\ x_2,\ x_3,\ x_4$$

$$\equiv (x)$$

Let $(\bar{x})$ be the coordinate of the polar sphere of a sphere whose coordinates are (x), then

$$(\bar{x}) = \frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\sin\frac{r}{k}$$

---

1 Coolidge, Non-Euclidean Geometry.

2 Non-Euclidean Geometry.

and

$$(xx)+(\bar{x}\bar{x})=1$$

And the coordinates of the null-sphere satisfy the following condition:

$$(xx)=0.$$

If θ be the angle of intersection of the two spheres whose coordinates are (x) and (y) respectively, then

$$\cos\theta=\frac{(xy)}{\sqrt{(xx)}\sqrt{(yy)}}.$$

The condition of orthogonal intersection will be

$$(xy)=0.$$

Next, if we put

$$i\sqrt{x_0^2+x_1^2+x_2^2+x_3^2+x_4^2}=x_5$$

then x_0, x_1, x_2, x_3, x_4, x_5 will also be the another coordinates of the sphere and

$$x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0.$$

Thus the condition of contact will be

$$x_0y_0+x_1y_1+x_2y_2+x_3y_3+x_4y_4+x_5y_5=0,$$

Let (x) and (y) be the coordinates of two spheres then the coordinates of the spheres which pass through the circles of intersection of these spheres will be

$$\lambda(x)+\mu(y)$$

where λ and μ are parameters.

II.

We may consider a circle as the intersection of two spheres whose centers are mutually orthogonal and one of whose radius is $\frac{\pi}{2}k$, i.e. the intersection of a sphere and a plane passing through the center of the sphere.

Then the determinants of the matrix

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{Vmatrix}$$

where

$$x_0y_0+x_1y_1+x_2y_2+x_3y_3=o$$

$$y_4=o$$

define the Study's coordinates of the circle.

If

$$(x)=\frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\cos\frac{r}{k}$$

$$(y)=\frac{b_0}{k},\ \frac{b_1}{k},\ \frac{b_2}{k},\ \frac{b_3}{k},\ o,$$

then the Study's coordinates will be

$$\left\|\begin{matrix} \frac{a_0}{k} & \frac{a_1}{k} & \frac{a_2}{k} & \frac{a_3}{k} & i\cos\frac{r}{k} \\ \frac{b_0}{k} & \frac{b_1}{k} & \frac{b_2}{k} & \frac{b_3}{k} & o \end{matrix}\right\|$$

where

$$(ab)=o.$$

Hence, if we take $(\mathfrak{X}_0,\ \mathfrak{X}_1,\ \mathfrak{X}_2,\ \mathfrak{X}_3,\ \mathfrak{X}_{01},\ \mathfrak{X}_{02},\ \mathfrak{X}_{03},\ \mathfrak{X}_{23},\ \mathfrak{X}_{31},\ \mathfrak{X}_{12})$ $\equiv(\mathfrak{X})$ be the Study's coordinates of the circle, then[1]

$$\mathfrak{X}_0\equiv i\frac{b_0}{k}\cos\frac{r}{k},\quad \mathfrak{X}_1\equiv i\frac{b_1}{k}\cos\frac{r}{k},\quad \mathfrak{X}_2\equiv i\frac{b_2}{k}\cos\frac{r}{k},\quad \mathfrak{X}_3\equiv i\frac{b_3}{k}\cos\frac{r}{k},$$

$$\mathfrak{X}_{01}\equiv\begin{vmatrix} \frac{a_0}{k} & \frac{a_1}{k} \\ \frac{b_0}{k} & \frac{b_1}{k} \end{vmatrix},\quad \mathfrak{X}_{02}\equiv\begin{vmatrix} \frac{a_0}{k} & \frac{a_2}{k} \\ \frac{b_0}{k} & \frac{b_2}{k} \end{vmatrix},\quad \mathfrak{X}_{03}\equiv\begin{vmatrix} \frac{a_0}{k} & \frac{a_3}{k} \\ \frac{b_0}{k} & \frac{b_3}{k} \end{vmatrix},$$

$$\mathfrak{X}_{23}\equiv\begin{vmatrix} \frac{a_2}{k} & \frac{a_3}{k} \\ \frac{b_2}{k} & \frac{b_3}{k} \end{vmatrix},\quad \mathfrak{X}_{31}\equiv\begin{vmatrix} \frac{a_3}{k} & \frac{a_1}{k} \\ \frac{b_3}{k} & \frac{b_1}{k} \end{vmatrix},\quad \mathfrak{X}_{12}\equiv\begin{vmatrix} \frac{a_1}{k} & \frac{a_2}{k} \\ \frac{b_1}{k} & \frac{b_2}{k} \end{vmatrix}$$

And it is evident that there exists the following relations:

$$\mathfrak{X}_{ij}=-\mathfrak{X}_{ji}\qquad (i,\ j=0,\ 1,\ 2,\ 3)$$

---

[1] T. Nishiuchi, loc. cit., p. 293.

$$\Omega_0(\mathscr{X}^k\mathscr{X}_{ij})\equiv\mathscr{X}_1\mathscr{X}_{23}+\mathscr{X}_2\mathscr{X}_{31}+\mathscr{X}_3\mathscr{X}_{12}=0,$$
$$\Omega_1(\mathscr{X}_k\mathscr{X}_{ij})\equiv\mathscr{X}_2\mathscr{X}_{30}+\mathscr{X}_3\mathscr{X}_{02}+\mathscr{X}_0\mathscr{X}_{23}=0,$$
$$\Omega_2(\mathscr{X}_k\mathscr{X}_{ij})\equiv\mathscr{X}_3\mathscr{X}_{01}+\mathscr{X}_0\mathscr{X}_{13}+\mathscr{X}_1\mathscr{X}_{30}=0,$$
$$\Omega_3(\mathscr{X}_k\mathscr{X}_{ij})\equiv\mathscr{X}_0\mathscr{X}_{12}+\mathscr{X}_1\mathscr{X}_{20}+\mathscr{X}_2\mathscr{X}_{01}=0,$$
$$(\mathscr{X}_{ij}\mid\mathscr{X}_{ij})\equiv\mathscr{X}_{01}\mathscr{X}_{23}+\mathscr{X}_{02}\mathscr{X}_{31}+\mathscr{X}_{03}\mathscr{X}_{12}=0.$$

It is easily seen that $\mathscr{X}_{ij}$ $(i, j=0,\ 1,\ 2,\ 3)$ represent the axial coordinates of the plane-axis[1] of the circle and also represent the radial coordinates of the space axis.[2]

III.

Suppose we have two circles Γ and Γ' with Study's coordinates $(\mathscr{X})$ and $(\mathscr{X}')$, then their mutual power[3] $P(\Gamma, \Gamma')$ will be

$$\begin{aligned}P(\Gamma\Gamma')&=(aa')(bb')-(ab')(a'b)-k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb')\\&=k^4[\Sigma\mathscr{X}_i\mathscr{X}'_i+\Sigma\mathscr{X}_{ij}\mathscr{X}'_{ij}]\\&\equiv k^4[\mathscr{X}\mathscr{X}']\end{aligned}$$

and the condition that two non-coplanar circles should be cospherical is that

$$(\mathscr{X}_{ij}\mid\mathscr{X}'_{ij})=0,$$
$$\Omega_l(\mathscr{X}_k\mathscr{X}'_{ij})=\Omega_l(\mathscr{X}'_k\mathscr{X}_{ij})$$
$$(l=0,\ 1,\ 2,\ 3).$$

The coordinates of the points of the intersection of two non-coplanar and non-cospherical circles are given by

$$\Omega_0(\mathscr{X}_k\mathscr{X}'_{ij})-\Omega_0(\mathscr{X}'_k\mathscr{X}_{ij}):\Omega_1(\mathscr{X}_k\mathscr{X}'_{ij})-\Omega_1(\mathscr{X}'_k\mathscr{X}_{ij}):$$
$$:\Omega_2(\mathscr{X}_k\mathscr{X}'_{ij})-\Omega_2(\mathscr{X}'_k\mathscr{X}_{ij}):\Omega_3(\mathscr{X}_k\mathscr{X}'_{ij})-\Omega_3(\mathscr{X}'_k\mathscr{X}_{ij}).$$

And the conditions that two circles intersect at two points will be

$$(\mathscr{X}_{ij}\mid\mathscr{X}'_{ij})=0$$
$$\Omega_l(\mathscr{X}_k\mathscr{X}'_{ij})-\Omega_l(\mathscr{X}'_k\mathscr{X}_{ij})=0$$
$$(l=0,\ 1,\ 2,\ 3).$$

These are the conditions that the two circles should be cospherical.

The coordinates of the center of the common orthagonal sphere

---

1, 2, 3 T. Nishiuchi, loc. cit.

of the two non-coplanar and non-cospherical circles are also

$$\lambda[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]$$

where λ is proportional factor and

$$l=0,\ 1,\ 2,\ 3.$$

Now let R be the radius of common orthogonal sphere, then

$$\cos\frac{R}{k}=\frac{\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a'_0 & a'_1 & a'_2 & a'_3 \\ b'_0 & b'_1 & b'_2 & b'_3 \end{vmatrix}}{k\sqrt{\begin{Vmatrix} a_0\cos\frac{r'}{k}-a'_0\cos\frac{r}{k} & \cdots\cdots\cdots & a_3\cos\frac{r'}{k}-a'_3\cos\frac{r}{k} \\ b_0 & \cdots\cdots\cdots & b_3 \\ b'_0 & & b'_3 \end{Vmatrix}^2}}$$

$$=\frac{(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})}{\sqrt{\sum[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]^2}}$$

and

$$\tan^2\frac{R}{k}=\frac{\sum[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]^2-(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2}{(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2}.$$

If we put

$$\sum[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]^2-(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2\equiv(\mathfrak{X}\mid\mathfrak{X}')^2$$

then

$$\tan^2\frac{R}{k}=\frac{(\mathfrak{X}\mid\mathfrak{X}'^2)}{(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2}.$$

The condition that two non-coplaner and non-cospherical circles should interest is that

$$(\mathfrak{X}\mid\mathfrak{X}')^2=0.$$

The condition that two circles should be in involution is

$$[\mathfrak{X}\mathfrak{X}']=\sum\mathfrak{X}_i\mathfrak{X}'_i+\sum\mathfrak{X}_{ij}\mathfrak{X}'_{ij}=0.$$

For bi-involution we shall have

$$(\mathfrak{X}_i\mathfrak{X}'_i)\equiv\sum\mathfrak{X}_i\mathfrak{X}'_i=0,$$

$$\sum \mathfrak{X}_{ij} \begin{vmatrix} \mathfrak{X}_i \mathfrak{X}_j \\ \mathfrak{X}' \mathfrak{X}'_j \end{vmatrix} = \sum \mathfrak{X}'_{ij} \begin{vmatrix} \mathfrak{X}_i \mathfrak{X}_j \\ \mathfrak{X}'_i \mathfrak{X}'_j \end{vmatrix} = o$$

and

$$(\mathfrak{X}_{ij} \mid \mathfrak{X}'_{ij}) + (\mathfrak{X}_i \mathfrak{X}_i)(\mathfrak{X}'_i \mathfrak{X}'_i) - (\mathfrak{X}_{ij} \mathfrak{X}_{ij})(\mathfrak{X}'_{ij} \mathfrak{X}'_{ij}) = o.$$

Let

$$\begin{Vmatrix} \frac{a_0}{k} & \frac{a_1}{k} & \frac{a_2}{k} & \frac{a_3}{k} & i\cos\frac{r}{k} \\ \frac{b_0}{k} & \frac{b_1}{k} & \frac{b_2}{k} & \frac{b_3}{k} & o \end{Vmatrix}$$

are the coordinates of a circle. Then the coordinates of the spheres passing through the circle are

$$\lambda\frac{a_0}{k} + \mu\frac{b_0}{k},\ \lambda\frac{a_1}{k} + \mu\frac{a_1}{k},\ \lambda\frac{a_2}{k} + \mu\frac{b_2}{k},\ \lambda\frac{a_3}{k} + \mu\frac{b_3}{k},\ \lambda i\cos\frac{r}{k},$$

where λ and μ are proportional factors.

Hence the coordinates of the foci of the circle (i.e. the null-sphere passing through the circle) are

$$(x) \equiv \frac{a_0}{k} + i\frac{b_0}{k}\sin\frac{r}{k},\ \frac{a_1}{k} + i\frac{b_1}{k}\sin\frac{r}{k},\ \ldots\ldots.,\ \frac{a_3}{k} + i\frac{b_3}{k}\sin\frac{r}{k},\ i\cos\frac{r}{k},$$

$$(y) \equiv \frac{a_0}{k} - i\frac{b_0}{k}\sin\frac{r}{k},\ \frac{a_1}{k} - i\frac{b_1}{k}\sin\frac{r}{k},\ \ldots\ldots.,\ \frac{a_3}{k} - \frac{b_3}{k}\sin\frac{r}{k},\ i\cos\frac{r}{k},$$

and that of the polar circles are

$$(\bar{x}) \equiv \frac{a_0}{x} + i\frac{b_0}{k}\cos\frac{r}{k},\ \frac{a_1}{k} + i\frac{b_1}{k}\cos\frac{r}{k},\ \ldots\ldots,\ \frac{a_3}{k} + i\frac{b_3}{k}\cos\frac{r}{k},\ i\sin\frac{r}{k},$$

$$(\bar{y}) \equiv \frac{a_0}{k} - i\frac{b_0}{k}\cos\frac{r}{k},\ \frac{a_1}{k} - i\frac{b_1}{k}\cos\frac{r}{k},\ \ldots\ldots.,\ \frac{a_3}{k} - i\frac{b_3}{k}\cos\frac{r}{k},\ i\sin\frac{r}{k}.$$

When two circles have the same plane-axis (consequently they have the same space-axis), it is evident that

$$\mathfrak{X}_{ij} = \mathfrak{X}'_{ij}$$

and they intersect one another on the absolute.

IV.

Definition—Two cospherical circles are said to be orthogonal when they are in involution.

Next we shall call *common perpendicular* to two non-cospherical and non-coplaner circles, a circle cospherical and orthogonal to both.

Let (x), (y) be the coordinates of foci of the first circle and (x') (y') be those of the second circles, whose coordinates are

$$\left\| \begin{matrix} \frac{a_0}{k} & \frac{a_1}{k} & \cdots\cdots & i\cos\frac{r}{k} \\ \frac{b_0}{k} & \frac{b_1}{k} & \cdots\cdots & 0 \end{matrix} \right\|$$

and

$$\left\| \begin{matrix} \frac{a'_0}{k} & \frac{a'_1}{k} & \cdots\cdots & i\cos\frac{r'}{k} \\ \frac{b'_0}{k} & \frac{b'_1}{k} & \cdots\cdots & 0 \end{matrix} \right\|$$

respectively.

If such a common perpendicular be determined by the sphere $\lambda(x)+\mu(y)$, $\lambda'(x')+\mu'(y')$, there must be some sphere through each circle that is orthogonal to each of these spheres. (Since the common perpendicular and each of circles are in involution). This will require that $\lambda(x)-\mu(y)$ be orthogonal to $\lambda'(x')+\mu'(y')$ and $\lambda'(x')-\mu'(y')$ orthogonal to $\lambda(x)+\mu(y)$. Hence we have

$$\lambda\lambda'(xx')-\mu\mu'(yy')=0$$
$$\mu\lambda'(yx')-\lambda\mu'(xy')=0.$$

Eliminating $\frac{\lambda'}{\mu'}$ and $\frac{\lambda}{\mu}$,

$$\lambda^2(xx')(xy')-\mu^2(yx')(yy')=0$$
$$\lambda'^2(xx')(yx')-\mu'^2(xy')(yy')=0.$$

Hence, we have two pairs of spheres

$$\sqrt{(yx')}\sqrt{(yy')}\,(x)\pm\sqrt{(xx')}\sqrt{(xy')}\,(y),$$
$$\sqrt{(y'x)}\sqrt{(yy')}\,(x')\pm\sqrt{(x'x)}\sqrt{(x'y)}\,(y')$$

whose intersections gives the common perpendiculars of two given circles. And they are orthogonal to one another.

Theorem. There are two pairs of common perpendiculars to two non-cospherical circles, and two circles of different pairs are in bi-involution.

We shall define as the *angles of two circles* those of two pairs of spheres.

If θ_1 and θ_2 are the angles of those spheres, we find

$$\cos\theta_1^2\cos\theta_2^2=\frac{\begin{vmatrix}(xx') & (xy')\\ (yx') & (yy')\end{vmatrix}^2}{(xy)^2(x'y')^2}$$

$$=\frac{1}{(xy)^2(x'y')^2}\left[\begin{Vmatrix}x_0\,x_1\,x_2\,x_3\,x_4\\ y_0\,y_1\,y_2\,y_3\,y_4\end{Vmatrix}\cdot\begin{Vmatrix}x'_0\,x'_1\,x'_2\,x'_3\,x'_4\\ y'_0\,y'_1\,y'_2\,y'_3\,y'_4\end{Vmatrix}\right]$$

$$=\frac{1}{\sin^4\frac{r}{k}\sin^4\frac{r'}{k}}\left[\begin{Vmatrix}\frac{a_0}{k} & \cdots\cdots & \frac{a_3}{k} & i\cos\frac{r}{k}\\ i\frac{b_0}{k}\sin\frac{r}{k} & \cdots\cdots i\frac{b_3}{k}\sin\frac{r}{k} & & 0\end{Vmatrix}\times\right.$$

$$\left.\times\begin{Vmatrix}\frac{a'_0}{k} & \cdots\cdots & i\cos\frac{r'}{k}\\ i\frac{b'_0}{k}\sin\frac{r'}{k} & \cdots\cdots & 0\end{Vmatrix}\right]^2$$

$$=\frac{1}{\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}\begin{vmatrix}\frac{(aa')}{k^2}-\cos\frac{r}{k}\cos\frac{r'}{k} & i\frac{(ab')}{k^2}\\ i\frac{(a'b)}{k^2} & -\frac{(bb')}{k^2}\end{vmatrix}^2$$

$$=\frac{\left[(aa')(bb')-(ab')(a'b)-k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb')\right]^2}{k^8\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}$$

$$=\frac{P(\Gamma,\Gamma')^2}{k^8\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}$$

$$=\frac{[\mathfrak{X}\mathfrak{X}']^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}.$$

$$\cos^2\theta_1+\cos^2\theta_2=\frac{2[(xx')(yy')+(yx')(xy')]}{(xy)(x'y')}.$$

$$\sin^2\theta_1\sin^2\theta_2=1-(\cos^2\theta_1+\cos^2\theta_2)+\cos^2\theta_1\cos^2\theta_1$$

$$=\frac{\begin{Vmatrix}x_0 & x_1 & x_2 & x_3 & x_4\\ y_0 & y_1 & y_2 & y_3 & y_4\\ x'_0 & x'_1 & x'_2 & x'_3 & x'_4\\ y'_0 & y'_1 & y'_2 & y'_3 & y'_4\end{Vmatrix}^2}{(xy)^2(x'y')^2}$$

$$=\frac{1}{\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}\left\|\begin{matrix}\frac{a_0}{k} & \frac{a_1}{k} & \cdots\cdots & \frac{a_3}{k} & i\cos\frac{r}{k}\\ \frac{b_0}{k} & \frac{b_1}{k} & \cdots\cdots & \frac{b_3}{k} & 0\\ \frac{a'_0}{k} & \frac{a'_1}{k} & \cdots\cdots & \frac{a'_3}{k} & i\cos\frac{r'}{k}\\ \frac{b'_0}{k} & \frac{b'_1}{k} & \cdots\cdots & \frac{b'_3}{k} & 0\end{matrix}\right\|^2$$

$$=\frac{1}{\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}\left[\left|\frac{a}{k}\ \frac{a'}{k}\ \frac{b}{k}\ \frac{b'}{k}\right|^2+\left\|\begin{matrix}\frac{a_0}{k}\cos\frac{r'}{k}-\frac{a'_0}{k}\cos\frac{r}{k} & \cdots\\ \frac{b_0}{k} & \cdots\cdots\\ \frac{b'_0}{k} & \cdots\cdots\end{matrix}\right\|^2\right]$$

$$=\frac{(\mathfrak{X}\mid\mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\cos^2\theta_1+\cos^2\theta_2=\frac{[\mathfrak{X}'\mathfrak{X}]^2-(\mathfrak{X}\mid\mathfrak{X}')^2+[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\theta_1+\sin^2\theta_2=\frac{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']-[\mathfrak{X}\mathfrak{X}']^2+(\mathfrak{X}\mid\mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

V.

Here, we shall discuss only about the oriented circles.

It is evident that two oriented circles have two common perpendiculars. Let θ_1 and θ_2 be their angles then we have

$$\cos^2\theta_1\cos^2\theta_2=\frac{[\mathfrak{X}\mathfrak{X}']^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']},$$

$$\sin^2\theta_1\sin^2\theta_2=\frac{(\mathfrak{X}\mid\mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']},$$

$$\cos^2\theta_1+\cos^2\theta_2=\frac{[\mathfrak{X}\mathfrak{X}']^2-(\mathfrak{X}\mid\mathfrak{X}')^2+[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\theta_1+\sin^2\theta_2=\frac{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']-[\mathfrak{X}\mathfrak{X}']^2+(\mathfrak{X}\mid\mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

Hence, $\cos\theta_1$, $\cos\theta_2$ and $\sin\theta_1$, $\sin\theta_2$ are the roots of the equations.

$$[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\cos^4\theta+\Big[(\mathfrak{X}\mid\mathfrak{X}')^2-[\mathfrak{X}\mathfrak{X}']^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big]\cos^2\theta$$
$$+[\mathfrak{X}\mathfrak{X}']^2=0$$

$$[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\sin^4\theta+\Big[[\mathfrak{X}\mathfrak{X}']^2-(\mathfrak{X}\mid\mathfrak{X}')^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big]\sin^2\theta$$
$$+(\mathfrak{X}\mid\mathfrak{X}')^2=0.$$

The condition that these angles should be equal or supplementary is

$$\Big[(\mathfrak{X}\mid\mathfrak{X}')^2-[\mathfrak{X}\mathfrak{X}']^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big]^2$$
$$-4[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}'][\mathfrak{X}\mathfrak{X}']^2=0.$$

or

$$\Big\{\Big[(\mathfrak{X}\mid\mathfrak{X}')+[\mathfrak{X}\mathfrak{X}']\Big]^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big\}$$
$$\Big\{\Big[(\mathfrak{X}\mid\mathfrak{X}')-[\mathfrak{X}\mathfrak{X}']\Big]^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big\}=0.$$

When two circles intersect on the absolute and touch one another, we shall call these circles are *parallel.*

These conditions will be expressed by the equations

$$\Big|\; a \;\; a' \;\; b \;\; b' \;\Big|^2=\Big\|\; a\cos\frac{r'}{k}-a'\cos\frac{r}{k} \;\; b \;\; b' \;\Big\|^2$$
$$=k^8\sin^2\frac{r}{k}\sin^2\frac{r'}{k}-\Big(P(\Gamma\Gamma')\Big)^2=0$$

or

$$(\mathfrak{X}\mid\mathfrak{X}')^2=(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})$$
$$=[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']-[\mathfrak{X}\mathfrak{X}']^2=0.$$

When two circles are parallel then their space axes and also plane axes intersect to each other and their points of intersection is on the absolute. Hence we have the theorem.

Theorem. When two circles are parallel their space-axis and the plane axes are respectively either parallel or psodoparallel.

From this theorem the condition of parallelism will be

$$\begin{vmatrix} a & a' & b & b' \end{vmatrix}^2 = \begin{vmatrix} a\cos\frac{r'}{k} - a'\cos\frac{r}{k} & b & b' \end{vmatrix}^2$$

$$= k^8 \sin^2\frac{r}{k}\sin^2\frac{r'}{k} - \left(P(\Gamma\Gamma')\right)^2$$

$$k^8 - \left[(aa')(bb') - (ab')(a'b)\right]^2 = 0.$$

or

$$(\mathfrak{X} \mid \mathfrak{X}')^2 = (\mathfrak{X}_{ij} \mid \mathfrak{X}'_{ij})$$

$$= [\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}'] - [\mathfrak{X}\mathfrak{X}']^2$$

$$= (\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij}) - (\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2 = 0.$$

From these conditions we have

$$\left[\pm k^4 - k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb')\right]^2 = k^8\sin\frac{r}{k}\sin^2\frac{r'}{k}$$

or

$$\left[\pm k^4 - k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb') - k^4\sin\frac{r}{k}\sin\frac{r'}{k}\right]$$

$$\left[\pm k^4 - k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb') + k^4\sin\frac{r}{k}\sin\frac{r'}{k}\right] = 0.$$

Hence we have the theorem :

Theorem. When two circles are parallel, if their radii is equal to zero or differ by πk, then the planes of circles are parallel or two circles are coplanar and vice versa.

d_1, d_2 which satisfy the following equations are called the *distances of two oriented circles* whose coordinate are $(\mathfrak{X})$, $(\mathfrak{X}')$

$$\cos^2\frac{d_1}{k}\cos^2\frac{d_2}{k} = \frac{[\mathfrak{X}\mathfrak{X}']}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\frac{d_1}{k}\sin^2\frac{d_2}{k} = \frac{(\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\cos^2\frac{d_1}{k} + \cos^2\frac{d_2}{k} = \frac{[\mathfrak{X}\mathfrak{X}'] - (\mathfrak{X} \mid \mathfrak{X}')^2 + [\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\frac{d_1}{k} + \sin^2\frac{d_2}{k} = \frac{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}'] - [\mathfrak{X}\mathfrak{X}']^2 + (\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

We thus get the following equations for the distances d of the two circles $(\mathfrak{X})$, $(\mathfrak{X}')$

$$
(A)\quad
\begin{aligned}
&[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\cos^4\frac{d}{k} \\
&\qquad+\Big[(\mathfrak{X}\mid\mathfrak{X}')^2-[\mathfrak{X}\mathfrak{X}']^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big]\cos^2\frac{d}{k}+[\mathfrak{X}\mathfrak{X}']^2=0, \\
&[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\sin^4\frac{d}{k} \\
&\qquad+\Big[[\mathfrak{X}\mathfrak{X}']^2-(\mathfrak{X}\mid\mathfrak{X}')^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big]\sin^2\frac{d}{k}+(\mathfrak{X}\mid\mathfrak{X}')^2=0.
\end{aligned}
$$

The square roots of the products of the roots of those two equations shall be called *commoment*[1] and *moment*[2] of two circles respectively.

If two circles intersect the moment must be zero and if their mutual powers are equal to zero the commoment must be vanished.

Two circles shall be said to be *paratactic*[3] when their distances and angles are all congruent.

If two circles be paratactic the condition may be expressed analytically by equating to zero the discriminant of either of our equations (A).

$$
\Big\{\Big[(\mathfrak{X}\mid\mathfrak{X}')+[\mathfrak{X}\mathfrak{X}']\Big]^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big\}
\times\Big\{\Big[(\mathfrak{X}\mid\mathfrak{X}')-[\mathfrak{X}\mathfrak{X}']\Big]^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\Big\}=0.
$$

This puts in evidence that intersecting circles cannot be paratatic unless they be parallel.

If

$$
\left(a+ib\sin\frac{r}{k}\quad a'+ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}
$$

$$
\left(a-ib\sin\frac{r}{k}\quad a'-ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}
$$

or

$$
\left(a+ib\sin\frac{r}{k}\quad a'-ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}
$$

$$
\left(a-ib\sin\frac{r}{k}\quad a'+ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}
$$

1, 2, 3 This word is adopted in analogy to that of in Line Geometry. See D'Ovidio, 'Studio sulla geometria proiettiva,' Annali di Mathematica, **6**, 1873, and see Study, 'Zur Nicht-Euclidischen und Linien- Geometrie,' Jahresber. D. M. Ver., **11**, 1902.

i.e. the focus of one is orthogonal to that of the other circle, then the equation above will be satisfied. Hence we get the theorem.

THEOREM. *If a focus of one circle is orthogonal to the other, two circles are paratactic.*

If the radius of the both circles be $\frac{\pi}{2}k$, then the circles will be two straight lines whose Plücker's coordinates are $(\mathfrak{X}_{ij})$ and $(\mathfrak{X}'_{ij})$ respectively, and the equations (A) will be

$$(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\sin^4\frac{d}{k} + \left[(\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2-(\mathfrak{X}_{ij} \mid \mathfrak{X}'_{ij})^2-(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\right]\sin^2\frac{d}{k} + (\mathfrak{X}_{ij} \mid \mathfrak{X}'_{ij})^2=0.$$

$$(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\cos^4\frac{d}{k} + \left[(\mathfrak{X}_{ij} \mid \mathfrak{X}_{ij}')^2-(\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2-(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\right]\cos^2\frac{d}{k} + (\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2=0.$$

These are the equations which give the distances between the two lines $(\mathfrak{X}_{ij})$, $(\mathfrak{X}'_{ij})$[1].

VI.

Lastly, we shall express the pseudo-moment[2] and pseudo-commonent[3] of two circles by the study's coordinates. Let d_1' and d_2' be the pseudo distances of two circles then the square of the pseudo moment will be

$$\sin^2\frac{d_1'}{k}\sin^2\frac{d_2'}{k}=\frac{|\, a \;\, a' \;\, b \;\, b' \,|^2\tan^4\frac{R}{k}}{\left(\sin^2\frac{r}{k}+\cos^2\frac{r}{k}\tan^2\frac{r}{k}\right)\left(\sin^2\frac{r'}{k}+\cos^2\frac{r'}{k}\tan^2\frac{R}{K}\right)}$$

$$=\frac{k^4\left[(\mathfrak{X} \mid \mathfrak{X}')^2\right]^2}{\begin{vmatrix}(\mathfrak{X}_{ij} \mid \mathfrak{X}_{ij}')^2 & (\mathfrak{X}_i\mathfrak{X}_i) \\ (\mathfrak{X} \mid \mathfrak{X}')^2 & [\mathfrak{X}\mathfrak{X}]\end{vmatrix}\cdot\begin{vmatrix}(\mathfrak{X}_{ij} \mid \mathfrak{X}_{ij}')^2 & (\mathfrak{X}_i'\mathfrak{X}_i') \\ (\mathfrak{X} \mid \mathfrak{X}')^2 & [\mathfrak{X}'\mathfrak{X}']\end{vmatrix}}$$

And the square of their pseudo-commoment will be

---

1 Coolidge, Non-Euclidean Geometry.

2 3 T. Nischiuchi, loc. cit.

$$\cos^2\frac{d_1'}{k}\cos^2\frac{d_2'}{k}=\frac{\left\{P(\Gamma\Gamma')+(bb')\cos\frac{r}{k}\cos\frac{r'}{k}\tan\frac{R}{k}\right\}^2}{\left\{\sin^2\frac{r}{k}+\cos^2\frac{r}{k}\tan^2\frac{R}{K}\right\}\left\{\sin^2\frac{r'}{k}\cos^2\frac{r'}{k}\tan^2\frac{R}{K}\right\}}$$

$$=\frac{k^4\begin{vmatrix}(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2 & (\mathfrak{X}_i\mathfrak{X}'_i)\\ (\mathfrak{X}\mid\mathfrak{X}')^2 & [\mathfrak{X}\mathfrak{X}']\end{vmatrix}^2}{\begin{vmatrix}(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2 & (\mathfrak{X}_i\mathfrak{X}_i)\\ (\mathfrak{X}\mid\mathfrak{X}')^2 & [\mathfrak{X}\mathfrak{X}]\end{vmatrix},\begin{vmatrix}(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2 & (\mathfrak{X}_i'\mathfrak{X}_i')\\ (\mathfrak{X}\mid\mathfrak{X}')^2 & [\mathfrak{X}'\mathfrak{X}']\end{vmatrix}}$$

In conclusion, I wish to discuss, in the next paper, the systems of circles and their differential Geometry in non-euclidean space by the aid of Study's coordinates.

The author wishes to express his sincere thanks to Prof. T. Nishiuchi for his kind guidance and valuable remarks.